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Duality for unbounded order convergence and applications

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Abstract Unbounded order convergence has lately been systematically studied as a generalization of almost everywhere convergence to the abstract setting of vector and Banach lattices. This paper presents a duality theory for unbounded order convergence. We define the unbounded order dual (or uo-dual) $X_{\simuo}$ of a Banach lattice $X$ and identify it as the order continuous part of the order continuous dual $X_{\simn}$. The result allows us to characterize the Banach lattices that have order continuous preduals and to show that an order continuous predual is unique when it exists. Applications to the Fenchel–Moreau duality theory of convex functionals are given. The applications are of interest in the theory of risk measures in Mathematical Finance.

Keywords Unbounded order dual · Order continuous dual · Order continuous predual · Monotonically complete Banach lattices · Representation of convex functionals

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1 Introduction

Let $X$ be a vector lattice. A net $(x_\alpha)$ in $X$ is said to order converge to $x \in X$, written as $x_\alpha \longrightarrow^\alpha x$, if there is another net $(y_\gamma)$ in $X$ such that $y_\gamma \downarrow 0$ and that for every $\gamma$, there exists $\alpha_0$ such that $|x_\alpha - x| \leq y_\gamma$ for all $\alpha \geq \alpha_0$. If $X$ is a lattice ideal of the space of all real-valued measurable functions $L^0(\Omega, \Sigma, \mu)$ on a measure space $(\Omega, \Sigma, \mu)$, then order convergence of a sequence in $X$ is equivalent to dominated almost everywhere convergence. For obvious reasons, both theoretically and in applications, a generalization of a.e. convergence to the abstract setting of vector lattices is of much interest. Motivated by this, the concept of unbounded order convergence or uo-convergence has recently been intensively studied in several papers [6,8,9]. A net $(x_\alpha)$ is said to unbounded order converge (uo-converge) to $x \in X$ if $|x_\alpha - x| \wedge y \longrightarrow 0$ for any $y \in X_+$. In this case, we write $x_\alpha \longrightarrow_{uo} x$. It is indeed easy to verify that if $X$ is a lattice ideal of $L^0(\Omega, \Sigma, \mu)$, then a sequence $(f_n)$ in $X$ uo-converges to $f \in X$ if and only if it converges a.e. to $f$.

When $X$ is a Banach lattice, there is a well known duality theory associated with order convergence. For instance, a linear functional $\phi$ on $X$ is said to be order continuous if $\phi(x_\alpha) \rightarrow 0$ for any net $(x_\alpha)$ in $X$ that order converges to 0. The set $X_\alpha^\circ$ of all order continuous linear functionals on $X$ is called the order continuous dual of $X$. It is a band (i.e., order closed lattice ideal) in $X^*$, and $X^* = X_\alpha^{\sim}$ if and only if $X$ is order continuous, i.e., $\|x_\alpha\| \rightarrow 0$ for any net $(x_\alpha)$ in $X$ that order converges to 0 ([13, Theorem 2.4.2]).

One of the aims of the present paper is to develop a duality theory for uo-convergence. Besides its intrinsic interest, a strong motivation derives from the representation theory of risk measures and convex functionals in Mathematical Finance. The following result was obtained by Gao and Xanthos in [10].

**Theorem 1.1** ([10, Theorem 2.4]). If an Orlicz space $L^\Phi$ is not equal to $L^1$, then the following statements are equivalent for every proper (i.e., not identically $\infty$) convex functional $\rho : L^\Phi \rightarrow (-\infty, \infty]$.

1. $\rho(f) = \sup_{g \in H^\Psi} \left( \int f g - \rho^*(g) \right)$ for any $f \in L^\Phi$, where $H^\Psi$ is the conjugate Orlicz heart, and $\rho^*(g) = \sup_{f \in L^\Phi} \left( \int f g - \rho(f) \right)$ for any $g \in H^\Psi$.
2. $\rho(f) \leq \liminf_n \rho(f_n)$, whenever $f_n \longrightarrow a.e.$ $f$ and $(f_n)$ is norm bounded in $L^\Phi$.

Condition (2) in Theorem 1.1 suggests the following definition. Let $X$ be a Banach lattice. A linear functional $\phi$ on $X$ is said to be boundedly uo-continuous if $\phi(x_\alpha) \rightarrow 0$ whenever $x_\alpha \longrightarrow_{uo} 0$ and $(x_\alpha)$ is norm bounded. The set of all boundedly uo-continuous functionals on $X$ is called the uo-dual of $X$ and is denoted by $X_{uo}^{\sim}$.

The definition of boundedly uo-continuous functionals is closely connected to that of order continuous functionals. Recall that a net $(x_\alpha)$ in a vector lattice $X$ is said to be order bounded if there exists $x \in X_+$ such that $|x_\alpha| \leq x$ for all $\alpha$. For any net $(x_\alpha)$ in $X$, it is easily seen that $x_\alpha \longrightarrow^\alpha 0$ if and only if $x_\alpha \longrightarrow_{uo} 0$ and a tail of $(x_\alpha)$
is order bounded. Thus a linear functional $\phi$ on $X$ is order continuous if and only if $\phi(x_\alpha) \to 0$ whenever $x_\alpha \uo \to 0$ and $(x_\alpha)$ is order bounded.

In Sect. 2, we show that $X\sim_\uo$ is precisely the order continuous part of $X\sim_n$. As a consequence, it is deduced that any Banach lattice can have at most one order continuous predual up to lattice isomorphism, namely, $X\sim_\uo$. In Sect. 3, a characterization is given of precisely when $X = (X\sim_\uo)^*$. In the final section, we apply these results to prove a theorem (Theorem 4.6) on dual representation of convex functionals in the general setting of Banach lattices with order continuous preduals. Theorem 4.6 generalizes Theorem 1.1 to the abstract setting.

We adopt [1,2] as standard references for unexplained terminology and facts on vector and Banach lattices. We will frequently use the following fact. Recall that a sublattice $Y$ of a vector lattice $X$ is said to be regular if $y_\alpha \downarrow 0$ in $Y$ implies $y_\alpha \downarrow 0$ in $X$. In this case, by [8, Theorem 3.2], for any net $(y_\alpha)$ in $Y$,

$$y_\alpha \uo \to 0 \text{ in } Y \iff y_\alpha \uo \to 0 \text{ in } X.$$ 

Ideals and order dense sublattices are regular. Order continuous norm closed sublattices of a Banach lattice are also regular.

2 Characterization of the uo-dual

**Definition 2.1** Let $X$ be a Banach lattice. A linear functional $\phi$ on $X$ is said to be boundedly uo-continuous if $\phi(x_\alpha) \to 0$ for any norm bounded uo-null net $(x_\alpha)$ in $X$. The set of all boundedly uo-continuous linear functionals on $X$ will be called the unbounded order dual, or uo-dual, of $X$, and will be denoted by $X\sim_\uo$.

The following proposition explains why the uo-dual is taken to be the set of all boundedly uo-continuous functionals rather than just the uo-continuous functionals. Recall first that a vector $x > 0$ in an Archimedean vector lattice $X$ is an atom if for any $u, v \in [0, x]$ with $u \wedge v = 0$, either $u = 0$ or $v = 0$. In this case, the band generated by $x$ is one-dimensional, namely, span$\{x\}$. Moreover, the band projection $P$ from $X$ onto span$\{x\}$ defined by

$$Pz = \sup_n (z^+ \wedge nx) - \sup_n (z^- \wedge nx)$$

exists, and there is a unique positive linear functional $\phi$ on $X$ such that $Pz = \phi(z)x$ for all $z \in X$. We call $\phi$ the coordinate functional of the atom $x$. Clearly, the span of any finite set of atoms is also a projection band. For any $\phi \in X\sim_n$, its null ideal and carrier are the bands of $X$ defined by

$$N_\phi := \{x \in X : |\phi||x|| = 0\} \quad \text{and} \quad C_\phi := N_\phi^d,$$

respectively. Note that $|\phi|$ acts as a strictly positive functional on $C_\phi$: if $0 < x \in C_\phi$, then $|\phi|(x) > 0$. 

Proposition 2.2 Let \( \phi \) be a nonzero linear functional on an Archimedean vector lattice \( X \) such that \( \phi(x_n) \rightarrow 0 \) whenever \( x_n \xrightarrow{uo} 0 \). Then \( \phi \) is a linear combination of the coordinate functionals of finitely many atoms.

Proof Note first that \( \phi \) is order continuous and is thus order bounded. We claim that \( C_\phi \) cannot contain an infinite disjoint sequence of nonzero vectors. Suppose otherwise that \( (u_n) \) is an infinite disjoint sequence of nonzero vectors in \( C_\phi \). Then \( |\phi|(|u_n|) > 0 \) for every \( n \geq 1 \). By Riesz–Kantorovich formula ([2, Theorem 1.18]), there exists \( v_n \in [-|u_n|, |u_n|] \) such that \( \phi(v_n) \neq 0 \). Since \( \left( \frac{v_n}{\phi(v_n)} \right) \) is disjoint and thus uo-null ([8, Corollary 3.6]), \( 1 = \phi \left( \frac{v_n}{\phi(v_n)} \right) \rightarrow 0 \), which is absurd. This proves the claim. It follows in particular that \( C_\phi \) contains at most finitely many disjoint atoms. Let \( A \) be the set of atoms of \( C_\phi \). Since \( A \) is a finite set (of atoms), the linear span \( B \) of \( A \) is a projection band in \( C_\phi \). If \( B \neq C_\phi \), there would exist \( 0 < x \in C_\phi \) such that \( x \perp B \). Since \( x \) is not an atom, there exist \( u_1, y \) such that \( 0 < u_1, y \leq x \) and \( u_1 \perp y \). Clearly, \( u_1, y \in C_\phi \). Since \( y \perp B \), \( y \) is not an atom, and thus there exist \( u_2, z \) such that \( 0 < u_2, z \leq y \) and \( u_2 \perp z \). Clearly, \( u_2, z \in C_\phi \). Repeating this process, we obtain an infinite disjoint sequence of nonzero vectors in \( C_\phi \), which contradicts the claim. Thus \( C_\phi = B \) is generated by finitely many atoms. The desired result follows immediately. \( \square \)

Since each order convergent net has a tail which is order bounded, and therefore, norm bounded, it is easy to see that \( X_{uo}^\sim \subset X_n^\sim \). The following theorem determines the precise position of \( X_{uo}^\sim \) in \( X_n^\sim \). Recall first that the order continuous part, \( X^a \), of a Banach lattice \( X \) is given by

\[
X^a = \{ x \in X : \text{ every disjoint sequence in } [0, |x|] \text{ is norm null} \}.
\]

For every \( x \in X \), it follows from applying [13, Corollary 2.3.6] to the ideal \( I = \bigcup_n [-n|x|, n|x|] \) of \( X \) that every disjoint sequence in \( [0, |x|] \) is norm null if and only if every increasing sequence in \( [0, |x|] \) is norm convergent. Thus

\[
X^a = \{ x \in X : \text{ every monotone sequence in } [0, |x|] \text{ is norm convergent} \}.
\]

By [13, Proposition 2.4.10], it is the largest norm closed ideal of \( X \) which is order continuous in its own right.

Theorem 2.3 Let \( X \) be a Banach lattice. Then \( X_{uo}^\sim \) is the order continuous part of \( X_n^\sim \). Specifically, for any \( \phi \in X_n^\sim \), the following statements are equivalent:

1. \( \phi \in X_{uo}^\sim \).
2. \( \phi(x_n) \rightarrow 0 \) for any norm bounded uo-null sequence \( (x_n) \) in \( X \).
3. \( \phi(x_n) \rightarrow 0 \) for any norm bounded disjoint sequence \( (x_n) \) in \( X \).
4. every disjoint sequence in \( [0, |\phi|] \) is norm null.

Observe that, since \( X_n^\sim \) is an ideal of \( X^* \), the interval \([0, |\phi|]\) is the same when taken in \( X_n^\sim \) or in \( X^* \) for any \( \phi \in X_n^\sim \).
Proof The implications $(1) \implies (2) \implies (3)$ are obvious since every disjoint sequence is uo-null by [8, Corollary 3.6]. Assume now that $(3)$ holds. By Riesz–Kantorovich formula, it can be easily verified that $|\phi(|x_n|)\to 0$ for any disjoint sequence $(x_n)$ in the closed unit ball $B_X$. Applying [2, Theorem 4.36] to the seminorm $|\phi(|\cdot|)$, the identity operator and $B_X$, we have that, for any $\varepsilon > 0$, there exists $u \in X_+$ such that

$$
\sup_{x \in B_X} |\phi((|x| - |u|) \wedge u) = \sup_{x \in B_X} |\phi((|x| - u)^+) \leq \varepsilon.
$$

If $(x_\alpha)$ is a uo-null net in $B_X$, then $|x_\alpha| \wedge u \overset{\alpha}{\to} 0$, so that $|\phi(|x_\alpha| \wedge u) \to 0$. Therefore,

$$
\limsup_{\alpha} |\phi(|x_\alpha|) \leq \varepsilon.
$$

By arbitrariness of $\varepsilon$, we have

$$
|\phi(x_\alpha)| \leq |\phi(|x_\alpha|) \to 0.
$$

This proves that $\phi \in X_\sim^{uo}$. Hence, $(3) \implies (1)$. The equivalence of $(3)$ and $(4)$ follows from [13, Theorem 2.3.3] with $A = B_X$ and $B = [-|\phi|, |\phi|]$, where the order interval is regarded as taken in $X^*$.

It follows, in particular, that $X^{\sim}_{uo}$ is a norm closed ideal of $X^{\sim}_n$ and of $X^*$ and is an order continuous Banach lattice itself.

Example 2.4 ($\ell^1)^{uo} = (\ell^\infty)^a = c_0$; $\ell^1)^{uo} = (\ell^1)^a = \ell^1$; $(\ell^\infty)^{uo} = (\ell^1)^a = \ell^1$.

Corollary 2.5 ([16, Theorem 5]). For a Banach lattice $X$, $X^{\sim}_{uo} = X^*$ iff $X$ and $X^*$ are both order continuous iff every norm bounded uo-null net is weakly null.

Proof $X^{\sim}_{uo} = X^*$ means precisely that every norm bounded uo-null net in $X$ is weakly null. Thus it suffices to prove the first “iff”. Assume that $X$ and $X^*$ are both order continuous. Then $X^{\sim}_n = X^*$, and $X^{\sim}_n$ is order continuous. By Theorem 2.3, $X^{\sim}_{uo} = X^{\sim}_n = X^*$. Conversely, assume that $X^{\sim}_{uo} = X^*$. In view of $X^{\sim}_{uo} \subset X^{\sim}_n \subset X^*$, it follows that $X^{\sim}_{uo} = X^* = X^{\sim}_n$. By the second equality, $X$ is order continuous. By the first equality and Theorem 2.3 again, $X^*$ is order continuous.

Recall that $X$ is identified as a norm closed sublattice of $X^{**}$ by the evaluation mapping

$$
j : X \to X^{**}; \ j(x)(\phi) = \phi(x),
$$

for any $x \in X$ and $\phi \in X^*$. In fact, $j(x) \in (X^*)_n^{\sim}$. Indeed, if $\phi_\alpha \downarrow 0$ in $X^*$, then $j(x)(\phi_\alpha) = \phi_\alpha(x) \to 0$ by [2, Theorems 1.18]. Thus $j(x)$ is order continuous on $X^*$ by [2, Theorem 1.56]. We identify $X$ as a norm closed sublattice of $(X^*)_n^{\sim}$. 

Proposition 2.6 For a Banach lattice $X$, $X \subset (X^*)_{uo}$ iff $X$ is order continuous iff $X = (X^*)_{uo}$. In particular, a Banach lattice has at most one order continuous lattice isomorphic predual, up to lattice isomorphism.

Proof It is well-known (and easy to verify) that a norm closed sublattice of an order continuous Banach lattice is order continuous in its own right. Thus by Theorem 2.3, if $X \subset (X^*)_{uo}$, then $X$ is order continuous. Suppose now that $X$ is order continuous. Since $X \subset (X^*)_n \subset X^{**}$ under the canonical identification, and $X$ is an ideal in $X^{**}$ [13, Theorem 2.4.2], it is also an ideal in $(X^*)_n$. Thus $X$ is contained in the order continuous part, $(X^*)_{uo}$ of $(X^*)_n$. Moreover, by [2, Theorem 1.70], $X$ is order dense in $(X^*)_n$, and therefore, in $(X^*)_{uo}$. This, together with order continuity of $(X^*)_{uo}$, implies that $X$ is also norm dense in $(X^*)_{uo}$. Therefore, since $X$ is norm closed in $(X^*)_{uo}$, $X = (X^*)_{uo}$.

For the last assertion, simply note that if $X$ is order continuous and $X^*$ is lattice isomorphic to $Y$, then $(X^*)_{uo}$ is lattice isomorphic to $Y_{uo}$. □

3 Banach lattices with order continuous preduals

In this section, we characterize the Banach lattices which have order continuous preduals. We begin with the following proposition which is of independent interest and generalizes [9, Theorem 4.7]. Recall first that a Banach lattice is said to be monotonically complete if every norm bounded positive increasing net has a supremum. A net $(x_\alpha)$ in a vector lattice $X$ is said to be order Cauchy, respectively, $uo$-Cauchy if the double net $(x_\alpha - x_\beta)_{\alpha,\beta}$ order converges to 0, respectively, $uo$-converges to 0 in $X$. Following [8], we say that a Banach lattice is boundedly $uo$-complete if every norm bounded $uo$-Cauchy net is $uo$-convergent.

Proposition 3.1 A monotonically complete Banach lattice $X$ is boundedly $uo$-complete. The converse is true if $X_n^-$ separates points of $X$.

Proof Let $X$ be a monotonically complete Banach lattice. Recall from [13, Proposition 2.4.19(i)] that $X$ is order complete and admits a constant $C$ such that

$$\|x\| \leq C \sup_\alpha \|x_\alpha\|$$

whenever $0 \leq x_\alpha \uparrow x$ in $X$.

Thus if $x_\alpha \oto x$ in $X$, then $y_\alpha := \inf_{\beta \geq \alpha} |x_\beta| \uparrow |x|$, and hence

$$\|x\| \leq C \sup_\alpha \|y_\alpha\| \leq C \sup_\alpha \|x_\alpha\|.$$

Now let $(x_\alpha)$ be any norm bounded $uo$-Cauchy net in $X$. By considering the positive and negative parts, respectively, we may assume that $x_\alpha \geq 0$ for each $\alpha$. For each $y \in X_+$, since $|x_\alpha \land y - x_\alpha \land y| \leq |x_\alpha - x_\alpha'| \land y$, the net $(x_\alpha \land y)$ is order Cauchy and hence order converges to some $u_y \in X_+$. The net $(u_y)_{y \in X_+}$ is directed upwards, and

$$\|u_y\| \leq C \sup_\alpha \|x_\alpha \land y\| \leq C \sup_\alpha \|x_\alpha\|$$
for all \( y \in X_+ \), by the preceding observation. Since \( X \) is monotonically complete, \((u, y)\) increases to an element \( u \in X \). Fix \( y \in X_+ \). For any \( \alpha, \alpha' \), define

\[
x_{\alpha, \alpha'} = \sup_{\beta \geq \alpha, \beta' \geq \alpha'} |x_{\beta} - x_{\beta'}| \wedge y.
\]

Since \((x_{\alpha})\) is uo-Cauchy, \(x_{\alpha, \alpha'} \downarrow 0\). Also, for any \( z \in X_+ \) and any \( \beta \geq \alpha, \beta' \geq \alpha' \),

\[
|x_{\beta} \wedge z - x_{\beta'} \wedge z| \wedge y \leq x_{\alpha, \alpha'}
\]

Taking order limit first in \( \beta' \) and then supremum over \( z \) in \( X_+ \), we obtain \( |x_{\beta} - u| \wedge y \leq x_{\alpha, \alpha'} \) for any \( \beta \geq \alpha \). This implies that \((x_{\alpha})\) uo-converges to \( u \).

For the second assertion, suppose that \( X \) is boundedly uo-complete and \( X_n^\sim \) separates points of \( X \). Let \((x_{\alpha})\) be a norm bounded increasing positive net in \( X \). Consider the evaluation mapping

\[
j : X \to (X_n^\sim)_n; \quad j(x)(\phi) = \phi(x),
\]

for any \( x \in X \) and \( \phi \in X_n^\sim \). By [2, Theorem 1.70], \( j \) is an (into) vector lattice isomorphism, and \( j(X) \) is an order dense, in particular regular, sublattice in \((X_n^\sim)_n \). Since \( \|j\| \leq 1 \), \((j(x_{\alpha}))\) is an increasing norm bounded positive net in \((X_n^\sim)_n \). By [13, Proposition 2.4.19], \((X_n^\sim)_n \) is monotonically complete, so that there exists \( x^{**} \in (X_n^\sim)_n \) such that

\[
j(x_{\alpha}) \uparrow x^{**} \quad \text{in} \quad (X_n^\sim)_n.
\]

In particular, \((j(x_{\alpha}))\) is order Cauchy, and therefore uo-Cauchy, in \((X_n^\sim)_n \). It follows from [8, Theorem 3.2] that \((j(x_{\alpha}))\) is uo-Cauchy in \( j(X) \). Since \( j \) is one-to-one and onto \( j(X), (x_{\alpha}) \) is uo-Cauchy in \( X \). Let \( x \) be the uo-limit of \((x_{\alpha})\) in \( X \). It is easy to check that \( x_{\alpha} \uparrow x \) in \( X \).

**Remark** Taylor [14, Theorem 2.6] has recently proved that the converse of Proposition 3.1 holds without the assumption that \( X_n^\sim \) separates points of \( X \).

We need two technical lemmas in preparation for the main result of this section.

**Lemma 3.2** Let \( X \) be a Banach lattice such that \( X_n^\sim \) separates points of \( X \), and let \( I \) be an ideal of \( X_n^\sim \). Then \( I \) is order dense in \( X_n^\sim \) iff it separates points of \( X \).

**Proof** The “only if” part is clear. For the “if” part, suppose that \( I \) separates points of \( X \) but is not order dense in \( X_n^\sim \). By [2, Theorem 1.36], there exists \( 0 < \phi \in X_n^\sim \) such that \( \phi \perp I \). By [2, Theorem 1.67], it follows that \( C_{\phi} \cap \bigcap_{\psi \in I} N_{\psi} = \{0\} \), where the last equality holds because \( I \) separates points of \( X \). Thus, \( C_{\phi} = \{0\} \), and \( N_{\phi} = C_{\phi} \) = \( X \), implying that \( \phi = 0 \), a contradiction. \( \square \)

The proof of the following lemma is inspired by that of [2, Theorem 1.65]. We provide the details for the convenience of the reader.
Lemma 3.3 If $Y$ is a norm closed order dense ideal of a Banach lattice $Z$, then $Z_n$ is lattice isometric to $Y_n$ via the restriction mapping.

**Proof** We put the restriction mapping $R : Z_n \rightarrow Y_n$ by $R\phi = \phi|_Y$ for any $\phi \in Z_n$. Since $Y$ is an ideal of $Z$, each order null net in $Y$ is also order null in $Z$. Thus $R\phi \in Y_n$ for any $\phi \in Z_n$, and $R$ indeed maps $Z_n$ into $Y_n$. For any $z \in Z$, by order denseness of $Y$, we can take two nets $(u_\alpha)$ and $(v_\alpha)$ in $Y$ such that $0 \leq u_\alpha \uparrow z^+$ and $0 \leq v_\alpha \uparrow z^-$ in $Z$. Let $y_\alpha = u_\alpha - v_\alpha$. Then $|y_\alpha| \leq |z|$ and $y_\alpha \rightarrow z$ in $Z$. It follows that

$$\|\phi\| = \sup_{z \in B_Z} |\phi(z)| = \sup_{y \in B_Y} |\phi(y)| = \|R\phi\|$$

for any $\phi \in Z_n$, where $B_Z$ and $B_Y$ are the closed unit balls of $Z$ and $Y$, respectively. This proves that $R$ is an isometry.

We now show that $R$ is surjective. Pick any $\psi \in Y_n$. By considering $\psi^\pm$, we may assume that $\psi \geq 0$. For any $z \in Z_+$, put

$$\phi(z) := \sup \{\psi(y) : y \in Y, 0 \leq y \leq z\};$$

the supremum is finite because $0 \leq \psi(y) \leq \|\psi\|\|y\| \leq \|\psi\|\|z\|$. It is clear that $\phi(z_1) + \phi(z_2) \leq \phi(z_1 + z_2)$ for any $z_1, z_2 \in Z_+$. Conversely, by Riesz Decomposition Theorem, it is easy to see that $\phi(z_1 + z_2) \leq \phi(z_1) + \phi(z_2)$. Thus

$$\phi(z_1) + \phi(z_2) = \phi(z_1 + z_2)$$

for any $z_1, z_2 \in Z_+$. By Kantorovich Extension Theorem ([2, Theorem 1.10]), $\phi$ determines a linear functional on $Z$ by setting

$$\phi(z) = \phi(z^+) - \phi(z^-)$$

for any $z \in Z$. We show that $\phi \in Z_n$. Take any $0 \leq z_\alpha \uparrow z$ in $Z$. Put $D = \{y \in Y_+ : y \leq z_\alpha \text{ for some } \alpha\}$. Then $D \uparrow z$, and $\sup_\alpha \phi(z_\alpha) = \sup \psi(D) \leq \phi(z)$. For any $y \in Y_+$ with $y \leq z$, $D \wedge y \uparrow z \wedge y = y$ in $Z$ and thus also in $Y$. Therefore, $\psi(y) = \sup \psi(D \wedge y) \leq \sup \psi(D)$. Hence, $\phi(z) \leq \sup \psi(D) = \sup_\alpha \phi(z_\alpha)$, and consequently, $\phi(z) = \sup_\alpha \phi(z_\alpha)$. It follows that $\phi \in Z_n$. Clearly, $R\phi = \psi$.

Finally, for any $\phi \in Z_n$, it is clear that $R\phi \geq 0$ iff $\phi \geq 0$. Therefore, $R$ is a lattice isomorphism, by [2, Theorem 2.15].

The next theorem is the main result of this section. It gives a characterization of Banach lattices $X$ that are canonically isomorphic to $(X_{uo})^*$. In light of Proposition 2.6, this is equivalent to characterizing when $X$ has an order continuous lattice isomorphic predual.

**Theorem 3.4** Let $X$ be a Banach lattice such that $X_{uo}$ separates points of $X$. The following statements are equivalent.

1. The mapping $j : X \rightarrow (X_{uo})^*$ is a surjective lattice isomorphism, where $j(x)(\phi) = \phi(x)$ for any $x \in X$ and any $\phi \in X_{uo}$. 
(2) $B_X$ is relatively $\sigma(X, X_{uo})$-compact in $X$.

(3) $X$ is boundedly uo-complete.

(4) $X$ is monotonically complete.

Proof The implication $(1) \implies (2)$ is immediate by Banach–Alaoglu Theorem.

Assume that $(2)$ holds. Clearly, $j$ is one-to-one and $\|j\| \leq 1$. Recall that $X_{uo}^\sim$ is an ideal of $X_n^\sim$. Note also that $(X_{uo}^\sim)^* = (X_{uo}^\sim)_n^\sim$ since $X_{uo}^\sim$ is order continuous by Theorem 2.3. Thus by [2, Theorem 1.70] again, $j$ is a bijective vector lattice isomorphism between $X$ and $j(X)$, and $j(X)$ is a regular sublattice in $(X_{uo}^\sim)^*$. Let $(x_\alpha)$ be a norm bounded uo-Cauchy net in $X$. Then $(j(x_\alpha))$ is uo-Cauchy in $j(X)$, and thus in $(X_{uo}^\sim)^*$, by [8, Theorem 3.2]. Since $(j(x_\alpha))$ is norm bounded in $(X_{uo}^\sim)^*$, by [6, Theorem 2.2], there exists $x^{**} \in (X_{uo}^\sim)^*$ such that

$$j(x_\alpha) \xrightarrow{uo} \sigma((X_{uo}^\sim)^*, X_{uo}^\sim) x^{**} \in (X_{uo}^\sim)^*.$$ 

On the other hand, by the assumption, there exist $x \in X$ and a subnet $(x_\beta)$ of $(x_\alpha)$ such that $x_\beta \xrightarrow{\sigma(X, X_{uo}^\sim)} x$, or equivalently,

$$j(x_\beta) \xrightarrow{\sigma((X_{uo}^\sim)^*, X_{uo}^\sim)} j(x).$$

It follows that $x^{**} = j(x) \in j(X)$, and $j(x_\alpha) \xrightarrow{uo} j(x)$ in $(X_{uo}^\sim)^*$. By [8, Theorem 3.2] again, we have $j(x_\alpha) \xrightarrow{uo} j(x)$ in $j(X)$, and consequently, $x_\alpha \xrightarrow{uo} x$ in $X$. This proves $(2) \implies (3)$. The implication $(3) \implies (4)$ follows from Proposition 3.1.

Suppose that $(4)$ holds. By [13, Theorem 2.4.22], $X$ is lattice isomorphic to $(X_n^\sim)^\sim$ via the evaluation mapping $e : X \to (X_n^\sim)_n^\sim$. Since $X_{uo}^\sim$ is order continuous, $(X_{uo}^\sim)^* = (X_{uo}^\sim)_n^\sim$. Let $R : (X_n^\sim)_n \to (X_{uo}^\sim)_n = (X_{uo}^\sim)^*$ be the restriction map $R \phi = \phi|X_{uo}^\sim$. By Lemma 3.2, $X_{uo}^\sim$ is order dense in $X_n^\sim$. Thus by Lemma 3.3, $R$ is a lattice isometry from $(X_n^\sim)_n$ onto $(X_{uo}^\sim)^*$. This proves $(4) \implies (1)$ since $j = Re$.

We point out that the mapping $j$ in Theorem 3.4 is an isometry iff $X_{uo}^\sim$ is norming on $X$, iff $X_n^\sim$ is norming, iff $0 \leq x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$ ([13, Theorem 2.4.21]), iff the closed unit ball $B_X$ is order closed.

4 Dual representations of convex functionals

In this section, we apply the general theory pertaining to the uo-dual developed in the previous sections to the representations of convex functionals on a Banach lattice $X$ with respect to the duality $(X, X_{uo}^\sim)$. The main motivation is Theorem 4.6, which gives a formulation of Theorem 1.1 in the general setting of Banach lattices. The main tool is the next theorem, whose conclusion should be compared with the $C$-property introduced in [3].

Theorem 4.1 Let $X$ be a vector lattice and $I$ be an ideal of $X^\sim$ containing a strictly positive order continuous functional $\phi$ on $X$. If $C$ is a convex subset of $X$ and $x \in \overline{C}^\sigma(X, I)$, then there exists a sequence $(x_n)$ in $C$ such that $x_n \xrightarrow{uo} x$ in $X$. 

Proof By Kaplan’s Theorem ([2, Theorem 3.50]), the topological dual of $X$ under $|\sigma|(X, I)$ is precisely $I$. Thus by Mazur’s Theorem, we have

$$\overline{C}^{\sigma}(X, I) = C^{|\sigma|(X, I)}.$$

Consequently, there exists a net $(x_\alpha)$ in $C$ such that $x_\alpha \xrightarrow{uo} x$. In particular, $\phi(|x_\alpha - x|) \to 0$. Choose $(\alpha_n)$ such that

$$\phi(|x_{\alpha_n} - x|) \leq \frac{1}{2^n}$$

for each $n \geq 1$. For the sake of convenience, write $x_n := x_{\alpha_n}$. It remains to be shown that $x_n \xrightarrow{uo} x$ in $X$. For any $y \in X$, put $\|y\|_L = \phi(|y|)$. Then $\|\cdot\|_L$ is a norm on $X$, and the norm completion $\tilde{X}$ of $(X, \|\cdot\|_L)$ is an $L^1$-space, in which $X$ sits as a regular sublattice; cf. [8, Section 4]. Since $\|x_n - x\|_L \leq \frac{1}{2^n}$, it follows that $\sum_1^\infty |x_n - x|$ converges in $\tilde{X}$. As

$$|x_n - x| \leq \sum_1^\infty |x_k - x| \downarrow 0 \text{ in } \tilde{X},$$

we have $x_n \xrightarrow{uo} x$ in $\tilde{X}$, so that $x_n \xrightarrow{uo} x$ in $\tilde{X}$, and in $X$ by [8, Theorem 4.1].

The following is an interesting special case of Theorem 4.1. Recall that a Banach function space over a $\sigma$-finite measure space admits strictly positive order continuous functionals (see, e.g., [8, Proposition 5.19]).

**Corollary 4.2** Let $X$ be a Banach function space over a $\sigma$-finite measure space. For any convex subset $C$ of $X$ and $f \in \overline{C}^{\sigma}(X, X_{uo}^\sim)$, there exists a sequence $(f_n)$ in $C$ such that $f_n \xrightarrow{uo} f$.

In line with order closures, we define the **bounded $uo$ closure** of a set $C$ in a Banach lattice $X$ by

$$\overline{C}^{buo} := \{ x \in X : x_\alpha \xrightarrow{uo} x \text{ in } X \text{ for some bounded net } (x_\alpha) \text{ in } C \}.$$

We similarly define the **bounded $uo$ sequential closure** $\overline{C}^{seq-buo}$ by replacing nets with sequences. We say that $C$ is boundedly $uo$ closed, respectively, boundedly $uo$ sequentially closed in $X$, if $C = \overline{C}^{buo}$, respectively, $C = \overline{C}^{seq-buo}$. It is clear that every $\sigma(X, X_{uo}^\sim)$-closed set is boundedly $uo$ (sequentially) closed, and for any set $C$,

$$\overline{C}^{seq-buo} \subset \overline{C}^{buo} \subset \overline{C}^{\sigma}(X, X_{uo}^\sim).$$

It is natural to wonder when the reverse inclusions hold. Namely, if $x \in \overline{C}^{\sigma}(X, X_{uo}^\sim)$, does there always exist a norm bounded net (sequence) in $C$ that $uo$-converges to $x$?
This is clearly a strengthened version of Theorem 4.1. Recall that a sequence \((x_n)\) in a vector lattice \(X\) is said to relatively uniformly converge to \(x \in X\) if there exist \(u \in X\) and a positive real sequence \(\varepsilon_n \to 0\) such that \(|x_n - x| \leq \varepsilon_n u\) for all \(n\). Clearly, every norm convergent sequence in a Banach lattice with limit \(x\) has a subsequence that relatively uniformly converges to \(x\). By [12, Theorem 16.2], the same subsequence order converges to \(x\).

**Proposition 4.3** Let \(X\) be a \(\sigma\)-order complete Banach lattice. The following statements are equivalent.

1. \(\overline{C}^{buo} = \overline{C}^\sigma(X, X_u)\) for every convex set \(C\).
2. \(X\) and \(X^*\) are both order continuous.

**Proof** Suppose that \(X\) and \(X^*\) are both order continuous. Let \(C\) be a convex set in \(X\). It suffices to show that \(\overline{C}^\sigma(X, X_u) \subset \overline{C}^{buo}\). By Corollary 2.5, \(X_u = X^*\), and \(\sigma(X, X_u)\) is the weak topology on \(X\). Thus by Mazur’s Theorem, \(\overline{C}^\sigma(X, X_u) = \overline{C}^{\| \cdot \|}\). By the remark preceding the proposition, every norm convergent sequence admits a subsequence order convergent to the same limit, it follows that \(\overline{C}^\sigma(X, X_u) \subset \overline{C}^{buo}\).

Conversely, suppose first that \(X\) is not order continuous. Then \(X\) has a lattice isomorphic copy of \(\ell^\infty\). A close look at the proof of [2, Theorem 4.51] shows that the copy of \(\ell^\infty\) can be chosen to be regular in \(X\). For convenience, we simply assume that \(\ell^\infty \subset X\). By Ostrovskii’s Theorem (cf. [11, Theorem 2.34]), there exist a subspace \(W\) of \(\ell^\infty\) and \(w \in \overline{W}^{\sigma(\ell^\infty, \ell^1)}\) such that \(w\) is not the \(\sigma(\ell^\infty, \ell^1)\)-limit of any sequence in \(W\).

By [8, Theorem 3.2], since \(\ell^\infty\) is regular in \(X\), every \(u\)-null net in \(\ell^\infty\) is also \(u\)-null in \(X\). Thus the restriction of each \(\phi \in X_u\) to \(\ell^\infty\) belongs to \((\ell^\infty)_u = \ell^1\). Therefore, \(w \in \overline{W}^{\sigma(X, X_u)}\). Suppose that \(W\) admits a bounded net \((x_\alpha)\) such that \(x_\alpha \overset{uo}{\longrightarrow} w\) in \(X\).

By [8, Theorem 3.2] again, \(x_\alpha \overset{uo}{\longrightarrow} w\) in \(\ell^\infty\). Clearly, we can select countably many \(\alpha_n\)’s such that \(x_{\alpha_n} \overset{uo}{\longrightarrow} w\) coordinatewise. Since \((x_{\alpha_n})\) is bounded in \(\ell^\infty\), it follows that \(x_{\alpha_n} \overset{\sigma(\ell^\infty, \ell^1)}{\longrightarrow} w\), contradicting the choice of \(w\).

Suppose now that \(X^*\) is not order continuous. Then by [2, Theorem 4.69], \(X\) has a lattice copy of \(\ell^1\). Since \(\ell^1\) is order continuous, it is easily seen that it is regular in \(X\). By Ostrovskii’s Theorem again, there exist a subspace \(W\) of \(\ell^1\) and \(w \in \overline{W}^{\sigma(\ell^1, c_0)}\) such that \(w\) is not the \(\sigma(\ell^1, c_0)\)-limit of any sequence in \(W\). Since \((\ell^1)_u = c_0\), the same arguments as above also lead to a contradiction. \(\square\)

While Proposition 4.3 asserts that the bounded \(uo\) closure and the \(\sigma(X, X_u)\)-closure of a general convex set rarely coincide, the situation is different for the coincidence of bounded \(uo\) closedness and \(\sigma(X, X_u)\)-closedness of convex sets, where an analogue of the Krein–Smulian property arises naturally. For any \(x \in X\), put

\[
\|x\|_u := \sup_{\phi \in X_u^\ast, \|\phi\| = 1} |\phi(x)|.
\]
Then the set
\[ B := \{ x \in X : \| x \|_{uo} \leq 1 \} \]
is \( \sigma(X, X_{uo}^\sim) \)-closed, and is thus boundedly \( uo \) closed. We say that \( \sigma(X, X_{uo}^\sim) \) satisfies the \textit{Krein–Smulian property} if every convex set of \( X \) is \( \sigma(X, X_{uo}^\sim) \)-closed whenever its intersections with \( kB \) are \( \sigma(X, X_{uo}^\sim) \)-closed for all \( k \geq 1 \).

**Corollary 4.4** Let \( X \) be a Banach lattice that admits a strictly positive boundedly \( uo \)-continuous functional. Suppose that \( X_{uo}^\sim \) is isomorphically norming on \( X \). For any convex set \( C \) in \( X \), the following are equivalent:

1. \( C \) is boundedly \( uo \) (sequentially) closed,
2. \( C \) is \( \sigma(X, X_{uo}^\sim) \)-sequentially closed,
3. \( C \cap kB \) is \( \sigma(X, X_{uo}^\sim) \)-closed for all \( k \geq 1 \).

Consequently, the following statements are equivalent to each other:

1'. Every boundedly \( uo \) (sequentially) closed convex set \( C \) is \( \sigma(X, X_{uo}^\sim) \)-closed,
2'. Every \( \sigma(X, X_{uo}^\sim) \)-sequentially closed convex set is \( \sigma(X, X_{uo}^\sim) \)-closed,
3' \( \sigma(X, X_{uo}^\sim) \) satisfies the Krein–Smulian property.

**Proof** Since \( B \) is norm bounded, the implication (1) \( \implies \) (3) is immediate by Theorem 4.1. The implication (3) \( \implies \) (1) is clear. Since \( X_{uo}^\sim \) isomorphically norms \( X \), \( X \) is isomorphic to a closed subspace of the Banach space \( (X_{uo}^\sim)^* \). Under this identification, every \( \sigma(X, X_{uo}^\sim) \)-convergent sequence is weak*-convergent and hence bounded. The implication (3) \( \implies \) (2) follows. Clearly, (2) implies the sequential version of (1) This proves that (1), (2) and (3) are equivalent. The equivalence of (1'), (2') and (3') now follows immediately. \( \Box \)

**Proposition 4.5** Let \( X \) be a Banach lattice that admits a strictly positive boundedly \( uo \)-continuous functional. Suppose that \( X_{uo}^\sim \) is isomorphically norming on \( X \) and \( \sigma(X, X_{uo}^\sim) \) satisfies the Krein–Smulian property (these conditions are satisfied when \( X \) is monotonically complete). Then for every proper convex functional \( \rho : X \rightarrow (-\infty, \infty] \), the following statements are equivalent to each other:

1. \( \rho(x) = \sup_{\phi \in X_{uo}^\sim} (\phi(x) - \rho^*(\phi)) \) for any \( x \in X \), where \( \rho^*(\phi) = \sup_{\phi \in X_{uo}^\sim} (\phi(x) - \rho(x)) \) for any \( \phi \in X_{uo}^\sim \),
2. \( \rho(x) \leq \liminf_{\alpha} \rho(x_{\alpha}) \) for any norm bounded net \( (x_{\alpha}) \) \( uo \)-converging to \( x \),
3. \( \rho(x) \leq \liminf_{n} \rho(x_{n}) \) for any norm bounded sequence \( (x_{n}) \) \( uo \)-converging to \( x \).

**Proof** By Fenchel–Moreau Formula ([4, Theorem 1.11]), (1) is equivalent to that \( \rho \) is \( \sigma(X, X_{uo}^\sim) \)-lower semicontinuous, i.e., the level set
\[ C_\lambda := \{ x : \rho(x) \leq \lambda \} \]
is \( \sigma(X, X_{uo}^\sim) \)-closed for any \( \lambda \in \mathbb{R} \). By Corollary 4.4, this is equivalent to that \( C_\lambda \) is boundedly \( uo \) (sequentially) closed for all \( \lambda \in \mathbb{R} \). A straightforward computation shows that this last condition is equivalent to (2), respectively, (3).
Finally, note that, since $X$ has a strictly positive $uo$-continuous functional, $X_{uo}^\sim$ separates points of $X$. Thus if $X$ is monotonically complete, then it is lattice isomorphic to $(X_{uo}^\sim)^*$, by Theorem 3.4. Consequently, $X_{uo}^\sim$ is isomorphically norming on $X$ and $\sigma(X, X_{uo}^\sim)$ satisfies the Krein–Smulian property. □

**Theorem 4.6** Let $Y$ be an order continuous Banach lattice with weak units, and let $X = Y^\ast$. For any proper convex functional $\rho : X \to (-\infty, \infty]$, the following statements are equivalent:

1. $\rho(x) = \sup_{y \in Y} (\langle x, y \rangle - \rho(y))$ for any $x \in X$, where $\rho^*(y) = \sup_{x \in X} (\langle x, y \rangle - \rho(x))$ for any $y \in Y$,
2. $\rho(x) \leq \liminf_{\alpha} \rho(x_\alpha)$ for any norm bounded net $(x_\alpha)$ $uo$-converging to $x$,
3. $\rho(x) \leq \liminf_{n} \rho(x_n)$ for any norm bounded sequence $(x_n)$ $uo$-converging to $x$.

**Proof** By Proposition 2.6, $Y = X_{uo}^\sim$. Thus $X_{uo}^\sim$ norms $X$ and $\sigma(X, X_{uo}^\sim)$, being the weak* topology on $X$, satisfies the Krein–Smulian property. Let $y \in Y_+$ be a weak unit in $Y$. Since $Y$ is order continuous, $z \land ny \to z$ in norm for any $z \in Y_+$. It follows that if $x \in X_+$ and $\langle x, y \rangle = 0$, then $\langle x, z \rangle = 0$ for all $z \in Y_+$, and hence $x = 0$. Therefore, $y$ acts as a strictly positive functional on $X$. The result now follows from Proposition 4.5. □

As was mentioned in the Introduction, a prime motivation for Theorem 4.6 comes from Theorem 1.1, which is concerned with Orlicz spaces. In the discussion below, we confine ourselves to probability spaces as they are the most important in applications to risk measures in Mathematical Finance. Fix a probability space $(\Omega, \Sigma, \mathbb{P})$. Assume that it admits an infinite disjoint sequence of measurable sets of positive probabilities; for otherwise the resulting spaces will be finite-dimensional. An Orlicz function is a convex nondecreasing function $\Phi : [0, \infty) \to \mathbb{R}$, not identically $0$, such that $\Phi(0) = 0$. For an Orlicz function $\Phi$, the Orlicz space $L^\Phi := L^\Phi(\Omega, \Sigma, \mathbb{P})$ is the space of all measurable functions $f$ such that

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int \Phi\left( \frac{|f(\omega)|}{\lambda} \right) \, d\mathbb{P} \leq 1 \right\}$$

is finite. The functional $\| \cdot \|_\Phi$ is a complete norm on $L^\Phi$, called the Luxemburg norm. The subspace of $L^\Phi$ consisting of all $f$ such that $\int \Phi(\frac{|f(\omega)|}{\lambda}) \, d\mathbb{P} < \infty$ for all $0 < \lambda < \infty$ is called the Orlicz heart $H^\Phi := H^\Phi(\Omega, \Sigma, \mathbb{P})$. Obviously, Orlicz spaces are Banach lattices in the almost everywhere order. Given an Orlicz function $\Phi$, its conjugate function $\Psi$ is defined by

$$\Psi(t) = \sup\{st - \Phi(s) : s \geq 0\} \text{ for } t \geq 0.$$ 

Then $\lim_{t \to \infty} \frac{\Psi(t)}{t} = \infty$. Moreover, $\Psi$ is an Orlicz function iff $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$, and in this case, the conjugate function of $\Psi$ is $\Phi$. Elements of the Orlicz space $L^\Psi$ (in particular, of $H^\Psi$) act on $L^\Phi$ (in particular, on $H^\Phi$), via the duality

$$\langle f, g \rangle = \int fg \, d\mathbb{P}, \quad g \in L^\Psi, \quad f \in L^\Phi.$$
We collect some basic facts concerning Orlicz spaces in the next proposition. Refer to [5, Chapter 2] and [15] for in-depth studies of these spaces.

**Proposition 4.7** Let $\Phi$ be an Orlicz function with conjugate $\Psi$. Then $H^\Phi$ is the order continuous part of $L^\Phi$. In particular, $H^\Phi$ is an order continuous Banach lattice. Moreover, $L^\Phi \neq L^1$ (as sets or as lattice isomorphic Banach lattices) if and only if $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$. In this case, $L^\Psi = (L^\Phi)_{\sim} = (H^\Phi)_{\sim} = (H^\Phi)^*$ (equality as sets and as lattice isomorphic Banach lattices).

Suppose that $X = L^\Phi$, where $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$. Set $Y = H^\Psi$. By Proposition 4.7 applied to the conjugate case, $Y$ is an order continuous Banach lattice with weak units (in fact it contains the constant functions), and $X = Y^*$. Thus the equivalence of (1) and (3) in Theorem 4.6 yields Theorem 1.1.

Proposition 4.7 identifies $L^\Psi$ as the order continuous duals of $L^\Phi$ and $H^\Phi$. When now $H^\Psi$ are their uo-duals. Indeed, by Proposition 4.7, $L^\Psi = (L^\Phi)_{\sim} = (H^\Phi)_{\sim}$ and $(L^\Psi)^a = H^\Psi$. Thus by Theorem 2.3, we obtain the following corollary, where equality again means equality as sets and with equivalent norms.

**Corollary 4.8** Let $\Phi$ be an Orlicz function with conjugate $\Psi$. Assume that $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$. Then $H^\Psi = (L^\Phi)_{uo} = (H^\Phi)_{uo}$.

When the Orlicz function $\Phi$ is an N-function and the underlying measure space is finite and separable, it is proved in [15, pp. 148, Proposition 6] that, for any $g \in H^\Psi$, $\int f_n g \to \int fg$ whenever $(f_n)$ is norm bounded in $L^\Phi$ and $f_n \overset{a.e.}{\longrightarrow} f$. In our notation, this means that $H^\Psi \subset (L^\Phi)_{uo}$. Corollary 4.8 shows that the technical assumptions in the result can be removed and also establishes the converse.

Finally, let us consider Proposition 4.3 and Corollary 4.4 in the context of Orlicz spaces. Assume that $(\Omega, \Sigma, \mathbb{P})$ is a nonatomic probability space and let $\Phi$ be an Orlicz function so that $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$. Set $X = H^\Phi$. Proposition 4.3 says that $C_{uo}^{\Phi} = C^\sigma(H^\Phi, H^\Psi)$ for every convex set $C$ in $H^\Phi$ if and only if $L^\Psi = (H^\Phi)^*$ is order continuous, or equivalently, $\Psi$ satisfies the $\Delta_2$-condition. On the other hand, Corollary 4.4 says that every boundedly uo closed convex set is $\sigma(H^\Phi, H^\Psi)$-closed if and only if $\sigma(H^\Phi, H^\Psi)$ has the Krein-Smulian property. In [7, Theorem 4.3], it is shown that the latter holds if and only if either $\Phi$ or $\Psi$ satisfies the $\Delta_2$-condition. The difference in the two assertions above points to the subtle fact that the bounded uo closure is not necessarily boundedly uo closed!

**References**

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