ON SWITCHING PROBABILITY MEASURES AND QUESTIONS OF KARDARAS

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Abstract. Let $\mathcal{K}$ be a convex bounded positive set in $L^1(\mathbb{P})$. Kardaras [6] asked the following two questions: (1) If the relative $L^0(\mathbb{P})$-topology is locally convex on $\mathcal{K}$, does there exist $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$? (2) If $\mathcal{K}$ is closed in the $L^0(\mathbb{P})$-topology and there exists $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on $\mathcal{K}$, does there exist $\mathbb{Q}' \sim \mathbb{P}$ such that $\mathcal{K}$ is $\mathbb{Q}'$-uniformly integrable? In the paper, we show that, no matter $\mathcal{K}$ is positive or not, the first question has a negative answer in general and the second one has a positive answer. In addition to answering these questions, we establish probabilistic and topological characterizations of existence of $\mathbb{Q} \sim \mathbb{P}$ satisfying these desired properties. We also investigate the peculiar effects of $\mathcal{K}$ being positive.

1. Introduction

The Fundamental Theorem of Asset Pricing establishes the prominent importance of working under an equivalent probability measure $\mathbb{Q}$ relative to the original physical probability measure $\mathbb{P}$. It is henceforth of great interest to study how certain analytical and probabilistic properties of a set behave when the underlying probability measure is switched from one to another. This line of research can be traced back to the remarkable work Brannath and Schachermayer [3] and is significantly expanded in two recent papers Kardaras and Žitković [7] and Kardaras [6].

Throughout the paper, let $(\Omega, \Sigma, \mathbb{P})$ stand for a nonatomic probability space. Let $L^0(\mathbb{P}) := L^0(\Omega, \Sigma, \mathbb{P})$ be the space of all random variables modulo a.s.-equality. By the $L^0(\mathbb{P})$-topology, we refer to the topology of convergence in the probability measure $\mathbb{P}$. A probability measure $\mathbb{Q}$ on $(\Omega, \Sigma)$ is equivalent to $\mathbb{P}$ if $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous with respect to one another. In this case, we write $\mathbb{Q} \sim \mathbb{P}$. It is well-known that if $\mathbb{Q} \sim \mathbb{P}$, then $L^0(\mathbb{Q}) = L^0(\mathbb{P})$ and the $L^0(\mathbb{Q})$- and $L^0(\mathbb{P})$-topologies coincide.

Given a sequence $(X_n)$ in $L^0(\mathbb{P})$, a forward convex combination (FCC) of $(X_n)$ is a sequence $(Y_k)$ such that $Y_k \in \text{co}(X_n)_{n=k}$ for each $k \in \mathbb{N}$. Here $\text{co}(\mathcal{A})$ is the convex hull of a set $\mathcal{A}$. For a convex set $\mathcal{K}$ in $L^0(\mathbb{P})$ and $X \in \mathcal{K}$, we say that the (relative) $L^0(\mathbb{P})$-topology on $\mathcal{K}$ is locally convex at $X$ if for any $L^0(\mathbb{P})$-neighborhood $\mathcal{U}$ of $0$, there

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exists a convex neighborhood \( \mathcal{W} \) of \( X \) in the relative \( L^{0}(\mathbb{P}) \)-topology on \( K \) such that 
\[ \mathcal{W} \subset (X + U) \cap K, \]
or equivalently, there exists a convex subset \( \mathcal{W}' \) of \( U \) containing 0 such that 
\[ (X + \mathcal{W}') \cap K \]
is a neighborhood of \( X \) in the relative \( L^{0}(\mathbb{P}) \)-topology on \( K \) (e.g., taking \( \mathcal{W}' = \mathcal{W} - X \), and \( \mathcal{W} = (X + \mathcal{W}') \cap K \), conversely). It is easily seen to be equivalent to that if \( (X_{n}) \) is a sequence in \( K \) that converges to \( X \) in probability, then every FCC of \( (X_{n}) \) also converges to \( X \) in probability. We say that the \( L^{0}(\mathbb{P}) \)-topology is \textit{locally convex on} \( K \) if it is locally convex at every point of \( K \).

The following theorem is part of the main result in [7].

**Theorem 1.1** ([7]). Let \( (X_{n}) \) be a sequence in \( L^{0}_{+}(\mathbb{P}) \) that converges in probability to a random variable \( X \in L^{0}_{+}(\mathbb{P}) \). The following are equivalent.

1. Every FCC of \( (X_{n}) \) converges to \( X \) in probability.
2. The \( L^{0}(\mathbb{P}) \)-topology is locally convex on the set \( K = \text{co} \left\{ (X_{n})_{n=1}^{\infty} \cup \{X\} \right\} \).
3. The \( L^{0}(\mathbb{P}) \)-topology is locally convex on the set \( K \), where the closure is taken in \( L^{0}(\mathbb{P}) \) with respect to the \( L^{0}(\mathbb{P}) \)-topology.
4. There exists \( Q \sim \mathbb{P} \) such that the \( L^{0}(Q) \) - and \( L^{1}(Q) \)-topologies agree on \( K \).

Theorem 1.1 is extended in Kardaras [6]. We say that a set \( A \) in \( L^{0}_{+}(\mathbb{P}) \) is \textit{positive solid} if \( Y \in A \) whenever there exists \( X \in A \) such that \( 0 \leq Y \leq X \). A subset \( A \) in \( L^{0}_{+}(\mathbb{P}) \) is \textit{bounded in probability} if \( \sup_{X \in A} \mathbb{P}(|X| > n) \to 0 \) as \( n \to \infty \).

**Theorem 1.2** ([6]). Let \( K \) be a convex, positive solid set in \( L^{0}_{+}(\mathbb{P}) \) that is bounded in probability. The following are equivalent.

1. The \( L^{0}(\mathbb{P}) \)-topology on \( K \) is locally convex at 0.
2. The \( L^{0}(\mathbb{P}) \)-topology is locally convex on \( K \).
3. There exists \( Q \sim \mathbb{P} \) such that the \( L^{0}(Q) \)- and \( L^{1}(Q) \)-topologies agree on \( K \).
4. There exists \( Q \sim \mathbb{P} \) such that \( K \) is \( Q \)-uniformly integrable.

Connections of these results to Mathematical Finance and Economics are also made in [7, 6]. We refer to the references therein for further connections.

Clearly, (4) \( \implies \) (3) \( \implies \) (2) in Theorem 1.2 hold for an arbitrary set \( K \) in \( L^{0}(\mathbb{P}) \). The following example, however, shows that Conditions (3) and (4) do not necessarily agree for any convex sets in \( L^{0}_{+}(\mathbb{P}) \) that are bounded in probability.

**Example** ([6]). Let \( K = \left\{ X \in L^{0}_{+}(\mathbb{P}) : \mathbb{E}[X] = 1 \right\} \). Then \( K \) is a convex set in \( L^{0}_{+}(\mathbb{P}) \) that is bounded in probability. It is well-known that the \( L^{0}(\mathbb{P}) \) - and \( L^{1}(\mathbb{P}) \)-topologies agree on \( K \). However, there is no \( Q \sim \mathbb{P} \) such that \( K \) is \( Q \)-uniformly integrable.

In view of these results, the following questions were raised in [6]. Let \( K \) be a convex set in \( L^{0}_{+}(\mathbb{P}) \) that is bounded in probability.

(Q1+) Is it true that if the \( L^{0}(\mathbb{P}) \)-topology is locally convex on \( K \), then there exists \( Q \sim \mathbb{P} \) such that the \( L^{0}(Q) \)- and \( L^{1}(Q) \)-topologies agree on \( K \)?

(Q2+) Assume that \( K \) is also closed in \( L^{0}(\mathbb{P}) \) with respect to the \( L^{0}(\mathbb{P}) \)-topology. If there exists \( Q \sim \mathbb{P} \) such that the \( L^{0}(Q) \)- and \( L^{1}(Q) \)-topologies agree on \( K \), does there exist \( Q' \sim \mathbb{P} \) such that \( K \) is \( Q' \)-uniformly integrable?
The “+” signs in the labels above remind us that these questions concern positive sets.

Brannath and Schachermayer [3] showed that if \( K \) is a convex positive set in \( L^0(\mathbb{P}) \) that is bounded in probability, then there exists \( \mathbb{P}' \sim \mathbb{P} \) such that \( K \) is bounded in \( L^1(\mathbb{P}') \). Thus we may assume that \( K \) is bounded in \( L^1(\mathbb{P}) \) in the first place. Hence we may ask the preceding questions for arbitrary convex bounded sets in \( L^1(\mathbb{P}) \). We will refer to these questions as (Q1) and (Q2), respectively. The validity of Theorem 1.1 for suitable nonpositive sequences was alluded to in [7, Remark 1.6].

We now describe the contributions of this paper with regard to the questions raised above. First, it is shown that (Q2) and hence (Q2+) have positive solutions. Precisely,

**Theorem 1.3.** Let \( K \) be a convex bounded subset of \( L^1(\mathbb{P}) \) that is closed in the \( L^0(\mathbb{P}) \)-topology. The following are equivalent.

1. There exists \( Q \sim \mathbb{P} \) such that the \( L^0(Q) \)- and \( L^1(Q) \)-topologies agree on \( K \).
2. There exists \( Q \sim \mathbb{P} \) such that \( K \) is \( Q \)-uniformly integrable.

Recall that a set \( K \) in a vector lattice is solid if \( Y \subseteq K \) whenever there exists \( X \subseteq K \) such that \( |Y| \leq |X| \). Let \( K \) be a convex bounded set in \( L^1(\mathbb{P}) \) and let \( S \) be a nonempty subset of \( K \). We say that the \( L^0(\mathbb{P}) \)-topology on \( K \) is uniformly locally convex-solid on \( S \) if for any \( L^0(\mathbb{P}) \)-neighborhood \( U \) of 0, there exists a convex-solid set \( W \subseteq U \) such that \((X + W) \cap K \) is a neighborhood of \( X \) in the relative \( L^0(\mathbb{P}) \)-topology on \( K \), for every \( X \in S \). If \( S = \{ X \} \) is a singleton set, then we simply say that the \( L^0(\mathbb{P}) \)-topology on \( K \) is locally convex-solid at \( X \). With this terminology, we obtain an intrinsic topological characterization of Condition (2) of Theorem 1.2.

**Theorem 1.4.** Let \( K \) be a convex bounded set in \( L^1(\mathbb{P}) \). The following are equivalent.

1. The \( L^0(\mathbb{P}) \)-topology on \( K \) is uniformly locally convex-solid on \( K \).
2. There exists \( Q \sim \mathbb{P} \) such that the \( L^0(Q) \)- and \( L^1(Q) \)-topologies agree on \( K \).

For a general convex bounded set \( K \) in \( L^1(\mathbb{P}) \), the condition that the \( L^0(\mathbb{P}) \)-topology on \( K \) is uniformly locally convex-solid on \( K \) is genuinely stronger than the plain local convexity. That is, (Q1) has a negative solution in general.

**Theorem 1.5 (Example A).** There exists a convex bounded circled set \( K \) in \( L^1[0,1] \) that is \( L^0[0,1] \)-compact, such that the \( L^0[0,1] \)-topology on \( K \) is locally convex but there does not exist a probability measure \( Q \) on \([0,1] \), equivalent to the Lebesgue measure, such that the \( L^0(Q) \)- and \( L^1(Q) \)-topologies agree on \( K \).

The construction of the example is based on an example of Pryce [10]. However, the set \( K \) in Example A is not contained in \( L^1[0,1] \). Nevertheless, it turns out that (Q1+) has a negative answer in general as well.

**Theorem 1.6 (Example B).** There exist a nonatomic probability space \((\Omega, \Sigma, \mathbb{P})\) and a convex bounded set \( K \) in \( L^1_+(\mathbb{P}) \) such that the \( L^0(\mathbb{P}) \)-topology on \( K \) is locally convex but there does not exist \( Q \sim \mathbb{P} \) such that the \( L^0(Q) \)- and \( L^1(Q) \)-topologies agree on \( K \).
Unlike in Example A, the set $K$ in Example B, as well as the underlying measure space, is nonseparable, neither is it closed in the $L^0(\mathbb{P})$-topology. Hence, the following modifications of $(Q1^+)$ are still open.

(Q1’) Let $K$ be a convex bounded set in $L^1_+(\mathbb{P})$. Assume that the $L^0(\mathbb{P})$-topology is locally convex on $K$. Is it true that if $K$ is closed, or separable, in $L^0(\mathbb{P})$, or if both conditions hold (in particular, if $K$ is compact in $L^0(\mathbb{P})$), then there exists $Q \sim P$ such that the $L^0(Q)$- and $L^1(Q)$-topologies agree on $K$?

Note that the $L^0(\mathbb{P})$-topology is metrizable and thus compact sets in $L^0(\mathbb{P})$ are both closed and separable. Note also that if $K$ is a separable set in $L^0_+(\mathbb{P})$, then there is a non-atomic separable sub-$\sigma$-algebra $\Sigma'$ of $\Sigma$ such that $K$ is $\Sigma'$-measurable.

Finally, concerning the problems $(Q1^+)$ and $(Q1')$, we have the following result in the positive direction that is somewhat surprising and complements Theorem 1.1.

**Theorem 1.7.** Let $(X_n)$ be a bounded sequence in $L^1_+(\mathbb{P})$ and let $K = \text{co}(X_n)$. The following are equivalent.

1. The $L^0(\mathbb{P})$-topology is locally convex on $K$.
2. There exists $Q \sim P$ such that the $L^0(Q)$- and $L^1(Q)$-topologies agree on $K$.

We also include alternative proofs of Theorems 1.1 and 1.2 in the spirit of the present paper as an appendix at the end.

2. “De-switching” probability measures

The main conditions of interest in Theorems 1.1 and 1.2 and in the questions (Q1) and (Q2) involve switching from a probability measure $\mathbb{P}$ to an equivalent one. It would be convenient to reformulate these conditions to remove the switching of probability measures. We begin with a simple lemma that is essentially an exhaustion technique.

**Lemma 2.1.** Let $\xi : \Sigma \to \{0,1\}$ be a function such that $\xi(A) \geq \xi(B)$ if $A \subseteq B$ and that $\xi(A \cup B) = 1$ if $\xi(A) = \xi(B) = 1$. Then there exists $C \in \Sigma$ such that

\[
\mathbb{P}(C) = \sup \left\{ \mathbb{P}(A) : A \in \Sigma, \xi(A) = 1 \right\} \quad \text{and} \quad \mathbb{P}(A \setminus C) = 0 \text{ if } \xi(A) = 1.
\]

**Proof.** Define

\[ a = \sup \left\{ \mathbb{P}(A) : A \in \Sigma, \xi(A) = 1 \right\} \]

Choose a sequence $(A_n)$ in $\Sigma$ such that $\xi(A_n) = 1$ for all $n \in \mathbb{N}$ and $\mathbb{P}(A_n) \to a$. Let $C = \bigcup_{n=1}^{\infty} A_n$. Note that $\xi(\bigcup_{m=1}^{n} A_m) = 1$ for all $n \in \mathbb{N}$. Hence, $\mathbb{P}(A_n) \leq \mathbb{P}(\bigcup_{m=1}^{n} A_m) \leq a$ for all $n$. It follows that $\mathbb{P}(C) = \lim_n \mathbb{P}(\bigcup_{m=1}^{n} A_m) = a$. Suppose that $A \in \Sigma$ and $\xi(A) = 1$. Since $A \setminus C \subseteq A$, $\xi(A \setminus C) \geq \xi(A) = 1$, implying that $\xi(A \setminus C) = 1$. If $\mathbb{P}(A \setminus C) > 0$, we can choose $n \in \mathbb{N}$ such that $\mathbb{P}(A_n) > a - \mathbb{P}(A \setminus C)$. Since $A_n \subseteq C$, $A_n$ and $A \setminus C$ are disjoint sets. Thus,

\[ \mathbb{P}(A_n \cup (A \setminus C)) = \mathbb{P}(A_n) + \mathbb{P}(A \setminus C) > a. \]

But we also have $\xi(A_n \cup (A \setminus C)) = 1$ since $\xi(A_n) = \xi(A \setminus C) = 1$. This contradicts the choice of $a$. Thus $\mathbb{P}(A \setminus C) = 0$, as desired. \(\square\)
Proposition 2.2. Let $\mathcal{K}$ be a convex bounded subset of $\mathbb{L}^1(\mathbb{P})$ and let $\mathcal{S}$ be a nonempty subset of $\mathbb{L}^1(\mathbb{P})$. The following are equivalent.

1. There exists $\mathbb{Q} \sim \mathbb{P}$ such that if $(X_n)$ is a sequence in $\mathcal{K}$ that converges in probability to some $X \in \mathcal{S}$, then $(X_n)$ converges to $X$ in $\mathbb{L}^1(\mathbb{Q})$.

2. For any $\varepsilon > 0$, there exists a measurable set $A$ with $\mathbb{P}(A) > 1 - \varepsilon$ such that if $(X_n)$ is a sequence in $\mathcal{K}$ that converges in probability to some $X \in \mathcal{S}$, then $E_\mathbb{P}[|X_n - X|^A] \to 0$.

3. For any measurable set $A$ with $\mathbb{P}(A) > 0$, there exists a measurable subset $B$ of $A$ with $\mathbb{P}(B) > 0$ such that if $(X_n)$ is a sequence in $\mathcal{K}$ that converges in probability to some $X \in \mathcal{S}$, then $E_\mathbb{P}[|X_n - X|^B] \to 0$.

Proof. (1) $\implies$ (2). Assume that (1) holds. Note that $Y := \frac{d\mathbb{Q}}{d\mathbb{P}} > 0$ a.s. Given $\varepsilon > 0$, choose $r > 0$ such that $A = \{Y \geq r\}$ satisfies $\mathbb{P}(A) > 1 - \varepsilon$. Suppose that $(X_n)$ is a sequence in $\mathcal{K}$ that converges in probability to some $X \in \mathcal{S}$. Then

$$E_\mathbb{P}[|X_n - X|^A] \leq \frac{1}{r} E_\mathbb{P}[1_A|X_n - X|^Y] \leq \frac{1}{r} E_\mathbb{Q}[|X_n - X|] \to 0.$$ 

(2) $\implies$ (3). Assume that (2) holds and let $A \in \Sigma$ be such that $\mathbb{P}(A) > 0$. By (2), choose a measurable set $C$ with $\mathbb{P}(C) > 1 - \mathbb{P}(A)$ such that if $(X_n)$ is a sequence in $\mathcal{K}$ that converges in probability to some $X \in \mathcal{S}$, then $E_\mathbb{P}[|X_n - X|^C] \to 0$. Since $\mathbb{P}(A) + \mathbb{P}(C) > 1$, there exists $B = A \cap C$ such that $\mathbb{P}(B) > 0$ and $\mathbb{P}(B) = 1$. By (2.1), $0 = \mathbb{P}(B \setminus C) = \mathbb{P}(B)$, where the second equality holds because $B \setminus C = B$. This contradicts the choice of $B$. Hence, $\mathbb{P}(C) = 1$.

Let $c = \sup_{X \in \mathcal{K}} E_\mathbb{P}[|X|]$ and let $\varepsilon > 0$ be given. Since $\mathbb{P}(C) = 1$, there is a sequence $(A_k)$ in $\Sigma$ such that $\mathbb{P}(A_k) \uparrow 1$ and that $\xi(A_k) = 1$ for all $n \in \mathbb{N}$. We may replace $A_k$ with $\cup_{j=1}^{k} A_j$, if necessary, to assume that $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$. We may also assume that $\Omega = \cup_{k=1}^{\infty} A_k$ since $\mathbb{P}(\cup_{k=1}^{\infty} A_k) = 1$. Set $A_0 = \emptyset$ and define $Y$ to be $\frac{1}{2^k}$ on the set $A_k \setminus A_{k-1}$ for any $k \in \mathbb{N}$. Then $Y$ is strictly positive and $\mathbb{Q} \sim \mathbb{P}$, where $d\mathbb{Q} = Y^{-1} d\mathbb{P}$. Suppose that $(X_n)$ is a sequence in $\mathcal{K}$ that converges in probability to some $X \in \mathcal{S}$. By Fatou’s Lemma, $E_\mathbb{P}[|X|] \leq \liminf_n E_\mathbb{P}[|X_n|] \leq c$. For any $k$, $0 \leq Y \leq \frac{1}{2k+1}$ on $A_k$. Hence, for any $n, k \in \mathbb{N}$,

$$E_\mathbb{P}[|X_n - X|^A] \leq \frac{1}{2k+1} (E_\mathbb{P}[|X_n|] + E_\mathbb{P}[|X|]) \leq \frac{c}{2k}.$$

Note that $Y \leq 1$ pointwise. Thus, for all $n$ and $k$,

$$E_\mathbb{Q}[|X_n - X|] = \frac{1}{E_\mathbb{P}[Y]} E_\mathbb{P}[|X_n - X|^A] + \frac{1}{E_\mathbb{P}[Y]} E_\mathbb{P}[|X_n - X|^B] \leq \frac{1}{E_\mathbb{P}[Y]} E_\mathbb{P}[|X_n - X|^A] + \frac{c}{2k E_\mathbb{P}[Y]}.$$
Since \( \xi(A_k) = 1 \), \( \mathbb{E}_P[|X_n - X|1_{A_k}] \rightarrow 0 \) as \( n \rightarrow \infty \). Therefore,

\[
\limsup_n \mathbb{E}_Q[|X_n - X|] \leq \frac{c}{2k\mathbb{E}_P[Y]}
\]

for any \( k \), so that \( \mathbb{E}_Q[|X_n - X|] \rightarrow 0 \). Condition (1) thus holds for \( Q \) as chosen. \( \square \)

Although not needed, we remark that \( \frac{dQ}{dP} \) is bounded for \( Q \) constructed above.

Before proceeding further, let us recall the well-known theorem of Komlós [8]. The result is applied to prove the crucial step (3) \( \implies \) (4) in Proposition 2.4 below.

**Lemma 2.3** ([8]). Let \( (X_n) \) be a bounded sequence in \( L^1(P) \). Then there exist a subsequence \( (X_{n_k}) \) of \( (X_n) \) and a random variable \( X \in L^1(P) \) such that for any further subsequence \( (X_{n_{kj}}) \) of \( (X_{n_k}) \),

\[
\lim_m \frac{1}{m} \sum_{j=1}^m X_{n_{kj}} = X \text{ a.s.}
\]

**Proposition 2.4.** Let \( K \) be a convex bounded subset of \( L^1(P) \). The following are equivalent.

1. There exists \( Q \sim P \) such that \( K \) is \( Q \)-uniformly integrable.
2. For any \( \varepsilon > 0 \), there exists a measurable set \( A \) with \( P(A) > 1 - \varepsilon \) such that if \( (X_n) \) is a sequence in \( K \) that is Cauchy in probability, then \( \mathbb{E}_P[|X_n - X_m|1_A] \rightarrow 0 \) as \( n, m \rightarrow \infty \).
3. For any measurable set \( A \) with \( P(A) > 0 \), there exists a measurable subset \( B \) of \( A \) with \( P(B) > 0 \) such that if \( (X_n) \) is a sequence in \( K \) that is Cauchy in probability, then \( \mathbb{E}_P[|X_n - X_m|1_B] \rightarrow 0 \) as \( n, m \rightarrow \infty \).
4. For any measurable set \( A \) with \( P(A) > 0 \), there exists a measurable subset \( B \) of \( A \) with \( P(B) > 0 \) such that \( K_B := \{X1_B : X \in K\} \) is \( P \)-uniformly integrable.

**Proof.** Let \( Q \) be a probability measure and suppose that \( (X_n) \) is a sequence of random variables that is Cauchy in probability and is \( Q \)-uniformly integrable. Then \( (X_n) \) converges in probability to some \( X \in L^1(Q) \). Since \( (X_n) \) is \( Q \)-uniformly integrable, \( X \) is \( Q \)-integrable and \( (X_n) \) converges to \( X \) in \( L^1(Q) \). Therefore, \( (X_n) \) is \( L^1(Q) \)-Cauchy. Using this observation, the implications (1) \( \implies \) (2) \( \implies \) (3) can be shown exactly as in the corresponding steps in Proposition 2.2.

The proof of (4) \( \implies \) (1) is also similar to the proof of (3) \( \implies \) (1) in Proposition 2.2. Define \( \xi : \Sigma \rightarrow \{0, 1\} \) by \( \xi(A) = 1 \) if \( K_A \) is \( P \)-uniformly integrable. Let \( C \) be obtained by applying Lemma 2.1 to \( \xi \). It follows from the assumption (4) that \( P(C) = 1 \). Take an increasing sequence of measurable sets \( (A_k) \) such that \( \xi(A_k) = 1 \) for all \( k \in \mathbb{N} \), \( P(A_k) \rightarrow 1 \), and \( \Omega = \cup_{k=1}^\infty A_k \). Set \( A_0 = \emptyset \) and define \( Y \) to be \( \frac{1}{2^k} \) on the set \( A_k \setminus A_{k-1} \) for any \( k \in \mathbb{N} \). Let \( dQ = \frac{Y}{\mathbb{E}_P[Y]}dP \). Then \( Q \sim P \). We claim that \( K \) is \( Q \)-uniformly integrable. Clearly, \( K \) is bounded in \( L^1(Q) \). Set \( c = \sup_{X \in K}\mathbb{E}_P[|X|] \). Let \( \varepsilon > 0 \) be given. Choose \( k \) large enough so that \( \frac{c}{2^k\mathbb{E}_P[Y]} \leq \varepsilon \). Since \( \xi(A_k) = 1 \), \( K_{A_k} \) is \( P \)-uniformly
integrable. Therefore, there exists \( \delta > 0 \) such that
\[
\sup_{X \in \mathcal{K}} \mathbb{E}_P[|X|1_B] < \frac{\varepsilon \mathbb{E}_P[Y]}{2} \quad \text{if } B \subseteq A_k \text{ and } \mathbb{P}(B) < \delta.
\]
Now, take any \( A \in \Sigma \) such that \( Q(A) < \frac{\delta}{2^k \mathbb{E}_P[Y]} \). Let \( B_1 = A \cap A_k \) and \( B_2 = A \setminus A_k \).
Since \( Y \geq \frac{1}{2^k} \) on \( A_k \), \( Q(B_1) = \frac{1}{\mathbb{E}_P[Y]} \mathbb{E}_P[Y1_{B_1}] \geq \frac{1}{2^k \mathbb{E}_P[Y]} \mathbb{P}(B_1) \), so that
\[
\mathbb{P}(B_1) \leq 2^k \mathbb{E}_P[Y]Q(B_1) \leq 2^k \mathbb{E}_P[Y]Q(A) < \delta.
\]
Thus, for any \( X \in \mathcal{K} \), \( \mathbb{E}_P [|X|1_{B_1}] < \frac{\varepsilon \mathbb{E}_P[Y]}{2} \). Moreover, note that \( 0 \leq Y \leq \frac{1}{2^k+1} \) on \( A_k \supset B_2 \). Thus, if \( X \in \mathcal{K} \), then
\[
\mathbb{E}_Q [|X|1_A] = \frac{1}{\mathbb{E}_P[Y]} \mathbb{E}_P [|X|Y1_{B_1}] + \frac{1}{\mathbb{E}_P[Y]} \mathbb{E}_P [|X|Y1_{B_2}]
\leq \frac{1}{\mathbb{E}_P[Y]} \mathbb{E}_P [|X|1_{B_1}] + \frac{c}{\mathbb{E}_P[Y]} 2^{k+1}
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
This proves that \( \mathcal{K} \) is \( Q \)-uniformly integrable, and thus (4) \( \Rightarrow \) (1).

Assume that (3) holds. Let \( A \) be a measurable set with \( \mathbb{P}(A) > 0 \). Choose a measurable subset \( B \) of \( A \) with \( \mathbb{P}(B) > 0 \) as in Condition (3). We aim to show that \( \mathcal{K}_B \) is \( \mathbb{P} \)-uniformly integrable. Suppose the contrary. By [2, Theorem 5.2.9], there exist a real number \( c' > 0 \) and a sequence \( (X_n) \) in \( \mathcal{K} \) such that for any \( n \in \mathbb{N} \) and any real numbers \( a_1, \ldots, a_n \),
\[
\mathbb{E}_P \left[ \left| \sum_{k=1}^n a_k X_k1_B \right| \right] \geq c' \sum_{k=1}^n |a_k|.
\]
Applying Komlós’ Theorem and relabeling, we may assume that the arithmetic means of \( (X_n) \) converge to some \( X \in L^0(\mathbb{P}) \) a.s. Put
\[
Y_n = \frac{1}{2^n} \sum_{k=1}^{2^n} X_k.
\]
Clearly, \( (Y_n) \subset \mathcal{K} \) is Cauchy in probability, and thus by choice of \( B \), \( (Y_n1_B) \) is Cauchy in \( L^1(\mathbb{P}) \). On the other hand, whenever \( n > m \),
\[
\mathbb{E}_P \left[ |Y_n1_B - Y_m1_B| \right] = \mathbb{E}_P \left[ \left| \sum_{k=1}^{2^m} \left( \frac{1}{2^m} - \frac{1}{2^n} \right) X_k1_B + \sum_{k=2^n+1}^{2^n} \frac{1}{2^n} X_k1_B \right| \right]
\geq c' \left( \sum_{k=1}^{2^m} \left( \frac{1}{2^m} - \frac{1}{2^n} \right) + \sum_{k=2^n+1}^{2^n} \frac{1}{2^n} \right)
= c' \left( 1 - \frac{2^m}{2^n} + \frac{2^n - 2^m}{2^n} \right) \geq c'.
\]
This contradiction completes the proof. \( \square \)

The next corollary clarifies the relationship between Conditions (3) and (4) of Theorem 1.2 and answers the questions (Q2) and (Q2+) in the positive.
Corollary 2.5. Let $\mathcal{K}$ be a convex bounded subset of $L^1(\mathbb{P})$. The following are equivalent.

1. There exists $Q \sim \mathbb{P}$ such that $\mathcal{K}$ is $Q$-uniformly integrable.
2. There exists $Q \sim \mathbb{P}$ such that $\overline{\mathcal{K}}$ is $Q$-uniformly integrable, where the closure is taken in the $L^0(\mathbb{P})$-topology.
3. There exists $Q \sim \mathbb{P}$ such that the $L^0(Q)$- and $L^1(Q)$-topologies agree on $\overline{\mathcal{K}}$.

Proof. Let $Q$ be as given in Condition (1). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\mathbb{E}_Q[|X|1_A] < \varepsilon$ if $X \in \mathcal{K}$ and $Q(A) < \delta$. Suppose that $X \in \overline{\mathcal{K}}$ and $Q(A) < \delta$. Choose a sequence $(X_n)$ in $\mathcal{K}$ that converges to $X$ in $L^0(\mathbb{P})$. Then it also converges to $X$ in $L^1(\mathbb{P})$. Thus by Fatou’s Lemma,
\[ \mathbb{E}_Q[|X|1_A] \leq \liminf_n \mathbb{E}_Q[|X_n|1_A] \leq \varepsilon. \]
Moreover, since $\mathcal{K}$ is bounded in $L^1(Q)$, a similar argument shows that $\overline{\mathcal{K}}$ is also bounded in $L^1(Q)$. Thus $\overline{\mathcal{K}}$ is $Q$-uniformly integrable. This proves $(1) \implies (2)$.

The implication $(2) \implies (3)$ is clear.

Assume that (3) holds. We apply Proposition 2.2 to $\overline{\mathcal{K}}$ with $S = \overline{\mathcal{K}}$. For any $\varepsilon > 0$, we obtain a measurable set $A$ with $\mathbb{P}(A) > 1 - \varepsilon$ such that if $(X_n)$ is a sequence in $\overline{\mathcal{K}}$ that converges to $X \in \mathcal{K}$ in probability, then $\mathbb{E}_P[|X_n - X|1_A] \to 0$. Now let $(X_n)$ be any sequence in $\overline{\mathcal{K}}$ that is Cauchy in probability. Since $\overline{\mathcal{K}}$ is closed in $L^0(\mathbb{P})$, $(X_n)$ converges in probability to some $X \in \overline{\mathcal{K}}$. Therefore, $\mathbb{E}_P[|X_n - X|1_A] \to 0$, and thus $\mathbb{E}_P[|X_n - X_m|1_A] \to 0$ as $n, m \to \infty$. We have thus verified Condition (2) of Proposition 2.4 for the set $\overline{\mathcal{K}}$ and therefore for the set $\mathcal{K}$. By the same result, Condition (1) holds. This proves $(3) \implies (1)$. \qed

Notice that Theorem 1.3 is an immediate consequence of Corollary 2.5.

3. Uniformly locally convex-solid topologies

In this section, we first characterize topologically the $L^1(\mathbb{P})$-bounded convex sets $\mathcal{K}$ on which there exists $Q \sim \mathbb{P}$ such that the $L^0(Q)$- and $L^1(Q)$-topologies agree. The condition, as indicated in Theorem 1.4, is precisely that the relative $L^0(\mathbb{P})$-topology is uniformly locally convex-solid on $\mathcal{K}$, which is introduced in Section 1. We begin our exploration with a result of the Hahn-Banach theorem spirit. Similar results of this type have been achieved in a recent paper [5], where the authors established a “localized” Hahn-Banach theorem on a vector space and applied it to study the uo-dual of a Banach lattice, resulting in a very transparent proof of Theorem 1.2. The following result is an extension of their approach, embracing solidity.

Proposition 3.1. Let $\mathcal{K}$ be a convex set in $L^1(\mathbb{P})$ and let $S$ be a nonempty subset of $\mathcal{K}$. Suppose that the relative $L^0(\mathbb{P})$-topology on $\mathcal{K}$ is uniformly locally convex-solid on $S$. Then for any measurable set $A$ with $\mathbb{P}(A) > 0$, there exists a nonzero random variable $Y \in L^\infty_+(\mathbb{P})$, supported in $A$, such that $\mathbb{E}_P[|X_n - X|Y] \to 0$ for any sequence $(X_n)$ in $\mathcal{K}$ that converges to some $X \in S$ in probability.
Proof. Since \( \mathbb{1}_A \neq 0 \), we can inductively choose \( L_0^0(\mathbb{P}) \)-neighborhoods \( V \) and \( U_k \) of 0 such that
\[
(V + U_1 + U_1) \cap (\mathbb{1}_A + V + U_1 + U_1) = \emptyset,
\]
\[
U_k + kU_k \subseteq U_{k-1}, \quad \text{if } k > 1.
\]

It is easily verified by induction that
\[
(V + U_1 + 2U_2 + \cdots + kU_k + U_k) \cap (\mathbb{1}_A + V + U_1 + 2U_2 + \cdots + U_k + U_k) = \emptyset
\]
for all \( k \geq 1 \). For each \( k \geq 1 \), choose a convex solid set \( W_k \subseteq U_k \) such that, for any \( X \in S \), \( (X + W_k) \cap K \) is a neighborhood of \( X \) in the relative \( L_0^0(\mathbb{P}) \)-topology on \( K \).

Replace \( W_k \) by \( W_k \cap L_1(\mathbb{P}) \), if necessary, to assume that \( W_k \subseteq L_1(\mathbb{P}) \). Let \( B_{L_1(\mathbb{P})} \) be the closed unit ball of \( L_1(\mathbb{P}) \). Since \( V \) is an \( L_0^0(\mathbb{P}) \)-neighborhood of 0, there exists \( r > 0 \) such that \( rB_{L_1(\mathbb{P})} \subseteq V \). Set
\[
C_k = rB_{L_1(\mathbb{P})} + W_1 + 2W_2 + \cdots + kW_k
\]
for each \( k \).

Since \( B_{L_1(\mathbb{P})} \) and each \( W_k \) are convex and solid in \( L_1(\mathbb{P}) \), \( C_k \) is also convex and solid in \( L_1(\mathbb{P}) \) for all \( k \in \mathbb{N} \) (solidity easily follows from the Riesz Decomposition Theorem [1, Theorem 1.13]). Moreover, \( kW_k \subseteq C_k \subseteq C_{k+1} \) for all \( k \). Let
\[
C = \bigcup_{k=1}^{\infty} C_k.
\]
Then it is easily seen that \( C \) is a convex solid set in \( L_1(\mathbb{P}) \). Since \( rB_{L_1(\mathbb{P})} \subseteq C \), \( C \) absorbs \( L_1(\mathbb{P}) \), that is, for any \( X \in L_1(\mathbb{P}) \), \( X \in tC \) whenever \( |t| \geq t_0 \) for some \( t_0 \in \mathbb{R} \).

Let \( \rho : L_1(\mathbb{P}) \to \mathbb{R} \) be the Minkowski functional for \( C \) defined by
\[
\rho(X) = \inf \left\{ \lambda > 0 : \frac{X}{\lambda} \in C \right\}.
\]
Then \( \rho \) is a seminorm on \( L_1(\mathbb{P}) \) (see, e.g., [12, Theorem 1.35]). Note that since \( C_k \) is solid, \( C_k = -C_k \), and thus \( C_k - C_k = 2C_k \) by convexity of \( C_k \). Since \( C_k \subseteq V + U_1 + 2U_2 + \cdots + kW_k + U_k \), \( C_k \cap (\mathbb{1}_A + C_k) = \emptyset \) by (3.1), so that \( \mathbb{1}_A \notin C_k - C_k = 2C_k \) for any \( k \in \mathbb{N} \). Thus \( \frac{1}{2} \notin C \), and therefore, \( \rho(\mathbb{1}_A) \geq 2 \). Define \( \phi_0 : \text{Span}\{\mathbb{1}_A\} \to \mathbb{R} \) by \( \phi_0(\alpha \mathbb{1}_A) = 2\alpha \).

Then \( \phi_0 \) is a linear functional on \( \text{Span}\{\mathbb{1}_A\} \) and \( \phi_0(\alpha \mathbb{1}_A) = 2\alpha \leq 2|\alpha| \leq \rho(\alpha \mathbb{1}_A) \) for any \( \alpha \in \mathbb{R} \). By the vector-space version of Hahn-Banach Theorem (see, e.g., [12, Theorem 3.2]), there is a linear functional \( \phi : L_1(\mathbb{P}) \to \mathbb{R} \) that extends \( \phi_0 \) and such that \( \phi(X) \leq \rho(X) \) for all \( X \in L_1(\mathbb{P}) \). In particular, \( \phi(\mathbb{1}_A) = \phi_0(\mathbb{1}_A) = 2 \neq 0 \).

As \( rB_{L_1(\mathbb{P})} \subseteq C \), \( \phi \) is bounded with respect to the \( L_1(\mathbb{P}) \)-norm. Hence there exists \( Y_0 \in L_\infty(\mathbb{P}) \) such that
\[
\phi(X) = E_\mathbb{P}[XY_0]
\]
for all \( X \in L_1(\mathbb{P}) \). In particular, \( E[\mathbb{1}_A Y_0] \neq 0 \) and hence \( Y_0 \mathbb{1}_A \neq 0 \). Set \( Y = |Y_0| \mathbb{1}_A \).

Then \( Y \) is a nonzero random variable in \( L_\infty^+(\mathbb{P}) \) that is supported in \( A \).

Suppose that \( (X_n) \) is a sequence in \( K \) that converges to some \( X \in S \) in the \( L_0^0(\mathbb{P}) \)-topology. Pick any \( k \in \mathbb{N} \). Since \( (X + W_k) \cap K \) is a neighborhood of \( X \) with respect to the relative \( L_0^0(\mathbb{P}) \)-topology on \( K \), there exists \( N \in \mathbb{N} \) such that \( X_n \in X + W_k \) if \( n \geq N \). Consider any \( n \geq N \). As \( W_k \) is solid, \( |X_n - X| \mathbb{1}_A \text{Sign}(Y_0) \in W_k \). Hence
k|X_n - X|₁₄Sign(Y₀) ∈ kW_k ⊆ C_k ⊆ ℂ. Therefore, ρ(k|X_n - X|₁₄Sign(Y₀)) ≤ 1. It follows that \( \mathbb{E}_P[k|X_n - X|₁₄Y₀] = \phi(k|X_n - X|₁₄Sign(Y₀)) \leq 1 \), and thus

\[
\mathbb{E}_P[|X_n - X|₁₄|Y₀|] \leq \frac{1}{k} \quad \text{for any } n \geq N.
\]

This proves that \( \mathbb{E}_P[|X_n - X|₁₄|Y₀|] \rightarrow 0 \). □

We now prove a slightly stronger version of Theorem 1.4. In the proof below, we use the specific metric \( d(X, Y) = \mathbb{E}_P[|X - Y| \wedge 1] \) to generate the \( \mathbb{L}^0(\mathbb{P}) \)-topology. Balls \( B(X, r) \) are taken with respect to this metric for any \( X \in \mathbb{L}^0(\mathbb{P}) \) and any \( r > 0 \).

**Theorem 3.2.** Let \( K \) be a convex bounded subset of \( \mathbb{L}^1(\mathbb{P}) \) and let \( S \) be a nonempty subset of \( K \). The following are equivalent.

1. The relative \( \mathbb{L}^0(\mathbb{P}) \)-topology on \( K \) is uniformly locally convex-solid on \( S \).
2. There exists \( Q \sim \mathbb{P} \) such that if \( (X_n) \) is a sequence in \( K \) that converges in probability to some \( X \in S \), then \( (X_n) \) converges to \( X \) in \( \mathbb{L}^1(Q) \).

**Proof.** Assume that (1) holds. Let \( A \) be a \( \mathbb{P} \)-measurable set with \( \mathbb{P}(A) > 0 \). By Proposition 3.1, there exists a nonzero random variable \( Y \in \mathbb{L}^\infty(\mathbb{P}) \), supported in \( A \), such that \( \mathbb{E}_P[|X_n - X|Y] \rightarrow 0 \) for any sequence \( (X_n) \) in \( K \) that converges to some \( X \in S \) in probability. There exists \( r > 0 \) such that \( B = \{ Y \geq r \} \) has positive \( \mathbb{P} \)-measure. By choice, \( B \subseteq A \). Also,

\[
\mathbb{E}_P[|X_n - X|₁₄B] \leq \frac{1}{r} \mathbb{E}_P[|X_n - X|Y] \rightarrow 0
\]

if \( (X_n) \) is a sequence in \( K \) that converges to some \( X \in S \) in probability. Thus Condition (3) of Proposition 2.2 is satisfied, and hence by the same result, (2) holds.

Assume that (2) holds. Let \( Q \) be given as in Condition (2), and write \( Y = \frac{Q_0}{Q^0} \). Let \( U \) be an \( \mathbb{L}^0(\mathbb{P}) \)-neighborhood of 0, and let \( r > 0 \) be such that \( B(0, r) \subseteq U \). Choose \( \delta > 0 \) such that \( \mathbb{P}(Y < \delta) < \frac{1}{2} \) and let \( s = \frac{r \delta}{2} \). Let

\[
\mathcal{W} = \left\{ X \in \mathbb{L}^1(\mathbb{P}) : \mathbb{E}_Q[|X|] = \mathbb{E}_P[|X|Y] < s \right\}.
\]

Obviously, \( \mathcal{W} \) is a convex solid set in \( \mathbb{L}^1(\mathbb{P}) \). If \( X \in \mathcal{W} \), then

\[
d(X, 0) = \mathbb{E}_P[|X| \wedge 1] \leq \mathbb{E}_P[|X|₁₄\{Y \geq \delta\}] + \mathbb{E}_P[₁₄\{Y < \delta\}]
\leq \frac{1}{\delta} \mathbb{E}_P[|X|Y₁₄\{Y \geq \delta\}] + \mathbb{P}(Y < \delta)
\leq \frac{s}{\delta} + \frac{r \delta}{2} = r.
\]

This proves that \( \mathcal{W} \subseteq B(0, r) \subseteq U \). Pick any \( X \in S \). Recall that the \( \mathbb{L}^0(Q) \)- and \( \mathbb{L}^0(\mathbb{P}) \)-topologies agree on \( \mathbb{L}^0(Q) = \mathbb{L}^0(\mathbb{P}) \). It follows easily from the assumption (2) that there exists \( t > 0 \) such that \( \mathbb{E}_Q[|X'' - X|] < s \) for all \( X'' \in B(X, t) \cap K \). This means that

\[
B(X, t) \cap K \subseteq (X + \mathcal{W}) \cap K,
\]

and hence \( (X + \mathcal{W}) \cap K \) is a neighborhood of \( X \) in the relative \( \mathbb{L}^0(\mathbb{P}) \)-topology on \( K \). Thus (1) holds, and the proof is completed. □
Clearly, Theorem 1.4 is an immediate consequence of Theorem 3.2 by taking $\mathcal{S} = \mathcal{K}$. Combining Theorem 1.4 and Corollary 2.5, we also obtain a measure-free characterization of uniform integrability in the sense of Kardaras [6].

**Corollary 3.3.** Let $\mathcal{K}$ be a convex bounded subset of $L^1(\mathbb{P})$. The following are equivalent.

1. The relative $L^0(\mathbb{P})$-topology on $\overline{\mathcal{K}}$ is uniformly locally convex-solid on $\overline{\mathcal{K}}$, where the closure is taken in the $L^0(\mathbb{P})$-topology.
2. There exists $\mathbb{Q} \sim \mathbb{P}$ such that $\mathcal{K}$ is $\mathbb{Q}$-uniformly integrable.

Example A, to be presented in the next section, shows that for the relative $L^0(\mathbb{P})$-topology on $\mathcal{K}$, being uniformly locally convex-solid is strictly stronger than being only local convex, for a general convex bounded set $\mathcal{K}$ in $L^1(\mathbb{P})$. However, we now show that in the presence of positivity, the equivalence of these two conditions may be established for some sets $\mathcal{K}$.

The main additional feature that positivity brings in is the following.

**Lemma 3.4** ([7, Lemma 2.4]). Let $(X_n)$ be a sequence in $L^0(\mathbb{P})$ and let $X \in L^0(\mathbb{P})$. Suppose that every FCC of $(X_n)$ converges to $X$ in probability. Then every FCC of $(|X_n - X|)$ converges to 0 in probability.

**Proof.** Note that $(X_n - X)^- \leq X$ for all $n \in \mathbb{N}$ and $(X_n - X)^- \rightarrow 0$ in probability. Let $d\mu = \frac{1}{1+X} d\mathbb{P}$. Then $\mu$ is a finite measure on $(\Omega, \Sigma)$, $\mu \sim \mathbb{P}$, and $X \in L^1(\mu)$. By Dominated Convergence Theorem, $(X_n - X)^- \rightarrow 0$ in the $L^1(\mu)$-norm, and consequently, any FCC of $((X_n - X)^-)_{n}$ converges to 0 in the $L^1(\mu)$-norm and thus also in the measure $\mu$. Note that the $L^0(\mu)$- and $L^0(\mathbb{P})$-topologies coincide. Hence, any FCC of $((X_n - X)^-)_{n}$ converges to 0 in the probability $\mathbb{P}$. The desired result now follows immediately from the equation $|X_n - X| = (X_n - X) + 2(X_n - X)^-$. \hfill $\square$

As remarked in Section 1, the next two results also hold if $\mathcal{K}$ is assumed to be a convex set in $L^1_+(\mathbb{P})$ that is bounded in probability. The solid hull $\text{so}(\mathcal{A})$ of a set $\mathcal{A}$ is defined by $\text{so}(\mathcal{A}) = \{ Y : |Y| \leq |X| \text{ for some } X \in \mathcal{A} \}$.

**Proposition 3.5.** Let $\mathcal{K}$ be a convex bounded set in $L^1_+(\mathbb{P})$. Assume that the relative $L^0(\mathbb{P})$-topology on $\mathcal{K}$ is locally convex on a countable subset $\mathcal{S}$ of $\mathcal{K}$. Then the relative $L^0(\mathbb{P})$-topology on $\mathcal{K}$ is uniformly locally convex-solid on $\mathcal{S}$.

**Proof.** We first establish the special case where $\mathcal{S}$ is a singleton set, say, $\mathcal{S} = \{X\}$. Again, we use the metric given by $d(X', X'') = \mathbb{E}[|X' - X''| \wedge 1]$ to generate the topology on $L^0(\mathbb{P})$. Let $U$ be a neighborhood of 0 in $L^0(\mathbb{P})$. Let $B_n$ be the ball of radius $\frac{1}{n}$ centered at 0 with respect to the metric $d$. Set

$$W_n = \text{co so} (B_n \cap (\mathcal{K} - X)).$$

Then $W_n$ is a convex solid set (again, one may apply the Riesz Decomposition Theorem to verify solidly), and $(X + W_n) \cap \mathcal{K}$ contains $(X + B_n) \cap \mathcal{K}$. Hence, $(X + W_n) \cap \mathcal{K}$ is a neighborhood of $X$ in the relative $L^0(\mathbb{P})$-topology on $\mathcal{K}$.\hfill $\square$
It remains to show that \( \mathcal{W}_n \subseteq \mathcal{U} \) for some \( n \in \mathbb{N} \). Assume the contrary. Then we can find consecutive finite subsets \( I_n \) of \( \mathbb{N} \) and random variables \( \sum_{k \in I_n} a_k Y_k \notin \mathcal{U} \), where, for any \( n \geq 1 \), \( (Y_k)_{k \in I_n} \subseteq \mathcal{B}_n \cap (\mathcal{K} - X) \), \( a_k \geq 0 \) for \( k \in I_n \), and \( \sum_{k \in I_n} a_k = 1 \). Take random variables \( (X_k)_{k \in I_n} \subseteq \mathcal{B}_n \cap (\mathcal{K} - X) \) such that \( |Y_k| \leq |X_k| \) for each \( k \). Clearly, \( X_k \to 0 \) in \( L^0(\mathbb{P}) \), and hence \( \mathcal{K} \ni X + X_k \to X \) in \( L^0(\mathbb{P}) \). By the local convexity of the relative \( L^0(\mathbb{P}) \)-topology on \( \mathcal{K} \) at \( X \), every FCC of \( (X + X_k) \) converges to \( X \) in probability. It then follows from Lemma 3.4 that every FCC of \( \{X_k\} \) converges to 0 in probability. In particular, \( \sum_{k \in I_n} a_k |X_k| \) converges to 0 in probability, and therefore so does \( \sum_{k \in I_n} a_k Y_k \), since \( \sum_{k \in I_n} a_k |X_k| \leq \sum_{k \in I_n} a_k |X_k| \). This contradicts that \( \sum_{k \in I_n} a_k Y_k \notin \mathcal{U} \) for all \( n \in \mathbb{N} \) and thus proves the special case.

Now we consider the general case. Enumerate the set \( \mathcal{S} \) as a sequence \( (Y_k) \). For each \( k \), by the special case and Theorem 3.2, there exists \( Q_k \sim \mathbb{P} \) such that if \( (X_n) \) is a sequence in \( \mathcal{K} \) that converges in probability to \( Y_k \), then \( (X_n) \) converges to \( Y_k \) in \( L^1(Q_k) \). Let \( \varepsilon > 0 \) be given. Using the equivalence of (1) and (2) in Proposition 2.2, for each \( k \), there exists a measurable set \( A_k \) with \( \mathbb{P}(A_k) > 1 - \frac{\varepsilon}{2^k} \) such that if \( (X_n) \) is a sequence in \( \mathcal{K} \) that converges in probability to \( Y_k \), then \( \mathbb{E}[|X_n - X| I_{A_k}] \to 0 \). Set \( A = \cap_{k=1}^{\infty} A_k \). Then \( A \) is a measurable set with \( \mathbb{P}(A) > 1 - \varepsilon \). By choice, if \( (X_n) \) is a sequence in \( \mathcal{K} \) that converges in probability to some \( Y \in \mathcal{S} \), then \( \mathbb{E}[|X_n - X| I_{A}] \to 0 \). By Proposition 2.2 and Theorem 3.2 again, the relative \( L^0(\mathbb{P}) \)-topology on \( \mathcal{K} \) is uniformly locally convex-solid on \( \mathcal{S} \).

Let \( \mathcal{K} \) be a convex set in \( L^0(\mathbb{P}) \). Say that a subset \( \mathcal{S} \) of \( \mathcal{K} \) is relatively internal in \( \mathcal{K} \) if for any \( X \in \mathcal{K} \), there exist \( Y \in \mathcal{S} \) and \( t > 0 \) such that \( Y + t(Y - X) \in \mathcal{K} \), or equivalently, if for any \( X \in \mathcal{K} \), there exist \( Z \in \mathcal{K} \) and \( 0 < \alpha < 1 \) such that \( \alpha X + (1 - \alpha)Z \in \mathcal{S} \). If \( \mathcal{S} \) is a singleton set, say, \( \mathcal{S} = \{Y\} \), then \( \mathcal{S} \) is relatively internal in \( \mathcal{K} \) if and only if 0 is an internal point of the set \( \text{span}(\mathcal{K} - Y) \) in the vector space \( \text{span}(\mathcal{K} - Y) \) in the usual sense [4, Definition V.1.6]. The next result gives a sufficient condition on the set \( \mathcal{S} \) in order that (Q1+) has an affirmative answer.

**Theorem 3.6.** Let \( \mathcal{K} \) be a convex bounded set in \( L^1(\mathbb{P}) \) that contains a countable relatively internal subset \( \mathcal{S} \). The following are equivalent.

1. The \( L^0(\mathbb{P}) \)-topology is locally convex on \( \mathcal{K} \).
2. There exists \( Q \sim \mathbb{P} \) such that the \( L^0(\mathbb{Q}) \)- and \( L^1(\mathbb{Q}) \)-topologies agree on \( \mathcal{K} \).

**Proof.** The implication (2) \( \implies \) (1) is trivial. We show that (1) \( \implies \) (2). By Proposition 3.5 and Theorem 3.2, choose \( Q \sim \mathbb{P} \) such that if \( (X_n) \subset \mathcal{K} \) converges in probability to some \( X \in \mathcal{S} \), then \( \mathbb{E}_Q[|X_n - X|] \to 0 \). Now let \( (X_n) \) be a sequence in \( \mathcal{K} \) that converges in probability to some \( X \in \mathcal{K} \). We aim to show that \( \mathbb{E}_Q[|X_n - X|] \to 0 \), which will complete the proof. Choose \( Z \in \mathcal{K} \) and \( \alpha \in (0, 1) \) such that \( \alpha X + (1 - \alpha)Z \in \mathcal{S} \). Then \( (\alpha X_n + (1 - \alpha)Z) \) is a sequence in \( \mathcal{K} \) that converges in probability to \( \alpha X + (1 - \alpha)Z \in \mathcal{S} \). Thus

\[
\alpha \mathbb{E}_Q[|X_n - X|] = \mathbb{E}_Q \left[ \left| (\alpha X_n + (1 - \alpha)Z) - (\alpha X + (1 - \alpha)Z) \right| \right] \to 0.
\]
Since \( \alpha > 0 \), it follows that \( \mathbb{E}_Q[|X_n - X|] \longrightarrow 0 \), as desired. \( \square \)

We are ready to prove Theorem 1.7, which complements Theorem 1.1.

**Proof of Theorem 1.7.** In light of Theorem 3.6, it suffices to show that \( K = \text{co}(X_n)_{n=1}^{\infty} \), where \( (X_n) \) is a bounded positive sequence in \( L^1(\mathbb{P}) \), contains a countable subset \( S \) that is relatively internal in \( K \). We claim that such a set is

\[
S = \left\{ \sum_{n=1}^{m} b_n X_n : m \in \mathbb{N}, \text{ each } b_n \geq 0 \text{ is rational, } \sum_{n=1}^{m} b_n = 1 \right\}.
\]

Obviously, \( S \) is a countable subset of \( K \). Suppose that \( X = \sum_{n=1}^{m} a_n X_n \in K \), where \( a_n \geq 0 \) for each \( 1 \leq n \leq m \) and \( \sum_{n=1}^{m} a_n = 1 \). Choose rational numbers \( b_n \geq \frac{a_n}{3} \) for each \( 1 \leq n \leq m \) such that \( b := \sum_{n=1}^{m} b_n \leq 1 \). Note that \( b \) is a rational number. Hence

\[
Y = \sum_{n=1}^{m} b_n X_n + (1 - b)X_{m+1} \in S.
\]

By direct computation,

\[
Y + \frac{1}{2}(Y - X) = \sum_{n=1}^{m} \left( \frac{3b_n}{2} - \frac{a_n}{2} \right) X_n + \frac{3}{2}(1 - b)X_{m+1}.
\]

By choice, \( \frac{3b_n}{2} - \frac{a_n}{2} \geq 0 \) for \( 1 \leq n \leq m \) and \( \frac{3}{2}(1 - b) \geq 0 \). Furthermore,

\[
\sum_{n=1}^{m} \left( \frac{3b_n}{2} - \frac{a_n}{2} \right) + \frac{3}{2}(1 - b) = \frac{3b}{2} - \frac{1}{2} + \frac{3}{2}(1 - b) = 1.
\]

Thus \( Y + \frac{1}{2}(Y - X) \in K \). This proves that \( S \) is relatively internal in \( K \). \( \square \)

4. **Construction of Example A**

In this section, we give an example which shows that for a general convex bounded set \( K \) in \( L^1(\mathbb{P}) \), the \( L^0(\mathbb{P}) \)-topology on \( K \) being uniformly locally convex-solid is strictly stronger than being locally convex. In fact, the set \( K \) we construct is even \( L^0[0,1] \)-compact and circled, i.e., \( K = -K \). (Note that Theorem 1.2 holds for general solid sets, not necessarily positive, and that circledness is a reasonable weakening of solidity).

The example is a modification of an example of Pryce [10]. Denote the Lebesgue measure on \([0,1]\) by \( m \). Let \( (X_n) \) be a sequence of independent random variables in \( L^0[0,1] \), each of which obeys the Cauchy distribution with pdf \( \frac{1}{\pi(1+t^2)}, t \in \mathbb{R} \). Fix \( 1 < p < 2 \). For any \( n \in \mathbb{N} \), let

\[
k_n = n(\log(n+2))^p,
\]

\[
\beta_n = \log(1+k_n^2).
\]

Define the function \( F_n : \mathbb{R} \to \mathbb{R} \) by

\[
F_n(t) = \frac{t}{\beta_n} \mathbb{I}_{[-k_n,k_n]}(t),
\]
and put

\[ Y_n = F_n(X_n). \]

It is easily checked that

\[ \mathbb{E}_m[|Y_n|] = \int_{\mathbb{R}} \frac{|F_n(t)|}{\pi(1 + t^2)} \, dt = \frac{1}{\pi} \]

for all \( n \). Now, set

\[ K = \left\{ \sum_{n=1}^{\infty} a_n Y_n : \sum_{n=1}^{\infty} |a_n| \leq 1 \right\}. \]

It is clear that \( K \) is a convex, circled, and bounded set in \( \mathbb{L}^1[0, 1] \).

We now proceed to verify that \( K \) satisfies the properties in Theorem 1.5.

Lemma 4.1. Let \( (a_i)_{i=1}^{\infty} \) be a sequence of real numbers and let \( b_i = \frac{a_i}{\beta_i} \) for all \( i \in \mathbb{N} \). Fix \( \varepsilon > 0 \). For any disjoint finite sets \( I \) and \( J \) in \( \mathbb{N} \), let

\[ P(I, J) = m\left( \left| \sum_{i \in I} a_i Y_i + \sum_{j \in J} b_j X_j \right| > \varepsilon \right). \]

If \( I \) and \( J \) are disjoint finite subsets of \( \mathbb{N} \) and \( i_0 \notin I \cup J \), then

\[ P(I \cup \{i_0\}, J) \leq \frac{2}{\pi k_{i_0}} P(I, J) + P(I, J \cup \{i_0\}). \]

The empty sum is conventionally regarded as 0. In particular, \( P(\emptyset, \emptyset) = m(\emptyset) = 0 \).

Proof. We have

\[ P(I \cup \{i_0\}, J) = m\left( \left\{ \left| \sum_{i \in I \cup \{i_0\}} a_i Y_i + \sum_{j \in J} b_j X_j \right| > \varepsilon \right\} \right. \]

\[ + m\left( \left\{ \left| \sum_{i \in I \cup \{i_0\}} a_i Y_i + \sum_{j \in J} b_j X_j \right| > \varepsilon \right\} \cap \{ |X_{i_0}| \leq k_{i_0} \} \right). \]

Since \( Y_{i_0} = 0 \) on the set \( \{ |X_{i_0}| > k_{i_0} \} \), the first term on the right is

\[ m\left( \left\{ \left| \sum_{i \in I \cup \{i_0\}} a_i Y_i + \sum_{j \in J} b_j X_j \right| > \varepsilon \right\} \cap \{ |X_{i_0}| > k_{i_0} \} \right) = P(I, J) \cdot m( |X_{i_0}| > k_{i_0} ) \]

by independence. Also,

\[ m( |X_{i_0}| > k_{i_0} ) = \frac{2}{\pi} \int_{k_{i_0}}^{\infty} \frac{1}{1 + t^2} \, dt \leq \frac{2}{\pi k_{i_0}}. \]

Hence, the first term on the right of (4.3) is \( \leq \frac{2}{\pi k_{i_0}} P(I, J) \). On the set \( \{ |X_{i_0}| \leq k_{i_0} \} \), \( a_{i_0} Y_{i_0} = b_{i_0} X_{i_0} \). Thus, the second term on the right in (4.3) is

\[ m\left( \left\{ \left| \sum_{i \in I \cup \{i_0\}} a_i Y_i + \sum_{j \in J} b_j X_j \right| > \varepsilon \right\} \cap \{ |X_{i_0}| \leq k_{i_0} \} \right) \leq P(I, J \cup \{i_0\}). \]

Combining the estimates above proves the lemma. \( \square \)
It is well-known that if \( J \) is a finite subset of \( \mathbb{N} \) and \( b_i, i \in J \), are real numbers, then \( \frac{1}{b} \sum_{j \in J} b_j X_j \) is Cauchy distributed, where \( b = \sum_{j \in J} |b_j| \). Hence, for \( \varepsilon > 0 \),

\[
(4.4) \quad m\left( \left| \sum_{j \in J} b_j X_j \right| > \varepsilon \right) = \frac{2}{\pi} \int_{t}^{\infty} \frac{1}{1 + t^2} dt \leq \frac{2b}{\pi \varepsilon}.
\]

**Lemma 4.2.** In the notation of Lemma 4.1, if \( I \) and \( J \) are disjoint finite subsets of \( \mathbb{N} \), then

\[
P(I, J) \leq \frac{2}{\pi \varepsilon} \prod_{i \in I} \left( 1 + \frac{2}{\pi k_i} \right) \sum_{j \in J} |b_j| + \frac{2}{\pi \varepsilon} \sum_{i \in I} \left| b_i \right| \prod_{i' \in I \setminus \{i\}} \left( 1 + \frac{2}{\pi k_{i'}'} \right).
\]

The product over an empty index set is conventionally regarded as 1.

**Proof.** The proof is by induction on the cardinality of \( I \). If \( I = \emptyset \), then the result holds by (4.4). Suppose that the result holds for a set \( I \) and let \( i_0 \notin I \cup J \). For convenience, let us write \( A_M = \prod_{i \in M} (1 + \frac{2}{\pi k_i}) \) for any finite subset \( M \) of \( \mathbb{N} \). By Lemma 4.1 and the inductive hypothesis, we have

\[
P(I \cup \{i_0\}, J)
\leq \frac{2}{\pi k_{i_0}} P(I, J) + P(I, J \cup \{i_0\})
\leq \frac{2}{\pi k_{i_0}} \frac{2}{\pi \varepsilon} \left[ A_I \sum_{j \in J} |b_j| + \sum_{i \in I} A_{I \setminus \{i\}} |b_i| \right] + \frac{2}{\pi \varepsilon} \left[ A_I \sum_{j \in J \cup \{i_0\}} |b_j| + \sum_{i \in I} A_{I \setminus \{i\}} |b_i| \right]
\leq \frac{2}{\pi \varepsilon} \left( 1 + \frac{2}{\pi k_{i_0}} \right) A_I \sum_{j \in J} |b_j| + \frac{2}{\pi \varepsilon} A_I |b_{i_0}| + \frac{2}{\pi \varepsilon} \sum_{i \in I} \left( 1 + \frac{2}{\pi k_{i_0}} \right) A_{I \setminus \{i\}} |b_i|
\leq \frac{2}{\pi \varepsilon} A_{I \cup \{i_0\}} \sum_{j \in J} |b_j| + \frac{2}{\pi \varepsilon} \sum_{i \in I \cup \{i_0\}} A_{(I \cup \{i_0\}) \setminus \{i\}} |b_i|.
\]

This completes the induction. \( \square \)

Taking \( \emptyset = \emptyset \) in Lemma 4.2 gives

**Lemma 4.3.** If \( I \) is a finite set in \( \mathbb{N} \) and \( a_i, i \in I \), are real numbers, then, for any \( \varepsilon > 0 \),

\[
m\left( \left| \sum_{i \in I} a_i Y_i \right| > \varepsilon \right) \leq \frac{2}{\pi \varepsilon} \sum_{i \in I} \frac{|a_i|}{\beta_i} \prod_{i' \in I \setminus \{i\}} \left( 1 + \frac{2}{\pi k_{i'}'} \right).
\]

**Proposition 4.4.** Any FCC of \((Y_1, -Y_1, Y_2, -Y_2, \ldots)\) converges to 0 in probability.

**Proof.** Let \( \varepsilon > 0 \), \( I \) be a finite subset of \( \mathbb{N} \) and \( a_i, i \in I \), be real numbers such that \( \sum_{i \in I} |a_i| \leq 1 \). Observe that by the choice of \((k_n)\), \( \sum_{n=1}^{\infty} \frac{1}{k_n} < \infty \) and hence \( \prod_{i=1}^{\infty} (1 + \frac{2}{\pi k_i}) \) converges to a nonzero finite number. Therefore, there exists a finite constant \( c \) such that \( \prod_{i \in J} (1 + \frac{2}{\pi k_i}) \leq c \) for any finite subset \( J \) of \( \mathbb{N} \). Let \( i_0 = \min I \). By Lemma 4.3 and the fact that \((\beta_i)\) is an increasing positive sequence,

\[
m\left( \left| \sum_{i \in I} a_i Y_i \right| > \varepsilon \right) \leq \frac{2c}{\pi \varepsilon} \sum_{i \in I} \frac{|a_i|}{\beta_i} \leq \frac{2c}{\pi \varepsilon \beta_{i_0}}.
\]
Observe that if $V \in \text{co}(Y_j, -Y_j, Y_{j+1}, -Y_{j+1}, \ldots)$, then there exists a finite set $I \subset \{j, j + 1, \ldots\}$ and real numbers $a_i$, $i \in I$, with $\sum_{i \in I}|a_i| \leq 1$ such that $V = \sum_{i \in I} a_i Y_i$. Thus, if $V \in \text{co}(Y_j, -Y_j, Y_{j+1}, -Y_{j+1}, \ldots)$, then

$$m(|V| > \varepsilon) \leq \frac{2c}{\pi \varepsilon \beta_j}.$$ 

This, together with $\beta_j \to \infty$, completes the proof of the lemma. 

We now proceed to a general result toward local convexity. Denote by $B_n$ the open ball of radius $\frac{1}{n}$ centered at 0 with respect to the metric $d(X', X'') = \mathbb{E}_\pi[|X' - X''| \wedge 1]$ on $\mathbb{L}^0(\mathbb{P})$.

**Lemma 4.5.** Let $(R_k)$ be a bounded sequence in $\mathbb{L}^1(\mathbb{P})$ such that any FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$ converges to 0 in probability. Set

$$\mathcal{L} = \left\{ \sum_{k=1}^{\infty} a_k R_k : \sum_{k=1}^{\infty} |a_k| \leq 1 \right\}.$$ 

Then for any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that if $G \in B_n \cap \mathcal{L}$, then $G = H + J$, where $\mathbb{E}_\pi[H] \leq \frac{1}{m}$ and $J \in \text{co}(R_k, -R_k)_{k=m+1}^\infty$.

**Proof.** Suppose otherwise. We can find $m \in \mathbb{N}$ and a sequence $(G_n)$ with $G_n \in B_n \cap \mathcal{L}$ such that $G_n$ cannot be decomposed as desired for any $n \in \mathbb{N}$. Write $G_n = \sum_{k=1}^{\infty} a_{nk} R_k$, where $\sum_{k=1}^{\infty} |a_{nk}| \leq 1$ for each $n$. By taking a subsequence if necessary, we may assume that $\lim_n a_{nk} = a_k$ exists for all $k \in \mathbb{N}$. Note that $\sum_{k=1}^{\infty} |a_k| \leq 1$. Set $k_0 = 1$. Take $m_1$ such that $\sum_{k=k_0}^{\infty} |a_{m_1,k} - a_k| \leq \frac{1}{2}$ and then take $k_1 > k_0$ such that $\sum_{k=k_1}^{\infty} |a_{m_1,k}| \leq \frac{1}{\mathcal{R}}$. Now take $m_2 > m_1$ such that $\sum_{k=k_1}^{\infty} |a_{m_2,k} - a_k| \leq \frac{1}{2\mathcal{R}}$ and then take $k_2 > k_1$ such that $\sum_{k=k_2}^{\infty} |a_{m_2,k}| \leq \frac{1}{\mathcal{R}}$. Repeating this process, we obtain a subsequence $(G_{m_n})$ of $(G_n)$ and a sequence $(k_n)$ in $\mathbb{N}$. Note that $G_{m_n} \in B_{m_n} \cap \mathcal{L} \subset B_n \cap \mathcal{L}$ for each $n \in \mathbb{N}$. Thus we abuse the notation to rewrite $(G_{m_n})$ as $(G_n)$. Then

$$\sum_{k=1}^{k_{n-1}} |a_{nk} - a_k| < \frac{1}{2^n} \quad \text{and} \quad \sum_{k=k_{n+1}}^{\infty} |a_{nk}| < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Let

$$U_n = \sum_{k=1}^{k_{n-1}} a_{nk} R_k, \quad V_n = \sum_{k=k_{n-1}+1}^{k_n} a_{nk} R_k \quad \text{and} \quad W_n = \sum_{k=k_{n+1}}^{\infty} a_{nk} R_k.$$ 

Then $G_n = U_n + V_n + W_n$. Clearly, $(U_n)$ converges to $\sum_{k=1}^{\infty} a_k R_k$ in $\mathbb{L}^1(\mathbb{P})$ and hence in $\mathbb{L}^0(\mathbb{P})$, and $(W_n)$ converges to 0 in $\mathbb{L}^1(\mathbb{P})$ and hence in $\mathbb{L}^0(\mathbb{P})$. Note that $(V_n)$ can be expressed as an FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$ and hence converges to 0 in $\mathbb{L}^0(\mathbb{P})$ by assumption. Since $(G_n)$ converges to 0 in $\mathbb{L}^0(\mathbb{P})$, $U_n = G_n - V_n - W_n \to 0$ in $\mathbb{L}^0(\mathbb{P})$ as well. Therefore, $\sum_{k=1}^{\infty} a_k R_k = 0$ a.s. Let $H_n = U_n + W_n$ and $J_n = V_n$. Then $G_n = H_n + J_n$, $(H_n)$ converges in $\mathbb{L}^1(\mathbb{P})$ to 0, and $J_n \in \text{co}(R_k, -R_k)_{k=k_{n-1}+1}^{\infty}$. For sufficiently large $n$, we see that

$$\mathbb{E}_\pi[|H_n|] \leq \frac{1}{m} \quad \text{and} \quad J_n \in \text{co}(R_k, -R_k)_{k=m+1}^{\infty}.$$
contrary to the choice of $G_n$’s. This establishes the lemma.

Recall that the convex-solid hull $\text{co so}(A)$ is convex and solid. Furthermore, it is an easy fact that the solid hull of a convex set in $\mathbb{L}^0_0(\mathbb{P})$ is also convex.

**Proposition 4.6.** Let $(R_k)$ be a bounded sequence in $\mathbb{L}^1(\mathbb{P})$ and let

$$\mathcal{L} = \left\{ \sum_{k=1}^{\infty} a_k R_k : \sum_{k=1}^{\infty} |a_k| \leq 1 \right\}.$$

1. If every FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$ converges to 0 in probability, then the $\mathbb{L}^0(\mathbb{P})$-topology on $\mathcal{L}$ is locally convex at 0.

2. If every FCC of $(|R_k|)$ converges to 0 in probability, then the $\mathbb{L}^0(\mathbb{P})$-topology on $\mathcal{L}$ is locally convex-solid at 0.

**Proof.** Let $r \in \mathbb{N}$ be given. We will find $n \in \mathbb{N}$ such that $\text{co}(\mathcal{B}_n \cap \mathcal{L}) \subseteq \mathcal{B}_r$ in Case (1) and $\text{co so}(\mathcal{B}_n \cap \mathcal{L}) \subseteq \mathcal{B}_r$ in Case (2), from which the desired conclusions follow. For Case (2), note that if $(U_k)$ is an FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$, then there is an FCC $(V_k)$ of $(|R_1|, |R_1|, |R_2|, |R_2|, \ldots)$ such that $|U_k| \leq V_k$ for all $k$. Hence in Case (2), every FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$ converges to 0 in probability as well. Therefore, Lemma 4.5 applies in both cases.

Choose $s \in \mathbb{N}$ such that $\mathcal{B}_s + \mathcal{B}_s \subseteq \mathcal{B}_r$. From the respective assumptions, there exists $m \in \mathbb{N}$ such that $\frac{1}{m} \mathcal{B}_{\mathbb{L}^1(\mathbb{P})} \subseteq \mathcal{B}_s$ and that

$$\text{co}(R_k, -R_k)_{k=m+1}^{\infty} \subseteq \mathcal{B}_s \quad \text{in Case (1)},$$

$$\text{co}(|R_k|)_{k=m+1}^{\infty} \subseteq \mathcal{B}_s \quad \text{in Case (2)}.$$

By Lemma 4.5, there exists $n$ such that

$$\mathcal{B}_n \cap \mathcal{L} \subseteq \frac{1}{m} \mathcal{B}_{\mathbb{L}^1(\mathbb{P})} + \text{co}(R_k, -R_k)_{k=m+1}^{\infty}.$$  

Since the right hand side of (4.7) is a convex set, in Case (1),

$$\text{co}(\mathcal{B}_n \cap \mathcal{L}) \subseteq \frac{1}{m} \mathcal{B}_{\mathbb{L}^1(\mathbb{P})} + \text{co}(R_k, -R_k)_{k=m+1}^{\infty} \subseteq \mathcal{B}_s + \mathcal{B}_s \subseteq \mathcal{B}_r.$$

In Case (2), note that $\text{co}(R_k, -R_k)_{k=m+1}^{\infty}$ is contained in $\text{co so}(|R_k|)_{k=m+1}^{\infty}$ and the latter set is convex and solid. It follows from (4.7) that

$$\mathcal{B}_n \cap \mathcal{L} \subseteq \frac{1}{m} \mathcal{B}_{\mathbb{L}^1(\mathbb{P})} + \text{so co}(|R_k|)_{k=m+1}^{\infty}$$

and that the right hand side is a convex solid set. Therefore, since $\mathcal{B}_s$ is solid,

$$\text{co so}(\mathcal{B}_n \cap \mathcal{L}) \subseteq \frac{1}{m} \mathcal{B}_{\mathbb{L}^1(\mathbb{P})} + \text{so co}(|R_k|)_{k=m+1}^{\infty} \subseteq \mathcal{B}_s + \text{so co} \mathcal{B}_s = \mathcal{B}_s + \mathcal{B}_s \subseteq \mathcal{B}_r.$$

This completes the proof of the proposition. □

We need one more technical lemma toward local convexity of the $\mathbb{L}^0[0, 1]$-topology on $\mathcal{K}$. 


Lemma 4.7. Let $\mathcal{A}$ be a convex circled set in a topological vector space $(\mathcal{X}, \tau)$. Then the relative $\tau$ topology on $\mathcal{A}$ is locally convex if and only if it is locally convex at 0.

Proof. Let $x_0 \in \mathcal{A}$ and let $\mathcal{V}$ be a $\tau$-neighborhood of 0. It suffices to produce a convex set $\mathcal{C}$ and a $\tau$-neighborhood $\mathcal{U}$ of 0 such that

$$(x_0 + \mathcal{U}) \cap \mathcal{A} \subseteq \mathcal{C} \subseteq (x_0 + \mathcal{V}) \cap \mathcal{A}.$$ 

Since the relative $\tau$ topology on $\mathcal{A}$ is locally convex at 0, there is a $\tau$-neighborhood $\mathcal{U}$ of 0 such that $\text{co}(\mathcal{U} \cap \mathcal{A}) \subseteq \mathcal{V}$. Thus $\text{co}(\mathcal{U} \cap 2\mathcal{A}) \subseteq \mathcal{V}$. Let

$$\mathcal{C} = \text{co}((x_0 + \mathcal{U}) \cap \mathcal{A}).$$

To complete the proof, we show that $\mathcal{C} \subseteq (x_0 + \mathcal{V}) \cap \mathcal{A}$. Let $x \in \mathcal{C}$. Write $x = \sum_{i=1}^{n} a_i(x_0 + x_i)$, where $a_i \geq 0$, $\sum_{i=1}^{n} a_i = 1$, $x_i \in \mathcal{U}$ and $x_0 + x_i \in \mathcal{A}$. Then

$$x_i = 2\left(\frac{x_0 + x_i}{2} - \frac{x_0}{2}\right) \in 2\mathcal{A}.$$ 

Hence $x_i \in \mathcal{U} \cap 2\mathcal{A}$. Therefore, $\sum_{k=1}^{n} a_k x_i \in \text{co}(\mathcal{U} \cap 2\mathcal{A}) \subseteq \mathcal{V}$. Thus $x = x_0 + \sum_{k=1}^{n} a_k x_i \in x_0 + \mathcal{V}$. Clearly, $\mathcal{C} \subseteq \mathcal{A}$. Hence, $x \in (x_0 + \mathcal{V}) \cap \mathcal{A}$, as desired. □

Combining Proposition 4.4, Proposition 4.6 and Lemma 4.7, we have

Proposition 4.8. The $\mathbb{L}^0[0,1]$-topology on $\mathcal{K}$ defined by (4.2) is locally convex on $\mathcal{K}$.

The following results conclude $\mathbb{L}^0[0,1]$-compactness of $\mathcal{K}$.

Proposition 4.9. Let $\mathcal{B}$ be the closed unit ball of $\ell^1$ with the relative $\sigma(\ell^1, c_0)$-topology (which coincides with the topology of coordinatewise convergence). Suppose that $(R_k)$ is a bounded sequence in $\mathbb{L}^1(\mathbb{P})$ such that every FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$ converges to 0 in probability. Define a map $T : \mathcal{B} \to \mathbb{L}^0[0,1]$ by $T((a_k)_k) = \sum_{k=1}^{\infty} a_k R_k$. Then $T$ is continuous and $T(\mathcal{B})$ is compact in $\mathbb{L}^0(\mathbb{P})$.

Proof. The second statement follows from the first one since $\mathcal{B}$ is $\sigma(\ell^1, c_0)$-compact. Note that the relative $\sigma(\ell^1, c_0)$-topology on $\mathcal{B}$ is metrizable. Let $(x_n)$ be a sequence in $\mathcal{B}$ that converges coordinatewise to $x \in \mathcal{B}$. It is enough to show that a subsequence of $(Tx_n)$ converges to $Tx$ in $\mathbb{L}^0(\mathbb{P})$. Write $x_n = (a_{nk})_k$ and $x = (a_k)_k$. By passing to a subsequence, we may assume that the inequalities (4.5) hold. Define $U_n, V_n, W_n$ and $U_n + W_n$ as in (4.6). Then $Tx_n = U_n + V_n + W_n((U_n + W_n)$ converges in $\mathbb{L}^1(\mathbb{P})$ to $\sum_{k=1}^{\infty} a_k R_k = Tx$, and $(V_n)$ converges to 0 in probability by assumption. It follows that $(Tx_n)$ converges to $Tx$ in probability, as desired. □

The following is now immediate from Proposition 4.4 and Proposition 4.9.

Corollary 4.10. The set $\mathcal{K}$ defined by (4.2) is a compact subset of $\mathbb{L}^0[0,1]$.

We need a final fact to complete the proof of Theorem 1.5.
Lemma 4.11. Let \((Y_n)\) be as defined in (4.1). For each \(n \in \mathbb{N}\), let

\[
Z_n = \frac{1}{n} \sum_{m=1}^{n} |Y_{n+m}|.
\]

Then \((Z_n)\) converges to \(\frac{1}{\pi}\) in \(L^2[0,1]\) and hence in probability.

Proof. Since \(E_m[|Y_n|] = \frac{1}{\pi}\) for all \(n\), \(E_m[Z_n] = \frac{1}{\pi}\) for all \(n \in \mathbb{N}\). Also, \((|Y_n|)\) is a sequence of independent random variables. Hence,

\[
E \left[ |Z_n - \frac{1}{\pi}|^2 \right] = \text{var}(Z_n) = \frac{1}{n^2} \sum_{m=1}^{n} \text{var}(|Y_{m+n}|)
\]

\[
\leq \frac{1}{n^2} \sum_{m=1}^{n} E_m[Y_{m+n}^2]
\]

\[
= \frac{1}{n^2} \sum_{m=1}^{n} \frac{2}{\beta_{m+n}^2} \int_{0}^{k_{m+n}} \frac{t^2}{\pi(1+t^2)} dt
\]

\[
\leq \frac{1}{n^2} \sum_{m=1}^{n} \frac{2k_{m+n}}{n \pi \beta_{m+n}^2} \leq \frac{2k_2}{n \pi \beta_n^2}.
\]

It is easy to see that \(\beta_n \geq 2 \log k_n \geq 2 \log n\). Therefore,

\[
E \left[ |Z_n - \frac{1}{\pi}|^2 \right] \leq \frac{(\log(2n+2))^p}{\pi(\log n)^2} \to 0
\]

as \(n \to \infty\), and the lemma is proved. \(\Box\)

Completion of proof of Theorem 1.5. By Proposition 4.8 and Corollary 4.10, it remains to verify that there does not exist \(Q \sim m\) such that the \(L^0(Q)\)- and \(L^1(Q)\)-topologies agree on \(K\). Suppose otherwise. Let \(Q\) be such. Let \(U\) be a \(L^0[0,1]\)-neighborhood of 0 such that \(\frac{1}{\pi} \not\in \overline{U}\), where the closure is taken in \(L^0[0,1]\). By Theorem 3.2, the \(L^0[0,1]\)-topology on \(K\) is locally convex-solid at 0. Hence there exists a convex solid set \(W \subseteq U\) such that \(W \cap K\) is a neighborhood of 0 for the relative \(L^0[0,1]\)-topology on \(K\). Note that \(Y_n \to 0\) in \(L^0[0,1]\) (see e.g. Proposition 4.4). Thus there exists \(n_0 \in \mathbb{N}\) such that \(Y_n \in W\) for all \(n > n_0\). Then \(Z_n \in W\) for all \(n > n_0\). But \(Z_n \to \frac{1}{\pi}\) in probability by Lemma 4.11. Hence, \(\frac{1}{\pi} \in \overline{W} \subseteq \overline{U}\), contrary to the choice of \(U\). This contradiction completes the proof. \(\Box\)

We include a remark on the importance of positivity in Theorems 1.1 and 1.7.

Remark 4.12. Put \(K' = \text{co} \left( \{0\} \cup (Y_n)_{n=1}^{\infty} \right)\). Then \(K' \subset K\), so that the relative \(L^0[0,1]\)-topology is also locally convex on \(K'\). The same arguments as in the above proof show that there does not exist \(Q \sim m\) such that the \(L^0(Q)\)- and \(L^1(Q)\)-topologies agree on \(K'\). Hence Theorem 1.7 fails without positivity. Since \(Y_n \to 0\) in \(L^0[0,1]\) and \(K' \subset K\), the main implication (3) \(\implies\) (4) in Theorem 1.1 fails without positivity as well.
5. Construction of Example B

In this section, we construct a convex bounded set \( K \) in \( L^1_+(\mathbb{P}) \) on which the relative \( L^0(\mathbb{P}) \)-topology is locally convex but there does not exist \( Q \sim \mathbb{P} \) such that the \( L^0(\mathbb{Q}) \)- and \( L^1(\mathbb{Q}) \)-topologies coincide on \( K \). This will establish our final result Theorem 1.6.

Let \((\Omega_0, \Sigma_0, \mathbb{P}_0)\) be the two-point probability space on \( \Omega_0 = \{0, 1\} \), where each point is given weight \( \frac{1}{2} \). Let \( \Gamma \) be an uncountable set and let \((\Omega, \Sigma, \mathbb{P})\) be the product probability space of \( \Gamma \times \mathbb{N} \)-copies of \((\Omega_0, \Sigma_0, \mathbb{P}_0)\):

\[
(\Omega, \Sigma, \mathbb{P}) = \prod_{\Gamma \times \mathbb{N}} (\Omega_0, \Sigma_0, \mathbb{P}_0);
\]

cf. [9, p.91]. Then

\[
\Omega = \prod_{\Gamma \times \mathbb{N}} \{0, 1\} = \{0, 1\}^{\Gamma \times \mathbb{N}},
\]

and a generic point in \( \Omega \) is a function \( \eta : \Gamma \times \mathbb{N} \to \{0, 1\} \).

For a subset \( \Theta \) of \( \Gamma \), let

\[
\Sigma_\Theta = \sigma \left( \{ \eta : \eta(\gamma, n) = 0 \} : (\gamma, n) \in \Theta \times \mathbb{N} \right).
\]

Then \( \Sigma_\Theta \subset \Sigma \), and \( \mathbb{P} \) is nonatomic on \( \Sigma_\Theta \). Furthermore, if \( \Theta \) and \( \Theta' \) are disjoint subsets of \( \Gamma \), and \( X \) and \( Y \) are two random variables that are \( \Sigma_\Theta \)- and \( \Sigma_\Theta' \)-measurable, respectively, then \( X \) and \( Y \) are independent. Finally, note that if \( A \in \Sigma \) and \( \mathbb{P}(A) > 0 \), then by the construction of \( \Sigma \) and \( \mathbb{P} \), it is easy to see that there exist a subset \( B \) of \( A \) and a countable subset \( \Theta \) of \( \Gamma \) such that \( B \in \Sigma_\Theta \) and \( \mathbb{P}(A \setminus B) = 0 \).

Let \( \gamma \in \Gamma \). Define random variables on \( \Omega \) by

\[
U_{\gamma,1} = 2^\mathbb{1}_{\{\eta(\gamma,1)=0\}},
\]

\[
U_{\gamma,n} = U_{\gamma,1} + 2^n \mathbb{1}_{\{\eta(\gamma,i)=0, 1 \leq i \leq n\}}, \quad \text{if } n \geq 2.
\]

Clearly, \( U_{\gamma,n} \in L^1_+(\Omega, \Sigma, \mathbb{P}) \) and \( \mathbb{E}_\mathbb{P}[U_{\gamma,n}] \leq 2 \) for any \( (\gamma, n) \in \Gamma \times \mathbb{N} \). If \( \Theta \subseteq \Gamma \), let

\[
K_\Theta = \text{co} \left\{ U_{\gamma,n} : \gamma \in \Theta, n \in \mathbb{N} \right\},
\]

and put

\[
(5.1) \quad K = K_\Gamma = \text{co} \left\{ U_{\gamma,n} : \gamma \in \Gamma, n \in \mathbb{N} \right\}.
\]

Clearly, every random variable in \( K_\Theta \) is \( \Sigma_\Theta \)-measurable. Note that if \( X \in K \), then there is a finite set \( \Theta \subseteq \Gamma \) such that \( X \in K_\Theta \). Moreover, for any set \( \Theta \subseteq \Gamma \), note that

\[
K = \text{co} \left( K_\Theta \cup K_{\Theta'} \right) = \left\{ \alpha X + (1 - \alpha)Y : 0 \leq \alpha \leq 1, X \in K_\Theta, Y \in K_{\Theta'} \right\}.
\]

We first disprove existence of any \( Q \) with the required properties for the set \( K \) constructed above.

**Proposition 5.1.** Let \( K \) be as in (5.1). There does not exist any \( Q \sim \mathbb{P} \) such that the \( L^0(\mathbb{Q}) \)- and \( L^1(\mathbb{Q}) \)-topologies agree on \( K \).
Proof. If the present proposition fails, then by Proposition 2.2, there is a measurable set $A$ with $\mathbb{P}(A) > 0$ such that $\mathbb{E}_\mathbb{P}[|X_n - X|1_A] \to 0$ for any sequence $(X_n)$ in $\mathcal{K}$ that converges in probability to some $X \in \mathcal{K}$. By replacing $A$ with a subset having the same measure, we may assume that there exists a countable subset $\Theta$ of $\Gamma$ such that $A \in \Sigma_\Theta$. Let $\gamma \in \Gamma \setminus \Theta$. Then $(U_{\gamma,n})_n$ is a sequence in $\mathcal{K}$ that converges to $U_{\gamma,1}$ in probability. Consider any $n \geq 2$. Let $B_n = \{\eta : \eta(\gamma, i) = 0, 1 \leq i \leq n\}$. Since $A \in \Sigma_\Theta$ and $B_n \in \Sigma_{\{\gamma\}}$, $A$ and $B_n$ are independent sets. Note that $U_{\gamma,n} - U_{\gamma,1} = 2^n$ on $B_n$ and $\mathbb{P}(B_n) = \frac{1}{2^n}$. Thus

$$\mathbb{E}_\mathbb{P}[|U_{\gamma,n} - U_{\gamma,1}|1_A] \geq \mathbb{E}_\mathbb{P}[|U_{\gamma,n} - U_{\gamma,1}|1_{A \cap B_n}] = 2^n \mathbb{P}(A \cap B_n) = 2^n \mathbb{P}(A) \mathbb{P}(B_n) = \mathbb{P}(A).$$

This contradicts the choice of the set $A$ and concludes the proof. \hfill \Box

We now turn to the proof that the $\mathbb{L}^0(\mathbb{P})$-topology on $\mathcal{K}$ is locally convex.

**Lemma 5.2.** Let $X$ and $Y$ be random variables such that $\mathbb{P}(|X + Y| > \varepsilon) < \varepsilon$ for some $\varepsilon > 0$. Assume that there exist a measurable set $A$ with $\mathbb{P}(A) = \frac{1}{2}$ and a real number $c$ such that $X \leq c$ on $A$ and $X \geq c$ on $A^c$ and that $1_A$ and $Y$ are independent. Then $\mathbb{P}(|Y + c| > \varepsilon) < 4\varepsilon$.

**Proof.** We have

$$\mathbb{P}(A) \mathbb{P}(Y < -c - \varepsilon) = \mathbb{P}(Y < -c - \varepsilon) \leq \mathbb{P}(X \leq c) \mathbb{P}(Y < -c - \varepsilon) \leq \mathbb{P}(X + Y < -\varepsilon) \leq \mathbb{P}(|X + Y| > \varepsilon) < \varepsilon.$$

Hence, $\mathbb{P}(Y < -c - \varepsilon) < 2\varepsilon$. Similarly, by considering $A^c$, we obtain that $\mathbb{P}(Y > -c + \varepsilon) < 2\varepsilon$. Combining these two inequalities gives the desired result. \hfill \Box

**Lemma 5.3.** Let $\Theta$ be a finite subset of $\Gamma$ and $X \in \mathcal{K}_\Theta$. Suppose that $(\alpha_kX_k + (1 - \alpha_k)Y_k)$ converges in probability to $X$, where $X_k \in \mathcal{K}_\Theta$, $Y_k \in \mathcal{K}_{\Theta^c}$ and $0 \leq \alpha_k \leq 1$ for all $k \in \mathbb{N}$. Then $(\alpha_kX_k)$ converges to $X$ in probability and $(1 - \alpha_k)Y_k$ converges to 0 in probability.

**Proof.** There is a sequence $(\varepsilon_k)$ decreasing to 0 such that

$$\mathbb{P}(|\alpha_kX_k - X + (1 - \alpha_k)Y_k| > \varepsilon_k) < \varepsilon_k$$

for all $k \in \mathbb{N}$. Since $\alpha_kX_k - X$ is $\Sigma_\Theta$-measurable and $\mathbb{P}$ is nonatomic on this $\sigma$-algebra, there exist a set $A_k \in \Sigma_\Theta$ with $\mathbb{P}(A_k) = \frac{1}{2}$ and a real number $c_k$ such that $\alpha_kX_k - X \leq c_k$ on $A_k$ and $\alpha_kX_k - X \geq c_k$ on $A^c_k$. By choice, $1_{A_k}$ and $Y_k$ are independent. Hence by Lemma 5.2, $\mathbb{P}(|(1 - \alpha_k)Y_k + c_k| > \varepsilon_k) < 4\varepsilon_k$ for all $k$. Therefore, $|(1 - \alpha_k)Y_k + c_k|$ converges to 0 in probability. It follows that $(\alpha_kX_k - c_k)$ converges to $X$ in probability. To complete the proof, it suffices to show that $c_k \to 0$. Observe that since $X \in \mathcal{K}_\Theta$ and $X_k \in \mathcal{K}_\Theta$ for all $k$, all $X_k$’s and $X$ vanish on the set

$$B = \cap_{\gamma \in \Theta} \{\eta : \eta(\gamma, 1) = 1\}.$$

Since $\Theta$ is finite, $\mathbb{P}(B) > 0$. Thus $-c_k1_B = (\alpha_kX_k - c_k)1_B \to X1_B = 0$ in $\mathbb{L}_0(\mathbb{P})$ implies that $c_k \to 0$, as desired. \hfill \Box
Lemma 5.4. Let $\mathcal{K}$ be as in (5.1). Then no sequence in $\mathcal{K}$ converges to 0 in probability.

Proof. Assume that some sequence $(X_k)$ in $\mathcal{K}$ converges to 0 in probability. Choose a countable subset $\Theta$ of $\Gamma$ such that $X_k \in \mathcal{K}_{\Theta}$ for all $k$. Enumerate $\Theta$ as $\{\gamma_1, \gamma_2, \ldots \}$ and express

$$X_k = \sum_{j=1}^{m_k} a_{kj}V_{kj},$$

where $m_k \in \mathbb{N}$, $a_{kj} \geq 0$, $\sum_{j=1}^{m_k} a_{kj} = 1$ and $V_{kj} \in \mathcal{K}_{\{\gamma_j\}}$ if $1 \leq j \leq m_k$. For convenience, let $a_{kj} = 0$ if $j > m_k$. Since $V_{kj} \in \mathcal{K}_{\{\gamma_j\}}$, $V_{kj} \geq U_{\gamma_j,1} \geq 0$. As a result, $(\sum_{j=1}^{m_k} a_{kj}U_{\gamma_j,1})_k$ converges to 0 in probability. Thus by taking a subsequence of $k$ terms and adjusting coefficients of the remaining terms, ending up with a new convex combination. Therefore, $(a_{kj})_k$ converges to 0 for each $j$. This allows us to perturb $\sum_{j=1}^{m_k} a_{kj}U_{\gamma_j,1}$ slightly, when $k$ is large, by removing the first few terms and adjusting coefficients of the remaining terms, ending up with a new convex combination. Thus by taking a subsequence of $k \in \mathbb{N}$ if necessary, we can find an FCC $(W_k)$ of $(U_{\gamma_j,1})_j$ such that

$$\mathbb{E}_p \left[ \sum_{j=1}^{m_k} a_{kj}U_{\gamma_j,1} - W_k \right] \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$ 

In particular, $(W_k)$ converges to 0 in probability. Being bounded above by the constant 2, $(U_{\gamma_j,1})_j$ is $\mathbb{P}$-uniformly integrable, and thus so is $(W_k)$. Therefore, $\mathbb{E}_p[W_k] \rightarrow 0$. However, since $\mathbb{E}_p[U_{\gamma_j,1}] = 1$ for all $j$, $\mathbb{E}_p[W_k] = 1$ for all $k$, a contradiction. \qed

We are ready to present the proof of the local convexity of the $L^0(\mathbb{P})$-topology on $\mathcal{K}$.

Proposition 5.5. For any $X \in \mathcal{K}$, there exists $Q_X \sim \mathbb{P}$ ($Q_X$ depending on $X$) such that if $(X_k)$ is a sequence in $\mathcal{K}$ that converges to $X$ in probability, then $(X_k)$ converges to $X$ in $L^1(Q_X)$. Consequently, the $L^0(\mathbb{P})$-topology on $\mathcal{K}$ is locally convex.

Proof. The second statement is easily deduced from the first. To prove the first statement, pick $X \in \mathcal{K}$. By Proposition 2.2, it is enough to show that for any $\varepsilon > 0$, there is a measurable set $A$ with $\mathbb{P}(A) > 1 - \varepsilon$ such that $\mathbb{E}_p[|X - X|1_A] \rightarrow 0$ for any sequence $(X_k)$ in $\mathcal{K}$ that converges to $X$ in probability. Let $\varepsilon > 0$ be given. Choose a finite set $\Theta \subseteq \Gamma$ such that $X \in \mathcal{K}_{\Theta}$. Let $n \in \mathbb{N}$ be so large that $\frac{\# \Theta}{2^n} < \varepsilon$. Set

$$B = \bigcup_{\gamma \in \Theta} \left\{ \eta : \eta(\gamma, i) = 0, 1 \leq i \leq n \right\},$$

$$A = B^c = \bigcap_{\gamma \in \Theta} \left\{ \eta : \eta(\gamma, i) = 1 \text{ for some } 1 \leq i \leq n \right\}.$$

Then $\mathbb{P}(B) < \varepsilon$ and hence $\mathbb{P}(A) > 1 - \varepsilon$. For any $\gamma \in \Theta$ and $k \geq n$, $1_{\{\eta(\gamma, i) = 0, 1 \leq i \leq k\}} = 0$ on $A$, and thus $0 \leq U_{\gamma,k}1_A \leq 2 + 2^{n-1}$ for any $\gamma \in \Theta$ and $k \in \mathbb{N}$, so that $0 \leq Y1_A \leq 2 + 2^{n-1}$ for any $Y \in \mathcal{K}_{\Theta}$. Therefore, $\{Y1_A : Y \in \mathcal{K}_{\Theta}\}$ is $\mathbb{P}$-uniformly integrable.

Let $(X_k)$ be a sequence in $\mathcal{K}$ that converges to $X$ in probability. Write

$$X_k = \alpha_kY_k + (1 - \alpha_k)Z_k,$$

where $Y_k \in \mathcal{K}_{\Theta}$, $Z_k \in \mathcal{K}_{\Theta^c}$ and $0 \leq \alpha_k \leq 1$ for all $k$.

By Lemma 5.3, $(\alpha_kY_k)$ converges to $X$ in probability and $((1 - \alpha_k)Z_k)$ converges to 0 in probability. By the above, $(Y_k1_A)$ is $\mathbb{P}$-uniformly integrable; hence, so is $(\alpha_kY_k1_A)$. 

Thus,
\[ \mathbb{E}_P[|\alpha_k Y_k - X|] \rightarrow 0. \]
If \( (\alpha_k) \) does not converge to 1, then, by considering a subsequence, we may assume that \( (\alpha_k) \) converges to some \( \alpha \) with \( 0 \leq \alpha < 1 \). Then \( Z_k = \frac{1}{1-\alpha_k}((1-\alpha_k)Z_k) \rightarrow \frac{1}{1-\alpha} \cdot 0 = 0 \) in probability, contrary to Lemma 5.4. Therefore, \( (\alpha_k) \) converges to 1. Thus since \( \mathbb{E}_P[Y] \leq 2 \) for any \( Y \in \mathcal{K} \),
\[ \mathbb{E}_P[|X_k - \alpha_k Y_k|] = \mathbb{E}_P[|1 - \alpha_k|Z_k] \rightarrow 0. \]
It follows that \( \mathbb{E}_P[|X_k - X|] \rightarrow 0 \), as desired. \( \square \)

Obviously, the set \( \mathcal{K} \) constructed is nonseparable in \( L^0(\mathbb{P}) \). Neither is \( \mathcal{K} \) closed in \( L^0(\mathbb{P}) \). Indeed, for any distinct sequence \( (\gamma_j) \), \( \left\{ \mathbb{1}_{\gamma_j>0} \right\} \) is an independent identically distributed sequence with expectation 1, and thus by Law of Large Numbers, \( \mathbb{1} \)-limit of the arithmetic averages of \( (U_{\gamma_j,1}) \), which all lie in \( \mathcal{K} \). But it is easy to see that \( \mathbb{1} \notin \mathcal{K} \). Thus the question \( (Q1') \) from \S 1, which is a restricted version of \( (Q1+) \), remains open.

**Appendix A. Alternative Proofs of Theorems 1.1 and 1.2**

We close by proving Theorems 1.1 and 1.2 in the spirit of the present paper, which we believe gives further insight into said theorems. We begin with one more lemma. Once again, we will use the metric \( d(X',X'') = \mathbb{E}_P[|X'-X''|] \) to generate the \( L^0(\mathbb{P}) \)-topology.

**Lemma A.1.** Let \( \mathcal{L} \) be a convex circled set in \( L^1(\mathbb{P}) \). Assume that the \( L^0(\mathbb{P}) \)-topology on \( \mathcal{L} \) is locally convex-solid at 0. Then it is uniformly locally convex solid on \( \mathcal{L} \).

**Proof.** Let \( \mathcal{U} \) be a \( L^0(\mathbb{P}) \)-neighborhood of 0. Then \( \frac{\mathcal{U}}{2} \) is also a \( L^0(\mathbb{P}) \)-neighborhood of 0. By assumption, there is a convex-solid set \( \mathcal{W} \subseteq \frac{\mathcal{U}}{2} \) such that \( \mathcal{W} \cap \mathcal{L} \) is a neighborhood of 0 in the relative \( L^0(\mathbb{P}) \)-topology on \( \mathcal{L} \). Thus there exists \( r > 0 \) such that \( \mathcal{B}(0,r) \cap \mathcal{L} \subseteq \mathcal{W} \cap \mathcal{L} \). Let \( X \in \mathcal{L} \). If \( Y \in \mathcal{B}(X,r) \cap \mathcal{L} \), then \( \frac{Y-X}{2} \in \mathcal{L} \), since \( \mathcal{L} \) is convex and circled, and \( \frac{Y-X}{2} \in \mathcal{B}(0,r) \), since \( \mathcal{B}(0,r) \) is solid. Hence, \( \frac{Y-X}{2} \in \mathcal{W} \cap \mathcal{L} \subseteq \mathcal{W} \). This shows that
\[ \mathcal{B}(X,r) \cap \mathcal{L} \subseteq (X + 2\mathcal{W}) \cap \mathcal{L}. \]
Hence, \( (X + 2\mathcal{W}) \cap \mathcal{L} \) is a neighborhood of \( X \) in the relative \( L^0(\mathbb{P}) \)-topology on \( \mathcal{L} \). Since \( 2\mathcal{W} \) is a convex-solid set contained in \( \mathcal{U} \), the proof is complete. \( \square \)

**Proof of Theorem 1.1.** The implications (4) \( \Rightarrow \) (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1) are immediate. Assume that (1) holds. WLOG, assume that \( (X_n) \cup \{X\} \) is bounded in \( L^1(\mathbb{P}) \). Let \( R_n = X_n - X \) for any \( n \in \mathbb{N} \) and
\[ \mathcal{L} = \left\{ \sum_{k=1}^{\infty} a_k R_k : \sum_{k=1}^{\infty} |a_k| \leq 1 \right\}. \]
Note that \( \mathcal{K} \subseteq X + \mathcal{L} \). By Lemma 3.4, every FCC of \( (|R_n|) \) converges to 0 in probability. By Proposition 4.6(2), the \( L^0(\mathbb{P}) \)-topology on \( \mathcal{L} \) is locally convex-solid at 0. By Lemma A.1, the \( L^0(\mathbb{P}) \)-topology on \( \mathcal{L} \) is uniformly locally convex-solid on
By Theorem 1.4, there exists $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on $\mathbb{L}$. If $(U_k)$ is an FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$, then there is an FCC $(V_k)$ of $([R_1], [R_1], [R_2], [R_2], \ldots)$ such that $|U_k| \leq V_k$ for all $k$. Hence every FCC of $(R_1, -R_1, R_2, -R_2, \ldots)$ also converges to 0 in probability. Therefore, it follows from Proposition 4.9 that $\mathbb{L}$ is compact in $L^0(\mathbb{P})$. In particular, $\overline{K} \subseteq X + \mathbb{L}$. Hence Condition (4) of Theorem 1.1 holds. This proves $(1) \implies (4)$.

Proof of Theorem 1.2. The implications $(4) \implies (3) \implies (2) \implies (1)$ are immediate. Assume that $(1)$ holds. Again, WLOG, assume that $\mathbb{K}$ is bounded in $L^1(\mathbb{P})$. By Proposition 3.5, the $L^0(\mathbb{P})$-topology on $\mathbb{K}$ is locally convex-solid at 0. Apply Theorem 3.2 with $\mathcal{S} = \{0\}$ to conclude that there exists $\mathbb{Q} \sim \mathbb{P}$ such that if $(X_n)$ is a sequence in $\mathbb{K}$ that converges to 0 in probability, then $(X_n)$ converges to 0 in $L^1(\mathbb{Q})$.

Let $\varepsilon > 0$ be given. By Proposition 2.2, there is a measurable set $A$ with $\mathbb{P}(A) > 1 - \varepsilon$ such that $\mathbb{E}_\mathbb{P}[|X_n|1_A] \to 0$ for any sequence $(X_n)$ in $\mathbb{K}$ that converges to 0 in probability. Let $(X_n)$ be a sequence in $\mathbb{K}$ that is Cauchy in probability. We want to show that $\mathbb{E}_\mathbb{P}[|X_n - X_m|1_A] \to 0$ as $n, m \to \infty$, which implies (4) by Proposition 2.4. Suppose otherwise. Then there exists $\delta > 0$ and natural numbers $n_1 < m_1 < n_2 < m_2 < \ldots$ such that

$$(A.1) \quad \mathbb{E}_\mathbb{P}[|X_{n_k} - X_{m_k}|1_A] > \delta \quad \text{for any } k \in \mathbb{N}.$$ 

On the other hand, clearly, $(X_{n_k} - X_{m_k})_k$ converges to 0 in probability, and hence so does the sequence $\left(\frac{|X_{n_k} - X_{m_k}|}{2}\right)_k$. Note that

$$0 \leq \frac{1}{2}|X_{n_k} - X_{m_k}| \leq \frac{1}{2}(X_{n_k} + X_{m_k}) \in \mathbb{K}.$$ 

Thus $\frac{|X_{n_k} - X_{m_k}|}{2} \in \mathbb{K}$, due to the positive solidity of $\mathbb{K}$. The choice of the set $A$ yields that $\mathbb{E}_\mathbb{P}[|X_{n_k} - X_{m_k}|1_A] \to 0$, contradicting (A.1).

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