SEPARABLE UNIVERSAL BANACH LATTICES

BY

DENNY H. LEUNG

Department of Mathematics, National University of Singapore  
Singapore 119076  
e-mail: matlhh@nus.edu.sg

AND

LEI LI

School of Mathematical Sciences and LPMC  
Nankai University, Tianjin 300071, China  
e-mail: leilee@nankai.edu.cn

AND

TIMUR OIKHBERG AND MARY ANGELICA TURSI

Department of Mathematics, University of Illinois  
Urbana IL 61801, USA  
e-mail: oikhberg@illinois.edu, gramcko2@illinois.edu

ABSTRACT
We construct separable universal injective and projective lattices for the class of all separable Banach lattices.

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1. Introduction

The object of this paper is to construct universal injective and projective objects for the class of separable (real) Banach lattices.

It is well known that $C[0, 1]$ is a universal injective Banach space for the class of all separable Banach spaces—that is, any separable Banach space embeds isometrically into $C[0, 1]$. Similarly, $\ell_1$ is a universal projective Banach space for the class of separable Banach spaces—every separable Banach space is a quotient of $\ell_1$. We construct similar objects in the lattice setting.


Suppose $E$ and $F$ are Banach lattices. We say that $u \in B(E, F)$ is a lattice homomorphism if it preserves lattice operations (it suffices to check that $u(x_1 \vee x_2) = ux_1 \vee ux_2$ for any $x_1, x_2 \in E$; note that $u$ is necessarily positive). An operator which is both an isometry and a lattice homomorphism is referred to as a lattice isometry.

We call $q \in B(E, F)$ a lattice quotient if there is an ideal $I \subset E$ so that $q$ identifies $F$ with $E/I$. Notice that $q$ is a lattice quotient if and only if it has the following properties: (i) $q$ maps the open ball of $E$ onto the open ball of $F$, and (ii) $q$ is a lattice homomorphism. Indeed, in this case the formal identity $i : E/I \to F$ is a lattice isometry; by [1], the same is true for $i^{-1}$.

Throughout, we work with real lattices. We make use of two compact metrizable sets—the Hilbert cube $\mathbb{H}$, and the Cantor set $\Delta$ (that can be regarded as $[0, 1]^\mathbb{N}$, respectively $\{0, 1\}^\mathbb{N}$, equipped with the product topology). We use $L_1$ as a shorthand for $L_1(0, 1)$.

The two theorems below represent the main results of this note.

**Theorem 1.1:** The Banach lattice $C(\Delta, L_1)$ is injectively universal for the class of separable Banach lattices. That is, any separable Banach lattice embeds lattice isometrically into $C(\Delta, L_1)$.

**Theorem 1.2:** There exists a separable Banach lattice $X$ which is projectively universal for the class of separable Banach lattices, that is, any separable Banach lattice is lattice isometric to a quotient of $X$ by a closed lattice ideal.

The proofs of Theorems 1.1 and 1.2 are given below.
Remark 1.1: As a separable Banach lattice can have infinitely many generators, no universal projective lattice can be finitely generated. However, the universal injective lattice $C(\Delta, L_1)$ can be generated by two elements. To verify this, we use a technique similar to [6, Theorem V.2.10]. Recall that $L_1$ is lattice isometric to $L_1(\Delta, \mu)$, where $\mu$ is the Haar measure on $\Delta$ (see [3, §14–15]). The measure $\mu$ can also be described as follows: consider $\nu = (\delta_0 + \delta_1)/2$ (a probability measure on $\{0, 1\}$); then $\mu = \nu^N$ is a probability measure on $\Delta = \{0, 1\}^N$. Note that the set $K = \Delta \times \Delta$ is homeomorphic to $\Delta$. Representing $\Delta$ as a compact subset of $\mathbb{R}$, and applying Stone’s Theorem (see [6, Theorem II.7.3]), we observe that $C(\Delta)$ is generated by the identity 1 and the coordinate function. Thus,

$$C(K) \cong C(\Delta)$$

has two generators. To show that $C(K)$ is dense in $C(\Delta, L_1(\Delta, \mu))$, note that any $f \in C(\Delta, L_1(\Delta, \mu))$ is uniformly continuous. Hence, the functions of the form $\sum_{k=1}^n \chi_{A_k} \otimes f_k$ (where $f_k \in L_1(\Delta, \mu)$, and $A_k$ is a clopen subset of $\Delta$) are dense in $C(\Delta, L_1(\Delta, \mu))$.

2. The proof of Theorem 1.1

Let $A_n, n \in \mathbb{N}$, be finite nonempty sets and let $\widehat{T}$ be the tree $\bigcup_{k=0}^{\infty} \prod_{n=1}^k A_n$, where, as usual, the product $\prod_{n=1}^k A_n$ is defined to be $\emptyset$ if $k = 0$. Suppose that $\sigma = (a_1, \ldots, a_k) \in \prod_{n=1}^k A_n$; we say that $\sigma$ has length $k$ and write $|\sigma| = k$. For any $b \in A_{k+1}$, we denote the element $(a_1, \ldots, a_k, b) \in \prod_{n=1}^{k+1} A_n$ by $(\sigma, b)$.

Let $E$ be a Banach lattice. A family $(x_\sigma)_{\sigma \in \widehat{T}}$ is said to be a finitely branching tree in $E_+$ if

(a) $x_\sigma \in E_+$ for all $\sigma \in \widehat{T}$,

(b) for any $\sigma \in \widehat{T}$ with $|\sigma| = k$, $(x_{(\sigma, b)})_{b \in A_{k+1}}$ is pairwise disjoint and

$$x_\sigma = \sum_{b \in A_{k+1}} x_{(\sigma, b)}.$$

Observe that if $(x_\sigma)_{\sigma \in \widehat{T}}$ is a finitely branching tree in $E_+$, then by (b), $\text{span}\{x_\sigma : \sigma \in \widehat{T}\}$ is a vector sublattice of $E$.

**Proposition 2.1:** Let $E$ be a Banach lattice. Suppose that there is a finitely branching tree $(x_\sigma)_{\sigma \in \widehat{T}}$ in $E_+$ so that $E$ is the closed linear span of $(x_\sigma)_{\sigma \in \widehat{T}}$. Then there exists a compact metric space $K$ so that $E$ is a lattice isometric to a closed sublattice of $C(K, L_1)$. 
Proof. Obviously, under the given assumption, \( E \) is a separable Banach lattice. Let \( K \) be the positive part of the closed ball of \( E^* \), endowed with the weak* topology. Then \( K \) is a compact metrizable topological space. By rescaling if necessary, we may assume that \( \|x_0\| \leq 1 \). For each \( \sigma \in \widehat{T} \), the function \( g_\sigma : K \to \mathbb{R} \) given by \( g_\sigma(x^*) = x^*(x_\sigma) \) is a nonnegative continuous function on \( K \). Furthermore, for all \( \sigma \in \widehat{T} \) with \( |\sigma| = k \), it follows from property (b) that

\[
g_\sigma = \sum_{b \in A_{k+1}} g(\sigma, b).
\]

We now define functions \( f_\sigma : K \to L_1, \sigma \in \widehat{T}, \) inductively as follows. Let \( f_\emptyset(x^*) = \chi_{[0, g_\emptyset(x^*)]} \). By the continuity of \( g_\emptyset \), we see that \( f_\emptyset \) is a continuous function from \( K \) into \( L_1 \). In general, assume that \( f_\sigma \) has been defined so that \( f_\sigma(x^*) = \chi_{[c(x^*), d(x^*)]} \), where \( c, d : K \to \mathbb{R} \) are nonnegative continuous functions so that \( d - c = g_\sigma \). Label the elements in \( A_{k+1} \) as \( b_1, \ldots, b_r \). Define \( f(\sigma, b_i)(x^*) \), \( 1 \leq i \leq r \), to be the characteristic function of the interval

\[
\left[ c(x^*) + \sum_{j=1}^{i-1} g(\sigma, b_j)(x^*), c(x^*) + \sum_{j=1}^{i} g(\sigma, b_j)(x^*) \right].
\]

By continuity of \( c \) and \( g(\sigma, b_j) \), \( f(\sigma, b_i) \) is a continuous function from \( K \) into \( L_1 \) for each \( i \). This completes the inductive definition of \( f_\sigma, \sigma \in \widehat{T} \). It follows from (1) that

\[
f_\sigma = \sum_{b \in A_{k+1}} f(\sigma, b) \quad \text{if } |\sigma| = k
\]

(equality in the \( L_1 \) sense at each \( x^* \in K \)). From (b) and (2), we see that the map \( x_\sigma \mapsto f_\sigma, \sigma \in \widehat{T} \), extends to a linear map \( u \) from \( \text{span}\{x_\sigma : \sigma \in \widehat{T}\} \) into \( C(K, L_1) \). By (b), for any \( y \in \text{span}\{x_\sigma : \sigma \in \widehat{T}\} \), one can derive that \( y \in \text{span}\{x_\sigma : |\sigma| = k\} \) for all sufficiently large \( k \). In particular, \( \text{span}\{x_\sigma : \sigma \in \widehat{T}\} \) is a sublattice of \( E \). Also, it is easy to check that if \( \sigma \) and \( \tau \) are distinct elements in \( \widehat{T} \) of the same length, then \( f_\sigma(x^*) \wedge f_\tau(x^*) = 0 \) (in \( L_1 \)) for each \( x^* \in K \). It follows that the map \( u \) is a lattice homomorphism. Next, we show that \( u \) is an (into) isometry. Let \( x \in \text{span}\{x_\sigma : \sigma \in \widehat{T}\} \). Write \( x = \sum_{|\sigma| = k} \sigma x_{\sigma} \) for some \( k \in \mathbb{N} \) and \( \sigma x_{\sigma} \in \mathbb{R} \). Then \( |x| = \sum_{|\sigma| = k} |\sigma| x_{\sigma} \) and

\[
|ux| = u|x| = \sum_{|\sigma| = k} |\sigma| f_\sigma.
\]
By construction, \( \| f_\sigma(x^*) \|_{L^1} = g_\sigma(x^*) = x^*(x_\sigma) \). Since \( K \) is the positive part of the ball of \( E^* \), one can derive that

\[
\| ux \| = \| ux \| = \sup_{x^* \in K} \left\| \sum_{|\sigma|=k} c_\sigma f_\sigma(x^*) \right\|_{L^1} = \sup_{x^* \in K} \sum_{|\sigma|=k} |c_\sigma| \| f_\sigma(x^*) \|_{L^1} = \sup_{x^* \in K} \sum_{|\sigma|=k} |c_\sigma| x^*(x_\sigma) = \sup_{x^* \in K} x^*(|x|) = \| x \|.
\]

Hence \( u \) is a lattice isometry from \( \text{span}\{x_\sigma : \sigma \in \hat{T}\} \) into \( C(K, L^1) \). As \( \text{span}\{x_\sigma : \sigma \in \hat{T}\} \) is dense in \( E \) by assumption, \( u \) extends to a lattice isometry from \( E \) into \( C(K, L^1) \).

**Proposition 2.2:** Let \( E \) be a separable Banach lattice, regarded as a closed sublattice of its bidual \( E^{**} \). There is a Banach lattice \( F \) such that \( E \subseteq F \subseteq E^{**} \), \( F_+ \) contains a finitely branching tree \( (x_\sigma)_{\sigma \in \hat{T}} \) and \( \text{span}\{x_\sigma : \sigma \in \hat{T}\} \) is dense in \( F \).

**Proof.** Let \( (e_i)_{i=1}^\infty \) be a countable dense subset of \( E \) consisting of nonzero vectors. We shall construct recursively a finitely branching tree \( (x_\sigma)_{\sigma \in \hat{T}} \subseteq E^{**}_+ \) so that, for any \( 1 \leq m \leq n \),

\[
\text{dist}(e_m, \text{span}\{x_\sigma : |\sigma| = n\}) < 2^{-n}.
\]

Then the proposition follows by taking \( F \) to be the closed linear span of \( (x_\sigma)_{\sigma \in \hat{T}} \) in \( E^{**} \).

Start the construction by setting \( A_0 = \emptyset \) and

\[
x_\emptyset = e = \sum_{i=1}^\infty \frac{|e_i|}{2^i \|e_i\|}.
\]

Suppose that \( n \in \mathbb{N} \cup \{0\} \) and the sets \( A_0, A_1, \ldots, A_n \) and vectors \( x_\sigma \in E^{**}_+ (|\sigma| \leq n) \) have already been selected so that condition (b) above is satisfied for every \( \sigma \) with \( |\sigma| < n \). In particular,

\[
\sum_{|\sigma|=n} x_\sigma = e.
\]
Since for all $1 \leq i \leq n + 1$, $e_i$ lies in the principal ideal generated by $e$ in $E^{**}$, by Freudenthal’s Spectral Theorem [5, Theorem 1.2.18] and its proof, there exist mutually disjoint $z_1, \ldots, z_N \in E_+^{**}$ so that $z_1 + \cdots + z_N = e$, and
\[
\text{dist}(e_m, \text{span}\{z_1, \ldots, z_N\}) < 2^{-(n+1)}
\]
for $1 \leq m \leq n + 1$. Denote by $P_i$ the band projection from $E^{**}$ onto the band generated by $z_i$ in $E^{**}$, $1 \leq i \leq N$. Let $A_{n+1} = \{1, \ldots, N\}$; for $\sigma \in \prod_{k=1}^{n} A_k$ and $i \in A_{n+1}$, let $x_{(\sigma,i)} = P_i x_\sigma$. Since $x_\sigma$ lies in the band $B$ generated by $e$ in $E^{**}$ and $\sum_{i=1}^{N} P_i$ is the band projection onto $B$, $x_\sigma = \sum_{i \in A_{n+1}} x_{(\sigma,i)}$. This completes the inductive construction of $(x_\sigma)_{\sigma \in \widehat{T}}$, where $\widehat{T} = \bigcup_{k=0}^{\infty} \prod_{n=1}^{k} A_n$. Clearly, $(x_\sigma)_{\sigma \in \widehat{T}}$ is a finitely branching tree in $E_+$. Furthermore, in the notation above,
\[
z_i = P_i e = \sum_{|\sigma|=n} P_i x_\sigma = \sum_{|\sigma|=n} x_{(\sigma,i)}.
\]
Thus, for $1 \leq m \leq n + 1$,
\[
\text{dist}(e_m, \text{span}\{x_\sigma : |\sigma| = n + 1\}) \leq \text{dist}(e_m, \text{span}\{z_1, \ldots, z_N\}) < 2^{-(n+1)}.
\]

Proof of Theorem 1.1. By Propositions 2.1 and 2.2, there are a compact metric space $K$ and a lattice isometry $u$ from $E$ into $C(K, L_1)$. It is well known that there exists a continuous surjection $\pi : \Delta \to K$. Then the map $j : E \to C(\Delta, L_1)$ given by $jx = ux \circ \pi$ is a lattice isometry.

3. The proof of Theorem 1.2

A few words of motivation before we begin the proof proper. Suppose that $X$ is a separable Banach lattice that is projectively universal for the class of separable Banach lattices. For any separable Banach lattice $E$, there is a lattice quotient $q$ from $X$ onto $E$. Then $q^* B_{E^*}$ is a $\sigma(X^*, X)$-closed convex solid subset of the $\sigma(X^*, X)$-compact metrizable space $B_{X^*}$. Let $H$ be the Hilbert cube $[0, 1]^N$. For each separable Banach lattice $E$, we will present $B_{E^*}$ as a closed convex solid subset of the ball of $M(H) = C(H)^*$ on a different copy of $H$. We then stitch these copies together to form a compact metric space, say $K$. The space $X$ is taken to be the completion of $C(K)$ normed by the union of the copies of $B_{E^*}$. 

If \( V \) is a solid subset of \( B_{M(\mathbb{H})} \), define a seminorm \( \rho_V \) on \( C(\mathbb{H}) \) by

\[
\rho_V(f) = \sup_{\mu \in V} \left| \int f \, d\mu \right|.
\]

Since \( V \) is solid, \( \rho_V \) is a lattice seminorm and \( \ker \rho_V \) is a vector lattice ideal of \( C(\mathbb{H}) \). Thus \( C(\mathbb{H}) / \ker \rho_V \) is a vector lattice. Clearly, \( \rho_V \) induces a lattice norm on \( C(\mathbb{H}) / \ker \rho_V \), which we denote by \( \tilde{\rho}_V \).

**Proposition 3.1:** Let \( E \) be a separable Banach lattice. Then there exists a \( \sigma(M(\mathbb{H}), C(\mathbb{H})) \)-closed convex solid subset \( V_E \) of \( B_{M(\mathbb{H})} \) such that \( E \) is lattice isometric to the completion of \( C(\mathbb{H}) / \ker \rho_{V_E} \) with respect to the lattice norm \( \tilde{\rho}_{V_E} \).

**Proof.** Choose a sequence \((x_n)\) in \( B_{E^+} \) that is dense in \( B_{E^+} \). Set \( x = \sum \frac{x_n}{2^n} \).

There are a compact Hausdorff space \( L \) and a vector lattice isomorphism \( i \) from \( C(L) \) onto the ideal \( E_x = \bigcup_k [-kx, kx] \) of \( E \). Furthermore, \( x = i1_L \), where \( 1_L \) is the constant function with value 1. Since \( x_n \in E_x \), \( x_n = if_n \) for some \( f_n \in C(L) \). Let \( F \) be the closed (with respect to the sup-norm) sublattice of \( C(L) \) generated by \((f_n) \cup \{1_L\} \). Since \( F \) is an AM-space with unit, there are a compact Hausdorff space \( K \) and a Banach lattice isomorphism \( j \) from \( C(K) \) onto \( F \) such that \( j1_K = 1_L \). The closed sublattice generated by a countable set is separable [4]; see also [6, p. 143, Exercise 5(c)]. Hence \( F \) is separable and thus \( K \) is metrizable. By [2, Theorem 4.14], there is an (into) homeomorphism \( \varphi : K \to \mathbb{H} \). Define \( q : C(\mathbb{H}) \to C(K) \) by \( qf = f \circ \varphi \). Then \( T = i \circ j \circ q : C(\mathbb{H}) \to E \) is a vector lattice homomorphism and, in particular, a bounded linear operator. Furthermore, \( TB_{C(\mathbb{H})} \subseteq [-x, x] \) and \( \|x\| \leq 1 \). Thus \( \|T\| \leq 1 \). Set \( V_E = T^*B_{E^+} \). Then \( V_E \) is a \( \sigma(M(\mathbb{H}), C(\mathbb{H})) \)-closed convex subset of \( B_{M(\mathbb{H})} \).

Next, we show that \( V_E \) is solid in \( M(\mathbb{H}) \). Suppose that \( |\nu| \leq |\mu| \), where \( \nu \in M(\mathbb{H}) \) and \( \mu \in V_E \). Choose \( x^* \in B_{E^*} \) so that \( \mu = T^*x^* \). For \( f \in C(\mathbb{H}) \), if \( |g| \leq |f| \) we have that \( \langle Tg, x^* \rangle \) which implies that

\[
|\langle f, \nu \rangle| \leq |\langle f, |\nu| \rangle| \leq |\langle f, |\mu| \rangle| = \sup_{|g| \leq |f|} |\langle Tg, x^* \rangle| \leq \langle T|f|, |x^*| \rangle \leq \|T|f|| \|x^*\| = \|Tf\| \|x^*\|.
\]
It follows that $y^* : T(C(H)) \to \mathbb{R}$ given by $y^*(Tf) = \langle f, \nu \rangle$ defines a bounded linear functional on the subspace $T(C(H))$ of $E$. Since $x_n \in T(C(H))$ for all $n$, $T(C(H))$ is a dense subspace of $E$. Thus $y^*$ extends uniquely to an element in $E^*$, which we denote still by $y^*$. By the computation above, $\|y^*\| \leq \|x^*\|$ and hence $y^* \in B_{E^*}$. Clearly, it follows from the definition that $T^*y^* = \nu$. Hence $\nu \in V_E$, as desired.

Finally, we show that the map $S : (C(H)/\ker \rho_{V_E}, \tilde{\rho}_{V_E}) \to E$ given by

$$S\tilde{f} = Tf$$

is a well-defined into lattice isometry. Since the image of $S$ is $T(C(H))$ and hence dense in $E$, the proof would be complete. If $f \in \ker \rho_{V_E}$, then $\langle f, T^*x^* \rangle = 0$ for all $x^* \in B_{E^*}$. Thus $Tf = 0$. This shows that $S$ is well-defined. Furthermore, for any $\tilde{f} \in C(H)/\ker \rho_{V_E}$,

$$\tilde{\rho}_{V_E}(\tilde{f}) = \rho_{V_E}(f) = \sup_{x^* \in B_{E^*}} |\langle f, T^*x^* \rangle| = \|Tf\| = \|S\tilde{f}\|.$$ 

Hence $S$ is an into isometry. Also,

$$|S\tilde{f}| = |Tf| = T|f| = S\tilde{f}.$$ 

Therefore, $S$ is a lattice homomorphism.

Proposition 3.2: Let $K$ be the set of all $\sigma(M(H), C(H))$-closed convex solid subsets of $B_{M(H)}$. Then $K$ is a closed subset of $C$. Consequently, $K$ is a compact set with respect to the Hausdorff metric $D$ generated by $d$. 

Since $C(H)$ is separable, we have that $B_{M(H)}$ is a compact metric space in the $\sigma(M(H), C(H))$-topology. Let $d$ be a metric on $B_{M(H)}$ that gives the $\sigma(M(H), C(H))$-topology. By a theorem of Hausdorff (see [2, Theorem 4.26]), the set $C$ of all $\sigma(M(H), C(H))$-closed subsets of $B_{M(H)}$ is compact with respect to the Hausdorff metric $D$ generated by $d$. Let $f \in C(H)$. Then there is a metric $d'$ on $B_{M(H)}$ that gives the $\sigma(M(H), C(H))$-topology and that

$$d'(\mu, \nu) \geq |\langle f, \mu \rangle - \langle f, \nu \rangle| \quad \text{for all } \mu, \nu \in B_{M(H)}.$$ 

Since $B_{M(H)}$ is $\sigma(M(H), C(H))$-compact, the formal identity map from $(B_{M(H)}, d)$ to $(B_{M(H)}, d')$ is a uniform homeomorphism. Thus, if $D'$ is the Hausdorff metric on $C$ generated by $d'$, then $D$ and $D'$ yield the same topology on $C$.

Proposition 3.2: Let $K$ be the set of all $\sigma(M(H), C(H))$-closed convex solid subsets of $B_{M(H)}$. Then $K$ is a closed subset of $C$. Consequently, $K$ is a compact set with respect to the Hausdorff metric $D$ generated by $d$. 

Proof. Suppose that $V_n \in \mathcal{K}$ for all $n$ and that $D(V_n, V) \to 0$ for some $V \in \mathcal{C}$. It is easy to see that $V$ is convex. Indeed, suppose that $a, b \in V$ and $0 \leq \alpha \leq 1$. There are sequences $(v_n)$ and $(w_n)$ so that $v_n, w_n \in V_n$ for each $n \in \mathbb{N}$ and that $d(v_n, a), d(w_n, b) \to 0$, i.e., $v_n \to a$ and $w_n \to b$ with respect to $\sigma(M(\mathbb{H}), C(\mathbb{H}))$. Then

$$\alpha v_n + (1 - \alpha)w_n \to \alpha a + (1 - \alpha)b \quad \text{with respect to } \sigma(M(\mathbb{H}), C(\mathbb{H})).$$

Since each $V_n$ is convex, $\alpha v_n + (1 - \alpha)w_n \in V_n$. Hence

$$d(\alpha v_n + (1 - \alpha)w_n, V) \leq D(V_n, V) \to 0.$$

Choose $u_n \in V$ such that $d(\alpha v_n + (1 - \alpha)w_n, u_n) \to 0$. Then $u_n \to \alpha a + (1 - \alpha)b$ with respect to $\sigma(M(\mathbb{H}), C(\mathbb{H}))$. Hence $\alpha a + (1 - \alpha)b \in V$. Similarly, one can show that $V$ is symmetric.

Next, we show that $V$ is solid (in $B_{M(\mathbb{H})}$). Suppose on the contrary that there are $a, b$ so that $|a| \leq |b|, b \in V$ and $a / \in V$. Since $V$ is convex, symmetric and $\sigma(M(\mathbb{H}), C(\mathbb{H}))$-closed, there exists $f \in C(\mathbb{H})$ so that

$$\langle f, a \rangle > \sup_{v \in V} |\langle f, v \rangle|.$$

As discussed above, there is a metric $d'$ on $B_{M(\mathbb{H})}$ so that its Hausdorff metric $D'$ generates the same topology on $\mathcal{C}$ and that

$$d'(v_1, v_2) \geq |\langle f, v_1 \rangle - \langle f, v_2 \rangle| \quad \text{for all } v_1, v_2 \in B_{M(\mathbb{H})}.$$ 

Let $w \in V_n$. Since $V_n$ is solid,

$$\langle |f|, |w| \rangle = \sup_{|u| \leq |w|} |\langle f, u \rangle|$$

$$\leq \sup_{u \in V_n} |\langle f, u \rangle|$$

$$\leq \sup_{v \in V} |\langle f, v \rangle| + D'(V_n, V).$$

Choose $(x_n)$ so that $x_n \in V_n$ for each $n$ and that $d'(x_n, b) \to 0$. For any $\varepsilon > 0$, there exists $g$ with $|g| \leq |f|$ such that

$$|\langle g, b \rangle| + \varepsilon \geq \langle |f|, |b| \rangle.$$

We have

$$|\langle g, b \rangle| = \lim |\langle g, x_n \rangle| \leq \limsup |\langle f, x_n \rangle|.$$
It follows that
\[ \langle f, a \rangle \leq \langle |f|, |a| \rangle \leq \langle |f|, |b| \rangle \]
\[ \leq \limsup_{n} \langle |f|, |x_{n}| \rangle \]
\[ \leq \limsup_{n} (\sup_{v \in V} |\langle f, v \rangle| + D'(V_{n}, V)) \]
\[ = \sup_{v \in V} |\langle f, v \rangle|, \]
contrary to the choice of \( f \). This proves that \( V \) is solid. 

Fix \( V \in \mathcal{K} \). Define \( q_{V} : C(\mathcal{K} \times \mathbb{H}) \to C(\mathbb{H}) \) by \( q_{V}(f) = f|_{\{ V \} \times \mathbb{H}} \). Let \( B \) be the set \( \bigcup_{V \in \mathcal{K}} q_{V}^{*}(V) \) and define \( \rho_{B} : C(\mathcal{K} \times \mathbb{H}) \to \mathbb{R} \) by
\[ \rho_{B}(F) = \sup_{\mu \in B} \left| \int F \, d\mu \right|. \]

**Lemma 3.3:** \( \rho_{B} \) is a lattice seminorm on \( C(\mathcal{K} \times \mathbb{H}) \). Thus \( C(\mathcal{K} \times \mathbb{H}) / \ker \rho_{B} \) is a vector lattice. Denote the lattice norm induced by \( \rho_{B} \) on \( C(\mathcal{K} \times \mathbb{H}) / \ker \rho_{B} \) by \( \tilde{\rho}_{B} \). The completion \( X \) of \( C(\mathcal{K} \times \mathbb{H}) / \ker \rho_{B} \) with respect to \( \tilde{\rho}_{B} \) is a separable Banach lattice.

**Proof.** Since \( \mathcal{K} \times \mathbb{H} \) is a compact metric space, \( C(\mathcal{K} \times \mathbb{H}) \) is separable with respect to the sup-norm. If \( V \in \mathcal{K} \), then \( V \subseteq B_{M(\mathbb{H})} \) and it is clear that \( q_{V}^{*}(V) \subseteq B_{M(\mathcal{K} \times \mathbb{H})} \). Hence \( B \subseteq B_{M(\mathcal{K} \times \mathbb{H})} \). It is then clear that \( \rho_{B} \leq \| \cdot \|_{\infty} \). Let \( A \) be a countable dense subset of \( C(\mathcal{K} \times \mathbb{H}) \) with respect to the sup-norm. Then \( \{ \tilde{F} : F \in A \} \) is a countable dense subset of \( C(\mathcal{K} \times \mathbb{H}) / \ker \rho_{B} \) with respect to \( \tilde{\rho}_{B} \). Thus \( X \) is separable. 

If \( V \in \mathcal{K} \), identify \( \{ V \} \times \mathbb{H} \) with \( \mathbb{H} \).

**Lemma 3.4:** Let \( E \) be a separable Banach lattice. The map \( Q : C(\mathcal{K} \times \mathbb{H}) \to C(\mathbb{H}) \) given by
\[ QF = F|_{\{ V_{E} \} \times \mathbb{H}} \]
has the following properties:

1. \( Q(\ker \rho_{B}) \subseteq \ker \rho_{V_{E}} \) and hence \( Q \) induces a map \( \tilde{Q} : C(\mathcal{K} \times \mathbb{H}) / \ker \rho_{B} \to C(\mathbb{H}) / \ker \rho_{V_{E}} \).
   \( \tilde{Q} \) is a lattice homomorphism.

2. \( \tilde{Q} \) maps the open ball in \( (C(\mathcal{K} \times \mathbb{H}) / \ker \rho_{B}, \tilde{\rho}_{B}) \) onto the open ball in \( (C(\mathbb{H}) / \ker \rho_{V_{E}}, \tilde{\rho}_{V_{E}}) \).
Proof. (1) Let $F \in \ker \rho_B$. Thus $\int F \, d\mu = 0$ for all $\mu \in B$. In particular, $\int F \, d\mu = 0$ for all $\mu \in q^*_V(V_E)$. Let $f = QF = F|_{\{V_E\} \times \mathbb{H}}$ and identify $\{V_E\} \times \mathbb{H}$ with $\mathbb{H}$. If $\nu \in V_E$, let $\mu = q^*_V(\nu)$. We have

$$0 = \int F \, d\mu = \int q_{V_E} F \, d\nu = \int f \, d\nu.$$ 

This shows that $\rho_{V_E}(QF) = 0$. Since $Q$ is obviously a lattice homomorphism, so is $\tilde{Q}$.

(2) Let $\tilde{F} \in C(K \times \mathbb{H})/\ker \rho_B$ with $\tilde{\rho}_B(\tilde{F}) < 1$. Then $F \in C(K \times \mathbb{H})$ and $\rho_B(F) < 1$. Let $f = F|_{\{V_E\} \times \mathbb{H}}$, identified as a function on $\mathbb{H}$. For any $\nu \in V_E$, $q^*_V(\nu) \in B$ and hence

$$\left| \int f \, d\nu \right| = \left| \int q_{V_E} F \, d\nu \right| \leq \rho_B(F) < 1.$$ 

Thus

$$\rho_{V_E}(f) = \sup_{\nu \in V_E} \left| \int f \, d\nu \right| < 1.$$ 

We claim that the function $V \in K \mapsto \rho_{V}(f) \in \mathbb{R}$ is continuous. As per the discussion preceding Proposition 3.2, there is a metric $d'$ on $B_M(\mathbb{H})$ so that

$$d'(\nu_1, \nu_2) \geq \left| \int f \, d\nu_1 - \int f \, d\nu_2 \right| \quad \text{for all } \nu_1, \nu_2 \in B_M(\mathbb{H})$$

and that the associated Hausdorff metric $D'$ generates the same topology as $D$ on $K$. Suppose that $V, W \in K$ and $D'(V, W) < \varepsilon$. Let $\nu \in V$. There exists $\nu' \in W$ such that

$$\left| \int f \, d\nu - \int f \, d\nu' \right| \leq d'(\nu, \nu') < \varepsilon.$$ 

It follows that $\rho_{V}(f) \leq \rho_{W}(f) + \varepsilon$. The claim follows by symmetry.

By continuity, there is an open neighborhood $O$ of $V_E$ in $K$ such that

$$\sup_{V \in O} \rho_{V}(f) < 1.$$ 

Choose a continuous function $h : K \to [0, 1]$ such that $h(V_E) = 1$ and that $h(V) = 0$ for all $V \notin O$. Let $G$ be the function on $K \times \mathbb{H}$ defined by

$$G(V, x) = h(V)f(x).$$

Then $G \in C(K \times \mathbb{H})$. We have

$$\rho_B(G) = \sup_{V \in K} \sup_{\nu \in V} \left| \int q_{V}(G) \, d\nu \right| = \sup_{V \in K} h(V) \rho_{V}(f).$$
If \( V \notin \mathcal{O} \), then \( h(V) = 0 \). Otherwise, \( 0 \leq h(V) \leq 1 \). Hence
\[
\rho_B(G) \leq \sup_{V \in \mathcal{O}} \rho_V(f) < 1.
\]
This proves that \( \tilde{G} \) belongs to the open ball of \( (C(K \times \mathbb{H})/ \ker \rho_B, \tilde{\rho}_B) \). Finally,
\[
\tilde{QG} = \tilde{QG} = (G|_{\{V_E\} \times \mathbb{H}}) = (h(V_E)f) = \tilde{f} = \tilde{F}.
\]

**Proof of Theorem 1.2.** Let \( X \) be the separable Banach lattice defined in Lemma 3.3. Let \( E \) be a separable Banach lattice. By Proposition 3.1, there exists \( V_E \in \mathcal{K} \) such that \( E \) is lattice isometric to the completion of \( (C(\mathbb{H})/ \ker \rho_{V_E}, \tilde{\rho}_{V_E}) \). We will identify \( E \) with the completion.

Define \( \tilde{Q} \) as in Lemma 3.4. By the lemma, \( \tilde{Q} \) extends uniquely to a lattice homomorphism \( Q \) that maps the open ball of \( X \) onto the open ball of \( E \). Hence \( Q \) is a lattice quotient from \( X \) onto \( E \). (See the Introduction.)

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**References**