Local operators on $C^p$ ∗

Denny H. Leung a,*, Ya-Shu Wang b

a Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Singapore
b Department of Mathematics, National Central University, Chungli 32054, Taiwan, ROC

A R T I C L E   I N F O

Article history:
Received 1 November 2010
Available online 6 April 2011
Submitted by K. Jarosz

Keywords:
Local operators
Vector-valued smooth functions

A B S T R A C T

Let $X$ be an open set in a Banach space $G$ on which there is a $C^p$ bump function with bounded derivatives. We present a complete characterization of local (i.e., support shrinking) linear operators $T: C^p(X, E) \to C^q(X, F)$, where $E$ and $F$ are Banach spaces.

© 2011 Elsevier Inc. All rights reserved.

The aim of the present paper is to characterize local operators from $C^p(X, E)$ to $C^q(X, F)$, where $p, q$ may be allowed to be finite. When $G$ is finite dimensional and $E = F = \mathbb{R}$, the problem was considered in [7]. (Actually, the more general case of disjointness preserving operators was studied there.) See also [2,4] for related results. The example below shows that the direct carry-over of Theorem 1 is no longer true. Specifically, while $T$ retains the form of the representation, $\alpha_k$ need no longer be $C^q$ as a map into the space $L(S^k(G, E), F)$.

Theorem 1. (See [11].) Suppose that $G$ supports a $C^\infty$ bump function with Lipschitz derivatives, then every local operator $T: C^\infty(X, E) \to C^\infty(X, F)$ has the form $Tf(x) = \sum_{k=0}^{\infty} \alpha_k(x)(D^k f(x))$, where $\alpha_k \in C^\infty(X, L(S^k(G, E), F))$ and the supports of the $\alpha_k$ form a locally finite collection.

Example. There is a non-$C^1$ map $\alpha: \mathbb{R} \to (\ell^2)^*$ so that $Tf(x) = \alpha(x)(f(x))$ defines a local operator from $C^1(\mathbb{R}, \ell^2)$ to $C^1(\mathbb{R}, \mathbb{R})$.

Proof. Let $(a_n)$, $(b_n)$ be positive null sequences with $b_{n+1} < a_n < b_n$ for all $n$ and let $(h_n)$ be a sequence of continuous real-valued functions on $\mathbb{R}$ so that $h_n$ is supported inside $(a_n, b_n)$, $\|h_n\| \leq 1$, $h_n(x_n) = 1$ for some $x_n \in (a_n, b_n)$, $\int_{a_n}^{b_n} h_n = 0$, and $\int_{a_n}^{b_n} |h_n| \leq 1$. Define $g_n$ by $g_n(x) = \int_0^x h_n$. Then $(g_n)$ is a sequence in $C^1(\mathbb{R})$, each $g_n$ is supported inside $(a_n, b_n)$, and...
\(|g_n(x)| \leq |x| \wedge 1\) for all \(n \in \mathbb{N}\) and \(x \in \mathbb{R}\). In particular, letting \((e_n)\) be the unit vector basis of \(\ell^2\), \(\sum g_n(x)e_n, x \in \mathbb{R}\), and \(\sum \frac{e_n}{n^2}e_n, x \neq 0\), are both contained in the ball of \(\ell^2\). Same goes for \(\sum g_n(x)e_n = \sum h_n(x)e_n\). Define \(\alpha : \mathbb{R} \to (\ell^2)^*\) by \(\alpha(x)(\sum a_ne_n) = \sum a_n g_n(x)\). If \(\alpha \in C^1(\mathbb{R}, (\ell^2)^*)\), then it is clear that 

\[
\alpha'(x) \left( \sum a_ne_n \right) = \sum a_n \frac{g_n'(x)}{x} \left( \sum a_n h_n(x) \right)
\]

and that \(\lim_{x \to 0} \alpha'(x) = \alpha'(0) = 0\) in the norm of \((\ell^2)^*\). However, \(\|\alpha'(x)\| = \|h_n(x)\| = 1\) for all \(n\). Thus \(\alpha\) is not \(C^1\). It remains to show that \(\alpha(x)f(x) \in C^1(\mathbb{R})\) for any \(f \in C^1(\mathbb{R}, \ell^2)\). Write \(f(x) = \sum a_n(x)e_n\). Then each \(a_n \in C^1(\mathbb{R})\) and \(\alpha(x)f(x) = \sum a_n(x)g_n(x)\). If \(x \neq 0\), there is a neighborhood \(U\) of \(x\) on which at most one term in the sum is not identically 0. Hence 

\[
\frac{d}{dx} \left[ \alpha(x)f(x) \right] = \sum \left[ a_n(x)g_n'(x) + a_n(x)g_n(x) \right],
\]

where the sum contains at most one nonzero term on \(U\). The continuity of \(\frac{d}{dx} [\alpha(x)f(x)]\) on \(\mathbb{R} \setminus \{0\}\) is then clear as well. We claim that the derivative of \(\alpha(x)f(x)\) at 0 is 0. Indeed, let \(\varepsilon > 0\) be given. Since \(f([-1, 1])\) is compact in \(\ell^2\), there exists \(n_0 \in \mathbb{N}\) so that \(\| \sum_{n=n_0+1}^{\infty} a_n(x)e_n \| < \varepsilon\) for all \(x \in [-1, 1]\). Observe that \(\alpha(0)f(0) = 0\). For \(x \in [-1, 1]\), \(x \neq 0\),

\[
\left| \frac{\alpha(x)f(x)}{x} \right| \leq \sum_{n=1}^{n_0} \frac{a_n(x)g_n(x)}{x} + \sum_{n=n_0+1}^{\infty} \frac{a_n(x)e_n}{x} = \sum_{n=1}^{n_0} \frac{a_n(x)g_n(x)}{x} + \varepsilon.
\]

But \(g_n'(0) = 0\). Hence \(\lim sup_{x \to 0} |\alpha(x)f(x)/x| \leq \varepsilon\). Since \(\varepsilon > 0\) is arbitrary, it follows that \(\frac{d}{dx}[\alpha(x)f(x)]|_{x=0} = 0\). The proof of the continuity of \(\frac{d}{dx}[\alpha(x)f(x)]\) at 0 follows along the same lines, recalling that \(\sum g_n(x)e_n\) and \(\sum g_n'(x)e_n\) both have norm at most 1, and noting that \(f([-1, 1])\) and \(f'([-1, 1])\) are compact in \(\ell^2\).}

Let \(N_p = \{ n \in \mathbb{N} \cup \{0\} : n \leq p \}\). Assume that there is a function \(\varphi \in C^p(G)\) so that \(\varphi(x) = 1\) on a neighborhood of 0, \(\varphi(x) = 0\) if \(\|x\| \geq 1\) and, for all \(i \in N_p\),

\[
C_i = \sup_{x \in G} \| D^i \varphi(x) \| < \infty.
\]

When \(p < \infty\), this assumption is possibly weaker than the assumption of a \(C^p\) bump function with Lipschitz derivatives; although we do not know of an example to distinguish the two. Good references for matters concerning differentiation on Banach spaces/manifolds are [5,6].

The main goal of the paper is Theorem 11. The proof is roughly divided into two parts. The first part (up to Proposition 5) is to obtain the representation of the local operator \(T\) (Theorem 11(d)). The proof of this part follows along the same path used in [11], via a concentration argument (Lemma 2). However, it seems to us that the proof of Lemma 1 in [11] is incomplete, since it does not take into account the case where all \(x_0\)'s are equal to \(x_0\). This gap is filled by Proposition 3. In the second half, we elaborate on the properties of the component maps of the representation. A key result is Proposition 9. Part of its proof depends on an application of the Closed Graph Theorem (Proposition 6). For the use of the Closed Graph Theorem in this context, see [1,3,4].

**Lemma 2.** Let \((x_n)\) be a sequence in \(X\) converging to a point \(x_0 \in X\). Suppose that \(\|x_n - x_0\| = r_n\), and that \(0 < 3r_{n+1} < r_n\) for all \(n \in \mathbb{N}\). Assume that \(0 < s_n \leq r_n/2\) for each \(n\) and let \(\varphi_n(x) = \varphi\left( \frac{1}{s_n} (x - x_0) \right)\). Let \((f_n)\) be a sequence in \(C^p(X, E)\) and let

\[
\eta_n = \sum_{j=0}^{k_n} \left( \frac{1}{s_n} \right)^j C_i \sup_{x \in B(x_n, s_n)} \| D^{k-j} f_n(x) \|
\]

for all \(n \in \mathbb{N}\) and all \(k \in N_p\). If

\[
\lim_{n \to \infty} \eta_n = 0, \quad 0 \leq k < p,
\]

(1)

and

\[
\lim_{n \to \infty} \eta_{np} = 0, \quad \text{in case } p < \infty,
\]

(2)

then the pointwise sum \(f = \sum \varphi_n f_n\) belongs to \(C^p(X, E)\), and \(D^k f(x_0) = 0, k \in N_p\).
Proof. Obviously, each \( \varphi_n f_n \) is \( C^p \). If \( x \neq x_0 \), then there is an open neighborhood of \( x \) on which \( f \) is equal to one of these terms or is identically 0. Thus \( f \) is \( C^p \) on \( X \setminus \{x_0\} \). We first prove by induction that \( D^k f (x_0) = 0 \) for all \( k \in \mathbb{N}_p \). The case \( k = 0 \) is trivial. Assume that the claim has been verified for some \( k \), \( 0 \leq k < p \). If \( x \notin \bigcup B(x_0, s_n) \), then \( D^k f (x) = 0 \). Suppose that \( x \in B(x_n, s_n) \) for some \( n \). Then

\[
\left\| D^k f (x) - D^k f (x_0) \right\| = \left\| D^k f (x) \right\| = \left\| D^k (\varphi_n f_n)(x) \right\|
\leq \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) \left\| D^j \varphi_n (x) \right\| \left\| D^{k-j} f_n(x) \right\|
\leq \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) \frac{C_j}{s_n^j} \left\| D^{k-j} f_n(x) \right\| \leq \eta_{nk}.
\]

Also, \( \|x - x_0\| \geq r_n - s_n = \frac{r_n}{2} \) for all \( x \in B(x_0, s_n) \). It follows from condition (1) that \( D^{k+1} f (x_0) = 0 \). If \( p < \infty \), we need to show that \( D^p f \) is continuous at \( x_0 \). But the same computation shows that \( \|D^p f (x) - D^p f (x_0)\| \leq \eta_{np} \) if \( x \in B(x_n, s_n) \) for some \( n \) and \( 0 \) otherwise. The continuity of \( D^p f \) at \( x_0 \) follows from condition (2). \( \Box \)

From hereon, consider a given local operator \( T : C^p (X, E) \rightarrow C^q (X, F) \), where \( 1 \leq p, q \leq \infty \).

Proposition 3. For each \( x_0 \in X \), there exists \( k_0 = k_0 (x_0) \in \mathbb{N}_p \), so that \( T f (x_0) = 0 \) for every \( f \in C^p (X, E) \) with \( D^j f (x_0) = 0, 0 \leq j \leq k_0 \).

Proof. Suppose that the proposition fails. For each \( n \in \mathbb{N} \), there exists \( f_n \in C^p (X, E) \) so that \( D^j f_n(x_0) = 0, 0 \leq j \leq n \land p \), and \( \|T f_n (x_0)\| > n \). Note that we have, for \( 0 \leq j \leq k \leq n \land p \),

\[
\lim_{x \to x_0} \frac{\|D^{k-j} f_n(x)\|}{\|x - x_0\|^p + j - k} = 0.
\]

Choose \( (r_n)_{n=1}^{\infty} \) so that \( r_n > 3r_{n+1} > 0 \) for all \( n \), \( \|T f_n(x)\| > n \) if \( x \in B(x_0, 3r_n/2) \), and

\[
\frac{\|D^{k-j} f_n(x)\|}{\|x - x_0\|^p + j - k} \leq \frac{1}{n}
\]

if \( 0 \leq j \leq k \leq n \land p \) and \( 0 < \|x - x_0\| < 3r_n/2 \). Finally, set \( s_n = r_n/2 \) and pick \( x_0 \) so that \( \|x_0 - x_0\| = r_n \). Thus \( \|x - x_0\| < 3r_n/2 \) if \( x \in B(x_n, s_n) \). With \( \eta_{nk} \) as defined in Lemma 2, we have the estimate

\[
\eta_{nk} \leq \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) C_j \frac{3r_n}{2}^{n \land p - k} \frac{1}{n}.
\]

Thus (1) and (2) of Lemma 2 are fulfilled. It follows that \( f = \sum \varphi_n f_n \in C^p (X, E) \). Since \( f = f_n \) on a neighborhood of \( x_0 \), \( T f (x_0) = T f_n (x_0) \). This implies that \( \|T f (x_0)\| > n \) for all \( n \), contradicting the continuity of \( T f \) at \( x_0 \). \( \Box \)

If \( x \in G \), let \( x^k \) be the element \( (\ldots, x, \ldots) \) \((k\) terms\) in \( G^k \).

Corollary 4. There are functions \( \Phi_k : X \times S^k (G, E) \rightarrow F, k \in \mathbb{N}_p \), so that

\[
T f (x) = \sum_k \Phi_k (x, D^k f (x))
\]

for each \( f \in C^p (X, E) \) and each \( x \in X \). For every \( x \in X \), \( \Phi_k (x, \cdot) : S^k (G, E) \rightarrow F \) is a linear operator, which is nonzero for only finitely many \( k \)'s (the number of nonzero terms depending on \( x \)).

Proof. Let \( x_0 \in X \) and \( k_0 = k_0 (x_0) \) be given by Proposition 3. For \( k \leq k_0 \), define \( \Phi_k (x_0, S) = T f_S (x_0) \), \( S \in S^k (G, E) \), where \( f_S (x) = S (x - x_0)^k/k! \). Set \( \Phi_k (x, \cdot) = 0 \) if \( k > k_0 \). For any \( f \in C^p (X, E) \), it follows from Proposition 3 that \( T f (x_0) = T f (x) \), where \( P \) is the \( k_0 \)-th Taylor polynomial of \( f \) at \( x_0 \), \( P (x) = \sum_{j=0}^{k_0} \frac{1}{j!} D^j f(x_0) (x - x_0)^j \). Thus \( T f (x_0) = \sum_k \Phi_k (x_0, D^k f (x_0)) \). \( \Box \)

Proposition 5. For each \( x_0 \in X \), there exist \( k_0 \in \mathbb{N}_p \) and \( \varepsilon > 0 \) so that \( \Phi_k (x, \cdot) = 0 \) for all \( x \in B(x_0, \varepsilon) \) and all \( k > k_0 \).
Proof. The statement being trivial if $p < \infty$, we may assume that $p = \infty$. Assuming that the proposition fails, we find a sequence $(x_n)$ in $\mathbb{N}$ diverging to $\infty$ and a sequence $(x_0)$ in $X$ converging to $x_0$ so that $\langle f_n, x_0 \rangle \neq 0$ for all $n$. By Corollary 4, no $x_n$ may be repeated an infinite number of times. Thus we may assume that $0 < 3r_{n+1} < r_n$ for all $n$, where $r_n = \|x_n - x_0\|$. Choose $S_n = \mathbb{D}^k(G, E)$ so that $\|\Phi_k(x_n, S_n)\| = 1$ for each $n$. Define $f_n(x) = S_n(x - x_0)^{k_n}$, $x \in X$. Then $f_n \in C^p(X, E)$. Pick $s_n$, $0 < s_n \leq r_n / 2$, so that $\lim_{n \to \infty} \|S_n\|^k_n s_n^{k_n + j - k} = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Using the estimate

$$\sup_{x \in B(x_0, s_n)} \|D^{k-j} f_n(x)\| \leq \|S_n\|^k_n s_n^{k_n + j - k}$$

and noting that $2/r_n \leq 1/s_n$, we obtain condition (1) of Lemma 2 from the choice of $s_n$. By Lemma 2, $f = \sum \psi_n f_n \in C^p(X, E)$. For each $u$, $(a)$, $(b)$, $(c)$, $(d)$ hold. Thus, for each $x$, $Tf(x_n) = Tf_n(x_n) = k_n! \Phi_k(x_n, S_n)$. As this sequence is unbounded, we have a contradiction to the continuity of $Tf$ at $x_0$. □

The results above give the representation of $T$. Next, we investigate the properties of the maps $\Phi_k$. Recall that $\mathbb{N}_p = \{n \in \mathbb{N} \cup \{0\} : n \leq p\}$. Define $\mathbb{N}_q$ similarly.

Proposition 6. Let $K$ be a compact, perfect subset of $X$. For any $i \in \mathbb{N}_p$, $j \in \mathbb{N}_q$, define $T_{ij} : \mathcal{S}^i(G, E) \to C(K, \mathcal{S}^j(G, F))$ by

$$T_{ij} = D^j(Tf_S)_k,$$

where $f_S$ is the function in $C^p(X, E)$ given by $f_S(x) = Sx^i$. The map $T_{ij}$ has closed graph and hence is bounded.

Proof. Suppose that $(m_n)$ is a sequence in $\mathcal{S}^i(G, E)$ converging to 0 so that $T_{ij}(m_n)$ converges to a function $g \in C(K, \mathcal{S}^j(G, F))$. Let $x_0 \in K$. It suffices to show that $g(x_0) = 0$. Since $K$ is perfect, there exists a sequence $(x_n)$ in $K$ converging to $x_0$ so that $0 < 3r_{n+1} < r_n < 1$ for all $n$, where $\|x_n - x_0\| = r_n$. Choose an increasing sequence $(m_n)$ so that

$$\lim_{n \to \infty} \frac{\|S_n\|}{r_n^{k+1}} = 0, \quad 0 \leq k \leq p.$$

Define $f_n(x) = n S_n x^i$ and set $s_n = r_n / 2$. If $x \in B(x_n, s_n)$ and $i \geq k + l$, we have the estimate

$$\|D^{k-l} f_n(x)\| \leq n ! \|S_n\| \|x\|^{i-k-l} \leq n ! \|S_n\| \left( \|x_0\| + \frac{3r_n}{2} \right)^{i-k+l}.$$

Thus

$$\eta_{nk} \leq \frac{n \|S_n\|}{r_n^{k+1}} \sum_{l=(k+1)}^k \frac{k!}{l!} 2^l C_l \left( \|x_0\| + \frac{3}{2} \right)^{i-k+l}.$$

Hence both (1) and (2) of Lemma 2 hold by choice of $(m_n)$ and thus $f = \sum \psi_n f_n \in C^p(X, E)$. Since $f = f_n$ on a neighborhood of $x_n$, $Tf = Tf_n$ on a neighborhood of $x_n$ and thus $D^j(Tf)(x_n) = nD^j(Tf_n)(x_n) = T_{ij}(S_m)$. Since $f_n$ converges uniformly to $g$ on $K$ and $(x_n)$ converges to $x_0$ in $K$, $(T_{ij}(S_m))$ converges to $g(x_0)$. Therefore, $g(x_0) = \lim_{n \to \infty} D^j(Tf)(x_n) / n = 0$. □

An important tool is Taylor’s Formula, which we recall below. See, e.g., [8, p. 11].

Proposition 7 (Taylor’s Formula). Let $E$ be a Banach space and let $f : X \to X$ be a $C^q$ function. If $x$ and $y$ are points in $X$ so that the segment joining them lies in $X$, then, for each $i \in \mathbb{N}_q$,

$$f(x + y) = \sum_{r=0}^{i-1} \frac{D^r f(x) y^r}{r!} + \int_0^1 (1 - t)^{i-1} \frac{D^i f(x + ty) y^i}{i!} dt.$$

Proposition 8. Let $E$ be a Banach space and suppose that $\Phi : X \times X \to F$ has the following properties:

(a) For each $u \in X$, $\Phi(\cdot, u) : X \to F$ belongs to $C^q(X, F)$. Denote the $j$-th derivative of this function by $\Phi^{(j)}(\cdot, u)$.

(b) For each $x \in X$ and each $j \in \mathbb{N}_q$, $\Phi^{(j)}(x, \cdot) : X \to \mathcal{S}^j(G, F)$ is a bounded linear operator.

(c) For each $x_0 \in X$ and each $j \in \mathbb{N}_q$, there exists $\varepsilon > 0$ so that

$$\sup_{x \in B(x_0, \varepsilon)} \sup_{|u| \leq 1} \|\Phi^{(j)}(x, u)\| < \infty.$$
If \( f : X \to X \) is a \( C^q \) function on \( X \), then \( \Psi : X \to F \) defined by \( \Psi(x) = \Phi(x, f(x)) \) belongs to \( C^q(X, F) \). For each \( n \in \mathbb{N}_q \) and each \( x \in X \),

\[
D^n\Psi(x)y^n = \sum_{j=0}^{n} \binom{n}{j} \Phi^{(j)}(x, D^{n-j}f(x)y^{n-j})y^j, \quad y \in G.
\]  

(3)

**Proof.** First, we prove Eq. (3) by induction on \( n \). The case \( n = 0 \) holds by definition. Assume that it holds for some \( n \), \( 0 \leq n < q \). Let \( x \in X \) be given. Choose \( \varepsilon > 0 \) and \( M < \infty \) so that

\[
\max_{0 \leq j \leq n+1} \sup_{\|z\| \leq \varepsilon} \sup_{\|u\| \leq 1} \|\Phi^{(j)}(x + z, u)\| \leq M.  
\]  

(4)

Suppose that \( \|z\| \leq \varepsilon \), \( 0 \leq j \leq n \) and \( \|y\| \leq 1 \). Then

\[
I \overset{\text{def}}{=} \|\Phi^{(j)}(x + z, D^{n-j}f(x)y^{n-j})y^j - \Phi^{(j)}(x, D^{n-j+1}f(x)y^{n-j})y^j\|
\]

\[
\leq M\|D^{n-j}f(x)z\|\|y\| \quad \text{by (4), for all } y \text{ with } \|y\| \leq 1
\]

Thus

\[
\lim_{z \to 0} \sup_{\|y\| \leq 1} \frac{1}{\|z\|} = 0.
\]  

(5)

By Taylor's Formula,

\[
II \overset{\text{def}}{=} \|\Phi^{(j)}(x + z, D^{n-j+1}f(x)y^{n-j})y^j - \Phi^{(j)}(x, D^{n-j+1}f(x)y^{n-j})y^j\|
\]

\[
\leq \int_0^1 \|\Phi^{(j+1)}(x + tz, D^{n-j+1}f(x)y^{n-j})z^j\| \, dt
\]

\[
\leq M\|D^{n-j+1}f(x)\|\|z\| \quad \text{by (4), for all } y \text{ with } \|y\| \leq 1
\]

\[
\leq M\|D^{n-j+1}f(x)\|\|z\|^2.
\]

Hence

\[
\lim_{z \to 0} \sup_{\|y\| \leq 1} \frac{II}{\|z\|} = 0.
\]  

(6)

Finally, consider the term

\[
III \overset{\text{def}}{=} \|\Phi^{(j)}(x + z, D^{n-j}f(x)y^{n-j})y^j - \Phi^{(j)}(x, D^{n-j}f(x)y^{n-j})y^j - \Phi^{(j+1)}(x, D^{n-j}f(x)y^{n-j})y^j\|
\]

If \( j = n \), then

\[
III = \|\Phi^{(n)}(x + z, f(x))y^n - \Phi^{(n)}(x, f(x))y^n - \Phi^{(n+1)}(x, f(x))zy^n\|.
\]

Thus \( \lim_{z \to 0} \sup_{\|y\| \leq 1} III/\|z\| = 0 \) by definition of \( \Phi^{(n+1)}(\cdot, f(x)) \). Suppose that \( j < n \). Then by Taylor's Formula,

\[
III \leq \int_0^1 (1 - t) \|\Phi^{(j+2)}(x + tz, D^{n-j}f(x)y^{n-j})z^j\| \, dt \leq M\|D^{n-j}f(x)\|\|z\|^2 \quad \text{by (4), for all } y \text{ with } \|y\| \leq 1.
\]

So, for \( 0 \leq j \leq n \), we have

\[
\lim_{z \to 0} \sup_{\|y\| \leq 1} \frac{III}{\|z\|} = 0.
\]  

(7)

Summing the terms I, II and III over the range \( 0 \leq j \leq n \), and using (5), (6) and (7) show that

\[
\lim_{z \to 0} \sup_{\|y\| \leq 1} \frac{D^n\Psi(x + z)y^n - D^n\Psi(x)y^n - \Theta(z, y)}{\|z\|} = 0,
\]

where

\[
\Theta(z, y) = \sum_{j=0}^{n} \binom{n}{j} \left[ \Phi^{(j)}(x, D^{n-j+1}f(x)y^{n-j})y^j + \Phi^{(j+1)}(x, D^{n-j}f(x)y^{n-j})zy^j \right].
\]

Therefore, we obtain formula (3) with \( n \) replaced by \( n + 1 \).
We also need to prove the continuity of $D^q\Psi$ when $q$ is finite. Choose $\varepsilon > 0$ so that condition (4) holds for $n + 1 = q$. Suppose that $0 \leq j \leq q$. Let $u = D^{(q-j)}(x+z)y^qj$ and $v = D^{q-j}f(x)y^qj$. For $\|z\| < \varepsilon$,

$$\sup_{\|y\| \leq 1} \|\Phi(j)(x+z, u - v)y^j\| \leq M \|D^{q-j}f(x+z) - D^{q-j}f(x)\| \to 0$$

(8)

as $z \to 0$. If $j = q$, then $v = f(x)$. Hence

$$\sup_{\|y\| \leq 1} \|\Phi(j)(x+z, v) - \Phi(j)(x, v)\|y^j\| = \|\Phi(q)(x+z, f(x)) - \Phi(q)(x, f(x))\| \to 0$$

(9)

as $z \to 0$. Finally, if $0 \leq j < q$, then

$$\sup_{\|y\| \leq 1} \|\Phi(j)(x+z, v) - \Phi(j)(x, v)\|y^j\| \leq \sup_{\|y\| \leq 1} \int_0^1 \|\Phi(j+1)(x+tz, v)y^j\|dt \leq M \|D^{q-j}f(x)\|\|z\| \to 0$$

(10)

as $z \to 0$. Combining (8) with (9) or (10) shows that, for any $j \leq q$,

$$\lim_{z \to 0} \Phi(j)(x+z, u) - \Phi(j)(x, v) = 0.$$

The continuity of $D^q\Psi$ follows from formula (3). \qed

Define $p - q = \infty$ if $p = \infty$ (regardless of $q$). We re-examine the operators $\Phi_k$ from Corollary 4.

**Proposition 9.** Suppose that $p \geq q$ and let $k \in \mathbb{N}_{p-q}$. For each $S \in \mathcal{S}^q(G, E)$, $\Phi_k(-, S) : X \to F$ belongs to $C^q(X, F)$. Denote its $j$-th derivative by $\Phi^{(j)}(\cdot, S)$. For every $x \in X$, $k \in \mathbb{N}_{p-q}$ and $j \in \mathbb{N}_q$, the linear operator $\Phi^{(j)}_k(x, \cdot) : \mathcal{S}^q(G) \to \mathcal{S}^j(G, F)$ is bounded. Moreover, for all $x_0 \in X$, all $k \in \mathbb{N}_{p-q}$, and $f \in \mathbb{N}_q$, there exists $\varepsilon > 0$ so that

$$\sup_{x \in B(x_0, \varepsilon)} \|\Phi^{(j)}_k(x, S)\| < \infty.$$  

(11)

For any $k \in \mathbb{N}_{p-q}$ and any $C^q$ function $g : X \to \mathcal{S}^k(G, E)$, the function $x \mapsto \Phi_k(x, g(x)) : X \to F$ is $C^q$ on $X$.

**Proof.** We prove the proposition by induction on $k \in \mathbb{N}_{p-q}$. Suppose that the proposition has been established up to and including $k - 1$. ($k = 0$ is allowed here, in which case the supposition is vacuous.) For any $S \in \mathcal{S}^k(G, E)$, let $f_S(x) = Sx^k$. Then,

$$(Tf_S)(x) = k!\Phi_k(x, S) + \sum_{i=0}^{k-1} \Psi_i(x),$$

(12)

where $\Psi_i(x) = \Phi_i(x, D^{(i)}f_S(x))$. Note that $D^{(i)}f_S \in C^q$. By the inductive hypothesis, $\Psi_i$ is $C^q$, $0 \leq i \leq k - 1$. Hence, $\Phi_k(-, S)$ belongs to $C^q$. Moreover, from Eq. (3) and condition (11) (with $i$ in place of $k$), one can deduce easily that for $0 \leq i \leq k - 1$, $j \in \mathbb{N}_q$, and any fixed $x$, the map $S \mapsto D^{(i)}\Psi_i(x)$ is a bounded linear operator, and that, for any $x_0$, there exists $\varepsilon > 0$ so that the norms of these maps are uniformly bounded for $x \in B(x_0, \varepsilon)$. By Proposition 6, for any compact perfect subset $K$ of $X$, there is a finite constant $M$ so that $\sup_{x \in K} \|D^{(i)}(Tf_S)(x)\| \leq M\|S\|$. It follows from (12) that for any compact perfect subset $K$ of $B(x_0, \varepsilon)$, $\sup_{x \in K} \|\Phi^{(j)}_k(x, \cdot)\| < \infty$. If condition (11) fails for $k$, there is a sequence $(x_i)$ in $X$ converging to $x_0$ such that $\sup_{x \in B(x_0, \varepsilon)} \|\Phi^{(j)}_k(x, S)\| = \infty$. We may assume that $x_i \in B(x_0, \varepsilon)$ for all $i$. For each $i$, let $[u_i, v_i]$ be a non-degenerate interval in $B(x_0, \varepsilon)$ containing $x_i$ with $\lim_{j \to \infty} \|u_j - v_j\| = 0$. Then $K = [x_0] \cup \bigcup_{i=1}^{\infty} [u_i, v_i]$ is a compact perfect subset of $B(x_0, \varepsilon)$. Hence $\sup_{x \in B(x_0, \varepsilon)} \|\Phi^{(j)}_k(x, S)\| = \sup_{x \in K} \|\Phi^{(j)}_k(x, \cdot)\| < \infty$, reaching a contradiction. Thus, we obtain condition (11) for $k$. Finally, it follows from Proposition 8 that the map $x \mapsto \Phi_k(x, g(x))$ is $C^q$ for any $C^q$ function $g : X \to \mathcal{S}^k(G, E)$. This completes the induction. \qed

**Proposition 10.** If $p \geq q$, then $\Phi_k = 0$ for all $k > p - q$. If $p < q$, then $\Phi_k = 0$ for all $k$.

**Proof.** If possible, suppose that $\Phi_{k_0} \neq 0$, where $k_0 \in \mathbb{N}_p$ and $k_0 > p - q$ if $p \geq q$. There are $x_0 \in X$ and $S \in \mathcal{S}^{k_0}(G, E)$ such that $\Phi_{k_0}(x_0, S) \neq 0$. Choose $\bar{q} \in \mathbb{N}_q$ so that $k_0 + \bar{q} > p$. Let $u$ be a normalized vector in $G$ and let $(r_n)$ be a real sequence so that $0 < 3r_{n+1} < r_n$ for all $n$. Set $s_n = r_n/4$, $x_n = x_0 + r_n u$, $f_n(x) = r_n^2 S(x-x_0)$, and $\psi_n(x) = \varphi(1/3n(x-x_0))$. Making use of the estimate

$$\sup_{x \in B(x_0, s_n)} \|D^{j-k}(f_n(x))\| \leq \begin{cases} \frac{r_n}{3n} \|S\|k_0 \|s_n^{k_0-k+j} & \text{if } j \geq k - k_0, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\sup_{x \in B(x_0, s_n)} \|\Phi^{(j)}_k(x, S)\| \leq M \|D^{j-k}(f_n(x))\| \to 0$$

as $n \to \infty$. This is a contradiction. Therefore, $\Phi_k = 0$ for all $k$.
Consequently, any local operator $T$. We see that the limit in (13) is equal to

$$\lim_{n \to \infty} \|Tf(x_n)\| = 0.$$  

(13)

However, $f = f_n$ on a neighborhood of each $x_n$ and hence $Tf(x_n)/r_n^q = Tf_n(x_n)/r_n^q = k_0!\Phi_{k_0}(x_n, S)$. By continuity of $\Phi_{k_0}(. S)$, we see that the limit in (13) is equal to $k_0!\|\Phi_{k_0}(x_n, S)\| \neq 0$. The contradiction thus obtained completes the proof.  

---

Endow $C^p(X, E)$ with the linear topology determined by the seminorms

$$\rho_{k, j}(f) = \sup_{x \in X} \sup_{0 \leq i \leq j} \|D^i f(x)\|,$$

where $K$ is a compact subset of $X$ and $j \in \mathbb{N}_p$. Topologize $C^q(X, F)$ in the same manner. The results above lead to the main theorem.

**Theorem 11.** Let $T : C^p(X, E) \to C^q(X, F)$ be a local operator, where $1 \leq p, q \leq \infty$. If $p < q$, then $T = 0$. If $p \geq q$, there are functions $\Phi_k : X \times S^q(G, E) \to F$, $k \in \mathbb{N}_{p-q}$, so that:

(a) $\Phi_k$ is a $C^q$ function of the first variable. Denote the $j$-th derivative of $\Phi_k(\cdot, S)$ by $\Phi_k^{(j)}(\cdot, S)$. For each $x$, $\Phi_k^{(j)}(x, \cdot) : S^q(G, E) \to S^j(G, E)$, $k \in \mathbb{N}_{p-q}$, $j \in \mathbb{N}_q$, is a bounded linear operator.

(b) For each $x_0 \in X$, there exist $k_0 \in \mathbb{N}_{p-q}$ and $\varepsilon > 0$ so that $\Phi_k(x_0, \cdot) = 0$ for all $x \in B(x_0, \varepsilon)$ and all $k > k_0$.  

(c) For each $x_0 \in X$, every $k \in \mathbb{N}_{p-q}$ and $j \in \mathbb{N}_q$, there exists $\varepsilon > 0$ so that

$$\sup_{x \in B(x_0, \varepsilon)} \sup_{\|S\| \leq 1} \|\Phi_k^{(j)}(x, S)\| < \infty.$$  

(d) $Tf(x) = \sum_{k=0}^{p-q} k_0!\Phi_k(x, D^k f(x))$ for all $f \in C^p(X, E)$ and all $x \in X$.

Consequently, any local operator $T : C^p(X, E) \to C^q(X, F)$ is continuous. Conversely, if $T$ is given by (d), where the functions $\Phi_k$ satisfy conditions (a)-(c), then $T$ is a local operator from $C^p(X, E)$ into $C^q(Y, F)$.

**References**


