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The Bourgain ℓ^1-index of mixed Tsirelson space

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Abstract

Suppose that $(\mathscr{F}_n)_{n=0}^{\infty}$ is a sequence of regular families of finite subsets of $\mathbb{N}$ such that $\mathscr{F}_0$ contains all singletons, and $(\theta_n)_{n=1}^{\infty}$ is a nonincreasing null sequence in $(0,1)$. The mixed Tsirelson space $T(\mathscr{F}_0,(\theta_n,\mathscr{F}_n)_{n=1}^{\infty})$ is the completion of c_{00} with respect to the implicitly defined norm

$$||x|| = \max\left\{||x||_{\mathscr{F}_0}, \sup_{n\in\mathbb{N}} \sup \theta_n \sum_{i=1}^{k} ||E_i x||\right\},$$

where $||x||_{\mathscr{F}_0} = \sup_{F\in\mathscr{F}_0}||Fx||_{\ell^1}$ and the last supremum is taken over all sequences $(E_i)_{i=1}^{k}$ in $[\mathbb{N}]^{<\infty}$ such that $\max E_i < \min E_{i+1}$ and $\{\min E_i : 1 \leqslant i \leqslant k\} \in \mathscr{F}_n$. In this paper, we compute the Bourgain ℓ^1-index of the space $T(\mathscr{F}_0,(\theta_n,\mathscr{F}_n)_{n=1}^{\infty})$. As a consequence, it is shown that if η is a countable ordinal not of the form ω^{ξ} for some limit ordinal ξ, then there is a Banach space whose ℓ^1-index is ω^{η}.

MSC: 46B20; 46B45

Keywords: Mixed Tsirelson space; Bourgain's ℓ^1-index; Schreier families

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1. Introduction

Mixed Tsirelson spaces were first introduced by Argyros and Deliyanni [4]. They furnished the first examples of asymptotic ℓ^1-spaces shown to be arbitrarily distortable. Since their introduction, their distortion and finite dimensional ℓ^1-structures have been studied extensively [3,5,6,15]. In the present paper, we focus our attention on the measure of ℓ^1-complexity given by Bourgain's ℓ^1-index [8]. The main results of the paper constitute a thorough investigation of the Bourgain ℓ^1-indices of mixed Tsirelson spaces. In the case of (Figiel and Johnson's version of) Tsirelson's space T [10], it was shown in [9] that every normalized block basis in T is equivalent to a subsequence of the unit vector basis. Even though this does not hold in general for mixed Tsirelson spaces, we show that the idea can be exploited by comparing block basic sequences in a mixed Tsirelson space with subsequences of the unit vector basis in a related mixed Tsirelson space (Proposition 3). In particular, we obtain in Corollary 8 that every normalized block basic sequence in a mixed Tsirelson space $T(\mathscr{F}_0, (\theta_n, \mathscr{S}_{\alpha_n})_{n=1}^{\ell})$ defined by finitely many Schreier families is equivalent to a subsequence of the unit vector basis in the same space. Our approach uses norming trees and may be considered as a descendant of that in [7].

In Section 3, the comparison result is used to obtain bounds on the ℓ^1-index. It follows from our work (see Corollary 16 below) that if η is a countable ordinal not of the form ω^{ξ} for some limit ordinal ξ, then there is a Banach space whose ℓ^1-index is ω^{η}. This answers Question 1 in [12]. In Section 4, we introduce a method of constructing ℓ^1-trees of large index. This is a two-step method whereby many $\ell^1(n)$-block basic sequences are first constructed (Lemma 21) and these are then condensed into ℓ^1-trees by a compactness argument (Lemma 22). In the final section, we obtain the precise value of the ℓ^1-index of a mixed Tsirelson space defined in terms of "standard" Schreier families.

We set the notation in the remainder of the section. Endow the power set of $\mathbb{N}$, identified with $2^{\mathbb{N}}$, with the product topology. Denote by $[\mathbb{N}]^{<\infty}$ the subspace consisting of all finite subsets of $\mathbb{N}$. A family $\mathscr{F} \subseteq [\mathbb{N}]^{<\infty}$ is said to be *hereditary* if $G \subseteq F \in \mathscr{F}$ implies $G \in \mathscr{F}$. It is *spreading* if whenever $F = \{n_1, \ldots, n_k\} \in \mathscr{F}$, $n_1 < \cdots < n_k$, and $m_1 < \cdots < m_k$ satisfy $m_i \geqslant n_i$, $1 \leqslant i \leqslant k$, then $\{m_1, \ldots, m_k\} \in \mathscr{F}$. In this case, we also say that $\{m_1, \ldots, m_k\}$ is a *spreading* of F. A *regular* family is one that is hereditary, spreading and compact (as a subset of the topological space $[\mathbb{N}]^{<\infty}$). Let c_{00} be the vector space of all finitely supported real sequences and let (e_k) be the standard unit vector basis of c_{00}. If $\mathscr{F}$ is regular, define the seminorm $\|\cdot\|_{\mathscr{F}}$ on c_{00} by $\|\sum a_k e_k\|_{\mathscr{F}} = \sup_{F \in \mathscr{F}} \sum_{k \in F} |a_k|$. For $E \in [\mathbb{N}]^{<\infty}$ and $x = \sum a_k e_k \in c_{00}$, let $Ex = \sum_{k \in E} a_k e_k \in c_{00}$. Given a sequence of regular families $(\mathscr{F}_n)_{n=0}^{\infty}$ such that $\mathscr{F}_0$ contains all singleton subsets of $\mathbb{N}$, and a nonincreasing null sequence $(\theta_n)_{n=1}^{\infty}$ in $(0,1)$, the *mixed Tsirelson space* $T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\infty})$ is the completion of c_{00} under the implicitly defined norm

$$\|x\| = \max\left\{ \|x\|_{\mathscr{F}_0}, \sup_{n \in \mathbb{N}} \sup \theta_n \sum_{i=1}^{k} \|E_i x\| \right\}, \tag{1}$$

where the last supremum is taken over all sequences $(E_i)_{i=1}^k$ in $[\mathbb{N}]^{<\infty}$ such that $\max E_i < \min E_{i+1}$ and $\{\min E_i : 1 \leqslant i \leqslant k\} \in \mathscr{F}_n$.

If M is an infinite subset of $\mathbb{N}$, denote the set of all finite, respectively infinite, subsets of M by $[M]^{<\infty}$, respectively $[M]$. If E and F are finite subsets of $\mathbb{N}$, we write $E < F$, respectively $E \leqslant F$, to mean $\max E < \min F$, respectively, $\max E \leqslant \min F$ ($\max \emptyset = 0$ and $\min \emptyset = \infty$). We abbreviate $\{n\} < E$ and $\{n\} \leqslant E$ to $n < E$ and $n \leqslant E$, respectively. Given $\mathscr{F} \subseteq [\mathbb{N}]^{<\infty}$, a sequence of finite subsets $\{E_1, \dots, E_n\}$ of $\mathbb{N}$ is said to be *$\mathscr{F}$-admissible* if $E_1 < \cdots < E_n$ and $\{\min E_1, \dots, \min E_n\} \in \mathscr{F}$. If $\mathscr{M}$ and $\mathscr{N}$ are regular subsets of $[\mathbb{N}]^{<\infty}$, we let

$$\mathscr{M}[\mathscr{N}] = \left\{ \bigcup_{i=1}^{k} F_i : F_i \in \mathscr{N} \text{ for all } i \text{ and } \{F_1, \dots, F_k\} \text{ is } \mathscr{M}\text{-admissible} \right\}.$$

Given a sequence of regular families $(\mathscr{M}_i)$, we define inductively $[\mathscr{M}_1, \mathscr{M}_2] = \mathscr{M}_1[\mathscr{M}_2]$ and $[\mathscr{M}_1, \dots, \mathscr{M}_{i+1}] = [\mathscr{M}_1, \dots, \mathscr{M}_i][\mathscr{M}_{i+1}]$. Also, let

$$(\mathscr{M}_1, \dots, \mathscr{M}_k) = \left\{ \bigcup_{i=1}^{k} M_i : M_i \in \mathscr{M}_i, M_1 < \cdots < M_k \right\}.$$

We abbreviate the k-fold construction $(\mathscr{M}, \dots, \mathscr{M})$ as $(\mathscr{M})^k$. Of primary importance are the Schreier classes as defined in [1]. We will need a slightly extended version of such classes. Suppose that $g : \mathbb{N} \to \mathbb{N}$ is a function increasing to ∞. Let $\mathscr{S}_0^g = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\mathscr{S}_1^g = \{F \subseteq \mathbb{N} : |F| \leqslant g(\min F)\}$. Here $|F|$ denotes the cardinality of F. The higher Schreier classes are defined inductively as follows. $\mathscr{S}_{\alpha+1}^g = \mathscr{S}_1^g[\mathscr{S}_\alpha^g]$ for all $\alpha < \omega_1$. If α is a countable limit ordinal, choose a sequence (α_n) strictly increasing to α and set

$$\mathscr{S}_\alpha^g = \{F : F \in \mathscr{S}_{\alpha_n}^g \text{ for some } n \leqslant g(\min F)\}.$$

If g is the identity function, then we obtain the usual Schreier classes, and we abbreviate $\mathscr{S}_\alpha^g$ to $\mathscr{S}_\alpha$. It is clear that $\mathscr{S}_\alpha^g$ is a regular family for all $\alpha < \omega_1$. If $M = (m_1, m_2, \dots)$ is a subsequence of $\mathbb{N}$, let $\mathscr{S}_\alpha(M) = \{\{m_i : i \in F\} : F \in \mathscr{S}_\alpha\}$. Since $\mathscr{S}_\alpha$ is spreading, $\mathscr{S}_\alpha(M) \subseteq \mathscr{S}_\alpha$.

The norm in a mixed Tsirelson space can be computed in terms of trees [7,15]. A *tree* in $[\mathbb{N}]^{<\infty}$ is a finite collection of elements (E_i^m), $0 \leqslant m \leqslant r$, $1 \leqslant i \leqslant k(m)$, in $[\mathbb{N}]^{<\infty}$ so that for each $m, E_1^m < E_2^m < \cdots < E_{k(m)}^m$, and that every E_i^{m+1} is a subset of some E_j^m. The elements E_i^m are called *nodes* of the tree. Any node E_i^m is said to be of *level m*. Nodes at level 0 are called *roots*. If $E_i^n \subseteq E_j^m$ and $n > m$, we say that E_i^n is a *descendant* of E_j^m and E_j^m is an *ancestor* of E_i^n. If, in the above notation, $n = m + 1$, then E_i^n is said to be an *immediate successor* of E_j^m, and E_j^m the *immediate predecessor* of E_i^n. Nodes with no descendants are called *terminal nodes* or *leaves* of the tree. Given a node E in a tree $\mathscr{T}$, denote by $\mathscr{T}_E$ the subtree consisting of the node E together with all its descendants. A tree (E_i^m), $0 \leqslant m \leqslant r$, $1 \leqslant i \leqslant k(m)$, is $(\mathscr{F}_n)$-

admissible if $k(0)=1$ and for every m and i, the collection (E_j^{m+1}) of all immediate successors of E_i^m is an $\mathscr{F}_n$-admissible collection for some $n\in\mathbb{N}$. Given an $(\mathscr{F}_n)$-admissible tree (E_i^m), we define the *history* of the individual nodes inductively as follows. Let $h(E_1^0)=(0)$. If $h(E_i^m)$ has been defined and the collection (E_j^{m+1}) of all immediate successors of E_i^m forms an $\mathscr{F}_n$-admissible collection, then define $h(E_j^{m+1})$ to be the $(m+2)$-tuple $(h(E_i^m),n)$ and let $n(E_j^{m+1})=n$ for each immediate successor E_j^{m+1} of E_i^m. Finally, assign ((θ_n)-compatible) *tags* to the nodes by defining $t(E_i^m)=\prod_{j=0}^m \theta_{n_j}$ if $h(E_i^m)=(n_0,n_1,\dots,n_m)(\theta_0=1)$. If $x\in c_{00}$ and $\mathscr{T}$ is an $(\mathscr{F}_n)$-admissible tree, let $\mathscr{T}x=\sum t(E)\|Ex\|$, where the sum is taken over all leaves in $\mathscr{T}$. It is easily observed that $\|x\|=\max\{\mathscr{T}x:\mathscr{T}$ is an $(\mathscr{F}_n)$-admissible tree$\}$. An $(\mathscr{F}_n)$-admissible tree is said to be *complete* (for a particular $x\in c_{00}$) if $\|Ex\|=\|Ex\|_{\mathscr{F}_0}$ for every leaf E in $\mathscr{T}$. Clearly, for every $x\in c_{00}$, there is a complete tree $\mathscr{T}$ such that $\|x\|=\mathscr{T}x$. Let us observe that if we define $\|x\|$ to be $\sup\sum t(E)\|Ex\|_{\mathscr{F}_0}$, where the sup is taken over all $(\mathscr{F}_n)$-admissible trees $\mathscr{T}$ and the sum is taken over all leaves E in $\mathscr{T}$, then the resulting norm satisfies the implicit equation (1).

Proposition 1. *Let $T(\mathscr{F}_0,(\theta_n,\mathscr{F}_n)_{n=1}^\infty)$ be as above. Choose a strictly increasing sequence of integers $(m_k)_{k=0}^\infty$ such that $m_0=0$ and $\theta_{m_{k+1}}\leqslant\frac{1}{2}\theta_{m_k}$ for all $k\in\mathbb{N}$. If $m_{k-1}<n\leqslant m_k$, let $\mathscr{G}_n=\{F\in\mathscr{F}_n : k\leqslant F\}\cup\mathscr{S}_0$. Then $T(\mathscr{F}_0,(\theta_n,\mathscr{F}_n)_{n=1}^\infty)$ is isomorphic to $T(\mathscr{F}_0,(\theta_n,\mathscr{G}_n)_{n=1}^\infty)$ via the formal identity.*

Proof. Denote the norms on $T(\mathscr{F}_0,(\theta_n,\mathscr{F}_n)_{n=1}^\infty)$ and $T(\mathscr{F}_0,(\theta_n,\mathscr{G}_n)_{n=1}^\infty)$ by $\|\cdot\|$ and $|||\cdot|||$ respectively. Clearly, $|||x|||\leqslant\|x\|$ for all $x\in c_{00}$. Given a fixed element $x\in c_{00}$, let $\mathscr{T}^{\mathscr{F}}$ denote a complete $(\mathscr{F}_n)$-admissible tree such that $\|x\|=\mathscr{T}^{\mathscr{F}}x$. If F is a node of $\mathscr{T}^{\mathscr{F}}$ other than the root, let $G_F=F\cap[k,\infty)$, where k is the unique integer such that $m_{k-1}<\max\{n_1,\dots,n_r\}\leqslant m_k, h(F)=(0,n_1,\dots,n_r)$. If F is the root of $\mathscr{T}^{\mathscr{F}}$, let $G_F=F$. Then $\mathscr{T}^{\mathscr{G}}=\{G_F : F\in\mathscr{T}^{\mathscr{F}}\}$ is a $(\mathscr{G}_n)$-admissible tree. For any $r\in\mathbb{N}$, let $\mathscr{L}_r$ be the set of level r leaves in $\mathscr{T}^{\mathscr{F}}$. Arrange the elements in $\mathscr{L}_r$ as $F_1<F_2<\cdots<F_\ell$. If $1\leqslant j\leqslant\ell$, write $h(F_j)=(0,n_{j,1},\dots,n_{j,r})$ and determine k_j such that $m_{k_j-1}<\max\{n_{j,1},\dots,n_{j,r}\}\leqslant m_{k_j}$. If $k_j\leqslant j$, then $k_j\leqslant j\leqslant F_j$. Thus $G_{F_j}=F_j\cap[k_j,\infty)=F_j$. Otherwise, $j<k_j$, and hence

$$t(F_j)\|F_jx\|_{\mathscr{F}_0}\leqslant\theta_{n_{j,1}}\dots\theta_{n_{j,r}}\|x\|_{\mathscr{F}_0}\leqslant\theta_1^{r-1}\theta_{m_{k_j-1}}\|x\|_{\mathscr{F}_0}\leqslant\theta_1^{r-1}\theta_{m_j}\|x\|_{\mathscr{F}_0}.$$

Therefore,

$$\begin{aligned}\sum_{F\in\mathscr{L}_r} t(F)\|Fx\|_{\mathscr{F}_0}&\leqslant\sum_{\{j:j<k_j\}}\theta_1^{r-1}\theta_{m_j}\|x\|_{\mathscr{F}_0}+\sum_{\{j:k_j\leqslant j\}} t(G_{F_j})\|G_{F_j}x\|_{\mathscr{F}_0}\\&\leqslant\theta_1^{r-1}\|x\|_{\mathscr{F}_0}\sum_{j=1}^\infty\theta_{m_j}+\sum_{F\in\mathscr{L}_r} t(G_F)\|G_Fx\|_{\mathscr{F}_0}.\end{aligned}$$

Finally,

$$\begin{aligned}||x|| = \mathscr{T}^{\mathscr{F}} x &= \sum_{r=1}^{\infty} \sum_{F \in \mathscr{L}_r} t(F) ||Fx||_{\mathscr{F}_0} \\ &\leqslant ||x||_{\mathscr{F}_0} \sum_{r=1}^{\infty} \theta_1^{r-1} \sum_{j=1}^{\infty} \theta_{m_j} + \sum_{r=1}^{\infty} \sum_{F \in \mathscr{L}_r} t(G_F) ||G_F x||_{\mathscr{F}_0} \\ &\leqslant \frac{2\theta_{m_1}}{1-\theta_1} |||x||| + |||x||| = \left(\frac{2\theta_{m_1}}{1-\theta_1} + 1 \right) |||x|||. \qquad \square\end{aligned}$$

If $\mathscr{F}$ is a closed subset of $[\mathbb{N}]^{<\infty}$, let $\mathscr{F}'$ be the set of all limit points of $\mathscr{F}$. Define a transfinite sequence of sets $(\mathscr{F}^{(\alpha)})_{\alpha<\omega_1}$ as follows: $\mathscr{F}^{(0)} = \mathscr{F}$, $\mathscr{F}^{(\alpha+1)} = (\mathscr{F}^{(\alpha)})'$ for all $\alpha<\omega_1$; $\mathscr{F}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathscr{F}^{(\beta)}$ if α is a countable limit ordinal. If $\mathscr{F}$ is regular, we let $\iota(\mathscr{F})$ be the unique ordinal α such that $\mathscr{F}^{(\alpha)} = \{\emptyset\}$. It is well known that $\iota(\mathscr{S}_\gamma) = \omega^\gamma$ for all $\gamma<\omega_1$ [1, Proposition 4.10]. The same is true if $\mathscr{S}_\gamma$ is replaced by any $\mathscr{S}_\gamma^g$.

From now on, we fix a sequence of regular families $(\mathscr{F}_n)_{n=0}^{\infty}$ such that $\mathscr{S}_0 \subseteq \mathscr{F}_0$, and a nonincreasing null sequence $(\theta_n)_{n=1}^{\infty}$ in $(0,1)$. Denote the mixed Tsirelson space $T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\infty})$ by X. Let $\alpha_n = \iota(\mathscr{F}_n)$, $n \in \mathbb{N} \cup \{0\}$. There is no loss of generality in assuming that $\alpha_n > 1$ for all $n \in \mathbb{N}$. Since $T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\infty})$ is obviously isometric to $T(\mathscr{F}_0, (\theta_n, \bigcup_{k=1}^{n} \mathscr{F}_k)_{n=1}^{\infty})$ via the formal identity, we may also assume that $(\alpha_n)_{n=1}^{\infty}$ is a nondecreasing sequence. In the notation of Proposition 1, $\iota(\mathscr{G}_n) = \iota(\mathscr{F}_n) = \alpha_n$, $n \in \mathbb{N}$. It is straightforward to check that $\bigcup_{n=1}^{\infty} \mathscr{G}_n$ is a regular family. Relabelling each $\mathscr{G}_n$ as $\mathscr{F}_n$, $n \in \mathbb{N}$, we may henceforth assume that $\mathscr{S}_0 \subseteq \mathscr{F}_n$ for all $n \in \mathbb{N}$ and that $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ is regular. Denote $\sup_{n\in\mathbb{N}} \alpha_n$ by α. Note that $\iota(\mathscr{F}) = \alpha$.

2. An estimate on the norm

Lemma 2. *Let $\mathscr{G}$ and $\mathscr{H}$ be regular families. Suppose $\bigcup_{j=1}^{k} F_j \in \mathscr{G}[\mathscr{H}]$, where $F_1 < F_2 < \cdots < F_k$. If $F_j \notin \mathscr{H}$ for all j, $1 \leqslant j \leqslant k$, then $\{\min F_1, \ldots, \min F_k\} \in \mathscr{G}$.*

Proof. For any nonempty set $G \in \mathscr{G}[\mathscr{H}]$, let $\mathscr{H}(G) = G \cap [1,n]$, where n is the largest integer in G such that $G \cap [1,n] \in \mathscr{H}$. There is a unique decomposition $G = \bigcup_{j=1}^{k} G_j$, where $G_1, \ldots, G_k \neq \emptyset$ and $G_1 = \mathscr{H}(G), G_{j+1} = \mathscr{H}(G \backslash (G_1 \cup \cdots \cup G_j)), 1 \leqslant j < k$. We claim that $\{\min G_1, \ldots, \min G_k\} \in \mathscr{G}$. To see this, note that since $G \in \mathscr{G}[\mathscr{H}]$, we can write $G = \bigcup_{i=1}^{\ell} H_i$, where $H_1 < \cdots < H_\ell$, $H_1, \ldots, H_\ell \in \mathscr{H}$, and $\{\min H_1, \ldots, \min H_\ell\} \in \mathscr{G}$. Clearly, $H_1 \subseteq G_1$. If $k \geqslant 2$, then $\min H_2 \leqslant \min G_2$. If $\max H_2 > \max G_2$, then $G_2 \subsetneq H_2 \subseteq G$. In particular, $G_3 \neq \emptyset$ and $\min G_3 \in H_2$. Therefore, $G_2 \cup \{\min G_3\} \in \mathscr{H}$, contrary to the fact that $G_2 = \mathscr{H}(G \backslash G_1)$. Thus $\max H_2 \leqslant \max G_2$. Continuing this argument, we conclude that $\max H_r \leqslant \max G_r$

for all $1 \leqslant r \leqslant k$. It follows that $\{\min G_1, \ldots, \min G_k\}$ is a spreading of $\{\min H_1, \ldots, \min H_k\} \in \mathcal{G}$. Hence $\{\min G_1, \ldots, \min G_k\} \in \mathcal{G}$.

Now suppose that $F_1, \ldots, F_k$ are as in the statement of the lemma. Let $G_j = \mathcal{H}(F_j)$, $1 \leqslant j \leqslant k$, and let $G = G_1 \cup \cdots \cup G_k$. Since $G_j \subseteq F_j$ for $1 \leqslant j \leqslant k$, $G \subseteq \bigcup_{j=1}^{k} F_j \in \mathcal{G}[\mathcal{H}]$. Note that $F_j \notin \mathcal{H}$ implies $G_j \subsetneqq F_j$. Therefore, $\mathcal{H}(G) = G_1$ and

$$\mathcal{H}(G \backslash (G_1 \cup \cdots \cup G_j)) = G_{j+1}, \quad 1 \leqslant j \leqslant k.$$

From the previous paragraph, we conclude that $\{\min G_1, \ldots, \min G_k\} \in \mathcal{G}$. Hence $\{\min F_1, \ldots, \min F_k\} = \{\min G_1, \ldots, \min G_k\} \in \mathcal{G}$. □

Proposition 3. *Suppose $\epsilon > 0$ and $\mathcal{G}$ is a regular family. Assume that there exists $m_0 \in \mathbb{N}$ such that for all $m \geqslant m_0$, there exist $n_1, \ldots, n_s \in \mathbb{N}$ such that $\theta_m < \epsilon \theta_{n_1} \ldots \theta_{n_s}$ and $\mathcal{F}_m \subseteq [\mathcal{G}, \mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$. Then there exists a constant $K = K(\epsilon, m_0) < \infty$ such that for any normalized block basic sequence $(x_k)_{k=1}^{p}$ in X and any real sequence $(a_k)_{k=1}^{p}$,*

$$\left\| \sum_{k=1}^{p} a_k x_k \right\| \leqslant K \left\| \sum_{k=1}^{p} a_k e_{i_k} \right\| + 2\epsilon \rho_1 \left(\sum_{k=1}^{p} a_k e_{i_k} \right) + 2\rho_2 \left(\sum_{k=1}^{p} a_k e_{i_k} \right) + 2\epsilon \sum_{k=1}^{p} |a_k|, \tag{2}$$

where $i_k = \max \operatorname{supp} x_k$, $1 \leqslant k \leqslant p$, and ρ_1 and ρ_2 are the norms on the mixed Tsirelson spaces $T(\mathcal{F}, (\theta_n, \mathcal{F}_n)_{n=1}^{\infty})$ and $T(\mathcal{G}, (\theta_n, \mathcal{F}_n)_{n=1}^{\infty})$, respectively. (Recall that $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$.)

Proof. With the given notation, let $x = \sum_{k=1}^{p} a_k x_k$ and $y = \sum_{k=1}^{p} a_k e_{i_k}$. Also let G_k be the integer interval $(i_{k-1}, i_k] (i_0 = 0)$. Since $x \in c_{00}$, there exists a complete $(\mathcal{F}_n)$-admissible tree $\mathcal{T}$ such that $\|x\| = \mathcal{T}x$. Each node $E \in \mathcal{T}$ may be assumed to be contained in the integer interval $[1, i_p]$. Call a node E *long* if $E \cap G_k \neq \emptyset$ for at least two values of k. Otherwise, term the node *short*. Let N be the smallest number such that $\theta_N \leqslant \epsilon$. Take $\mathcal{E}_1$ to be the collection of all minimal elements in the set of all long nodes $E \in \mathcal{T}$ such that $n(E) > N$. Minimality is taken with respect to the order (reverse inclusion) in the tree $\mathcal{T}$. Similarly, let $\mathcal{E}_2$ be the collection of all minimal elements of the set of all short nodes that are not in $\cup\{\mathcal{T}_E : E \in \mathcal{E}_1\}$. Then let $\mathcal{E}_3$ be the set of all leaves in $\mathcal{T}$ that are not in $\cup\{\mathcal{T}_E : E \in \mathcal{E}_1 \cup \mathcal{E}_2\}$. Observe that

$$\mathcal{T}x \leqslant \sum_{j=1}^{3} \sum_{E \in \mathcal{E}_j} t(E) \|Ex\|$$

The proof of the proposition is completed by combining Lemmas 4–7 below. □

Lemma 4. $\sum_{E\in\mathscr{E}_1} t(E)\|Ex\| \leqslant 2\epsilon \sum_{k=1}^{p} |a_k|$.

Proof. Arrange the nodes in $\mathscr{E}_1$ as $E_1 < \cdots < E_r$. Since $n(E_j) > N, t(E_j) < \theta_N \leqslant \epsilon$. For $1 \leqslant j \leqslant r$, let $J_j = \{k : G_k \cap E_j \neq \emptyset\}$. Then $J_1 \leqslant \cdots \leqslant J_r$, and $|J_j| \geqslant 2$ for all j. Hence $\sum_{j=1}^{r} \sum_{k\in J_j} |a_k| \leqslant 2 \sum_{k=1}^{p} |a_k|$. It follows that

$$\sum_{E\in\mathscr{E}_1} t(E)\|Ex\| \leqslant \sum_{j=1}^{r} t(E_j) \sum_{k\in J_j} |a_k| \leqslant \epsilon \sum_{j=1}^{r} \sum_{k\in J_j} |a_k| \leqslant 2\epsilon \sum_{k=1}^{p} |a_k|. \qquad \square$$

Lemma 5. $\sum_{E\in\mathscr{E}_3} t(E)\|Ex\| \leqslant 2\|\sum_{k=1}^{p} a_k e_{i_k}\|$.

Proof. Since any node $E \in \mathscr{E}_3$ is a leaf in the complete tree $\mathscr{T}$ for x, $\|Ex\| = \|Ex\|_{\mathscr{F}_0}$. Choose $E_0 \in \mathscr{F}_0$ such that $E_0 \subseteq E$ and $\|Ex\| = \|E_0 x\| = \|E_0 x\|_{\ell^1}$. Let $J_E = \{k : G_k \cap E_0 \neq \emptyset\}$. For each $k \in J_E$, choose $j_k \in G_k \cap E_0$ and set $z = \sum_{E\in\mathscr{E}_3} \sum_{k\in J_E} a_k e_{j_k}$. Because each $E \in \mathscr{E}_3$ is a long node, each k belongs to at most two J_E. It follows that $\|z\| \leqslant 2\|\sum_{k=1}^{p} a_k e_{i_k}\|$. Now

$$\begin{aligned}
\sum_{E\in\mathscr{E}_3} t(E)\|Ex\| &= \sum_{E\in\mathscr{E}_3} t(E)\|E_0 x\| \\
&\leqslant \sum_{E\in\mathscr{E}_3} t(E) \sum_{k\in J_E} |a_k| \quad \text{(by triangle inequality)} \\
&\leqslant \sum_{E\in\mathscr{E}_3} t(E)\|E_0 z\|_{\ell^1} \leqslant \sum_{E\in\mathscr{E}_3} t(E)\|Ez\|_{\mathscr{F}_0} \\
&\leqslant \|z\| \leqslant 2\left\|\sum_{k=1}^{p} a_k e_{i_k}\right\|. \qquad \square
\end{aligned}$$

Observe that any ancestor F of any node in $\mathscr{E}_2$ must be a long node such that $n(F) \leqslant N$. Subdivide $\mathscr{E}_2$ into two parts $\mathscr{E}_{21}$ and $\mathscr{E}_{22}$ according to whether the node E in question satisfies $n(E) > N$ or $n(E) \leqslant N$.

Lemma 6. $\sum_{E\in\mathscr{E}_{21}} t(E)\|Ex\| \leqslant \frac{2}{\theta_{m_0}}\|y\| + 2\epsilon\rho_1(y) + 2\rho_2(y)$.

Proof. Let $\mathscr{D}$ be the set of all nodes that are immediate predecessors of some node in $\mathscr{E}_{21}$. Let us first show that any two distinct nodes D and D' in $\mathscr{D}$ are mutually incomparable. Indeed, suppose that D is an ancestor of D'. Let E and E' be immediate successors of D and D' respectively that are in $\mathscr{E}_{21}$. Consider the immediate successor D'' of D such that $D'' \supseteq D'$. Since D'' and E are both immediate successors of D, $n(D'') = n(E)$. But $n(D'') \leqslant N$ since D'' is an ancestor of $E' \in \mathscr{E}_{21}$, while $n(E) > N$ by definition of $\mathscr{E}_{21}$. Thus D and D' must be mutually incomparable. List the elements in $\mathscr{D}$ as $D_1 < D_2 < \cdots < D_r$. If $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant p$, let $\mathscr{D}_{jk} = \{E \in \mathscr{E}_{21} : E \subseteq D_j \cap G_k\}$ and $J_j = \{k : \mathscr{D}_{jk} \neq \emptyset\}$. By the preceding argument, each E in

$\bigcup_{k\in J_j}\mathscr{D}_{jk}$ is an immediate successor of D_j. Given $k\in J_j$, choose $E_{jk}\in\mathscr{D}_{jk}$ and $\ell_{jk}\in E_{jk}$. As in the proof of Lemma 5, note that each k belongs to at most two J_j because each D_j is a long node. Hence $\|w\|\leqslant 2\|y\|$ and $\rho_i(w)\leqslant 2\rho_i(y)$, $i=1,2$, where $w=\sum_{j=1}^r\sum_{k\in J_j}a_k e_{\ell_{jk}}$. For each j, let $m=m(j)$ be the common value of $n(E)$ for all $E\in\bigcup_{k\in J_j}\mathscr{D}_{jk}$. In particular, $\bigcup_{k\in J_j}\mathscr{D}_{jk}$ is $\mathscr{F}_m$-admissible. Consider the set $M=\{j: m(j)<m_0\}$. If $j\in M$, then

$$\begin{aligned}\sum_{k\in J_j}\sum_{E\in\mathscr{D}_{jk}}t(E)\|Ex\| &= \sum_{k\in J_j}t(D_j)\theta_m\sum_{E\in\mathscr{D}_{jk}}\|Ex\|\leqslant t(D_j)\sum_{k\in J_j}|a_k|\\ &\leqslant\frac{t(D_j)}{\theta_{m_0}}\theta_m\sum_{k\in J_j}\sum_{E\in\mathscr{D}_{jk}}\|E(D_jw)\|_{\mathscr{S}_0}\\ &\leqslant\frac{t(D_j)}{\theta_{m_0}}\|D_jw\|.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{j\in M}\sum_{k\in J_j}\sum_{E\in\mathscr{D}_{jk}}t(E)\|Ex\| &\leqslant\frac{1}{\theta_{m_0}}\sum_{j\in M}t(D_j)\|D_jw\|\\ &\leqslant\frac{1}{\theta_{m_0}}\|w\|\leqslant\frac{2}{\theta_{m_0}}\|y\|.\end{aligned}\tag{3}$$

If $j\notin M$, choose $n_1,\ldots,n_s\in\mathbb{N}$ as in the hypothesis of Proposition 3. Note that $I_j=\{\ell_{jk}:k\in J_j\}\in\mathscr{F}_m$. Partition J_j into J_j' and J_j'' so that J_j' consists of all $k\in J_j$ such that $\mathscr{D}_{jk}$ is $[\mathscr{F}_{n_1},\ldots,\mathscr{F}_{n_s}]$-admissible and $J_j''=J_j\backslash J_j'$. Set $I_j'=\{\ell_{jk}:k\in J_j'\}$. Then

$$\begin{aligned}\sum_{k\in J_j'}\sum_{E\in\mathscr{D}_{jk}}t(E)\|Ex\| &= t(D_j)\sum_{k\in J_j'}\theta_m\sum_{E\in\mathscr{D}_{jk}}\|Ex\|\\ &\leqslant\epsilon t(D_j)\sum_{k\in J_j'}\theta_{n_1}\ldots\theta_{n_s}\sum_{E\in\mathscr{D}_{jk}}\|Ex\|\\ &\leqslant\epsilon t(D_j)\sum_{k\in J_j'}|a_k|\leqslant\epsilon t(D_j)\|I_j'(D_jw)\|_{\ell^1}\\ &\leqslant\epsilon t(D_j)\|D_jw\|_{\mathscr{F}_m}\leqslant\epsilon t(D_j)\|D_jw\|_{\mathscr{F}}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{j\notin M}\sum_{k\in J_j'}\sum_{E\in\mathscr{D}_{jk}}t(E)\|Ex\| &\leqslant\epsilon\sum_{j=1}^r t(D_j)\|D_jw\|_{\mathscr{F}}\\ &\leqslant\epsilon\rho_1(w)\leqslant 2\epsilon\rho_1(y).\end{aligned}\tag{4}$$

On the other hand, since $\bigcup_{k\in J_j''}\mathscr{D}_{jk}$ is $\mathscr{F}_m$- and thus $[\mathscr{G},\mathscr{F}_{n_1},\dots,\mathscr{F}_{n_s}]$-admissible, while $\mathscr{D}_{jk}$ is not $[\mathscr{F}_{n_1},\dots,\mathscr{F}_{n_s}]$-admissible for all $k\in J_j''$,

$$\left\{\min\bigcup_{E\in\mathscr{D}_{jk}} E : k\in J_j''\right\}\in\mathscr{G}$$

by Lemma 2. Thus $I_j''=\{\ell_{jk}:k\in J_j''\}\in\mathscr{G}$. Consequently,

$$\begin{aligned}\sum_{k\in J_j''}\sum_{E\in\mathscr{D}_{jk}} t(E)\|Ex\| &= t(D_j)\sum_{k\in J_j''}\theta_m\sum_{E\in\mathscr{D}_{jk}}\|Ex\|\leqslant t(D_j)\sum_{k\in J_j''}|a_k|\\ &\leqslant t(D_j)\|I_j''(D_jw)\|_{\ell^1}\leqslant t(D_j)\|D_jw\|_{\mathscr{G}}.\end{aligned}$$

Therefore,

$$\sum_{j\notin M}\sum_{k\in J_j''}\sum_{E\in\mathscr{D}_{jk}} t(E)\|Ex\|\leqslant\sum_{j=1}^{r} t(D_j)\|D_jw\|_{\mathscr{G}}\leqslant\rho_2(w)\leqslant 2\rho_2(y). \tag{5}$$

Combining inequalities (3), (4) and (5) completes the proof. □

Lemma 7. $\sum_{E\in\mathscr{E}_{22}} t(E)\|Ex\|\leqslant\frac{2}{\theta_N}\|y\|$.

Proof. For $1\leqslant k\leqslant p$, let $\mathscr{E}_{22}(k)=\{E\in\mathscr{E}_{22}:E\subseteq G_k\}$. If $\mathscr{E}_{22}(k)\neq\emptyset$, denote by $\mathscr{P}_k$ the collection of all minimal elements in the set of all nodes that are immediate predecessors of some node in $\mathscr{E}_{22}(k)$. Observe that if $P\in\mathscr{P}_k$, then P is a long node and $P\cap G_k\neq\emptyset$. Hence $|\mathscr{P}_k|\leqslant 2$. For each $P\in\mathscr{P}_k$, choose an immediate successor E_P of P such that $E_P\in\mathscr{E}_{22}(k)$, then fix $j_P\in E_P$. Note that the nodes in $\{E_P:P\in\bigcup_{k=1}^{p}\mathscr{P}_k\}$ are pairwise disjoint. Set $v=\sum_{k=1}^{p}a_k\sum_{P\in\mathscr{P}_k}e_{j_P}$. Since $|\mathscr{P}_k|\leqslant 2, \|v\|\leqslant 2\|y\|$. Notice that $t(E_P)=\theta_{n(E_P)}t(P)\geqslant\theta_N t(P)$ since $E\in\mathscr{E}_{22}$ implies $n(E_P)\leqslant N$. Now

$$\begin{aligned}\sum_{E\in\mathscr{E}_{22}} t(E)\|Ex\| &= \sum_{k=1}^{p}\sum_{E\in\mathscr{E}_{22}(k)} t(E)\|Ex\| = \sum_{k=1}^{p}\sum_{P\in\mathscr{P}_k}\sum_{\substack{E\in\mathscr{E}_{22}(k)\\ E\subseteq P}} t(E)\|Ex\|\\ &\leqslant\sum_{k=1}^{p}\sum_{P\in\mathscr{P}_k} t(P)\|P(G_kx)\|\leqslant\sum_{k=1}^{p}\sum_{P\in\mathscr{P}_k} t(P)|a_k|\\ &=\sum_{k=1}^{p}\sum_{P\in\mathscr{P}_k} t(P)\|E_Pv\|_{\mathscr{S}_0}\leqslant\frac{1}{\theta_N}\sum_{k=1}^{p}\sum_{P\in\mathscr{P}_k} t(E_P)\|E_Pv\|_{\mathscr{F}_0}\\ &\leqslant\frac{1}{\theta_N}\|v\|\leqslant\frac{2}{\theta_N}\|y\|.\qquad\square\end{aligned}$$

Observe that in the preceding proof, the hypothesis of Proposition 3 (that is, the existence of the family $\mathscr{G}$) is used only in Lemma 6. One may consider mixed

Tsirelson spaces $Z = T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\ell})$ determined by finitely many regular families, defined in the obvious way. For such spaces, it is worthwhile to observe the following corollary of the proof of Proposition 3. For any $n \in \mathbb{N}$, define $\mathscr{A}_n$ to be the family of all subsets of $\mathbb{N}$ of cardinality $\leqslant n$.

Corollary 8. *Let the space Z be as above. There exists a constant $K < \infty$ such that for any normalized block basic sequence $(x_k)_{k=1}^{\infty}$ in Z and any $(a_k) \in c_{00}$,*

$$\left\| \sum_{k=1}^{\infty} a_k x_k \right\| \leqslant K \left\| \sum_{k=1}^{\infty} a_k e_{i_k} \right\|,$$

where $i_k = \max \operatorname{supp} x_k, 1 \leqslant k < \infty$. Moreover, if $\mathscr{F}_n[\mathscr{A}_3] \subseteq (\mathscr{F}_n)^2$ for all $n \in \mathbb{N}$, then the sequences (x_k) and (e_{i_k}) are equivalent.

Proof. In the notation of the proof of Proposition 3, take $N = \ell$. Then $\mathscr{E}_1 = \mathscr{E}_{21} = \emptyset$. In particular, the hypothesis in Proposition 3 is no longer required since Lemma 6 is not needed any more. Lemmas 5 and 7 give the desired result.

For the "moreover" part, observe that if $j_k = \min \operatorname{supp} x_k, 1 \leqslant k < \infty$, then

$$\frac{1}{3} \left\| \sum_{k=1}^{\infty} a_k e_{i_k} \right\| \leqslant \left\| \sum_{k=1}^{\infty} a_k e_{j_k} \right\| \leqslant \left\| \sum_{k=1}^{\infty} a_k x_k \right\|$$

by Proposition 9 below. □

Suppose $(\mathscr{F}_n)_{n=0}^{\infty}$ is a sequence of regular families that satisfies $\mathscr{F}_n[\mathscr{A}_3] \subseteq (\mathscr{F}_n)^2$ for all $n \in \mathbb{N}$ and $\mathscr{S}_0 \subseteq \mathscr{F}_n$ for all $n \in \mathbb{N} \cup \{0\}$. Define a sequence of norms $(\|\cdot\|_m)$ as follows: $\|x\|_0 = \sup_{F \in \mathscr{F}_0} \|Fx\|_{\ell^1}$ and for $m \in \mathbb{N} \cup \{0\}$,

$$\|x\|_{m+1} = \max\left(\|x\|_m, \sup_n \theta_n \sup\left\{ \sum_{i=1}^{r} \|E_i x\|_m : (E_i) \text{ is } \mathscr{F}_n\text{-admissible} \right\} \right).$$

It is well known that $\|x\| = \lim_m \|x\|_m$ gives the norm on the mixed Tsirelson space $T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\infty})$. The idea of the proof of the next proposition comes from [9, Lemma 2]. Let $(i_k), (j_k) \in [\mathbb{N}]$ be such that $j_k \leqslant i_k < j_{k+1}$ for all $k \in \mathbb{N}$. For $E \subseteq \mathbb{N}$, let the left shift of E be the set $L_E = \{j_k : i_k \in E\}$.

Proposition 9. *For all $(a_k) \in c_{00}, m \in \mathbb{N}$, there exist $E_1 < E_2 < E_3$ such that*

$$\|x\|_m \leqslant \|E_1 y\|_m + \|E_2 y\|_m + \|E_3 y\|_m,$$

where $x = \sum a_k e_{i_k}$ and $y = \sum a_k e_{j_k}$.

Proof. The proof is by induction on m. When $m = 0, \|x\|_0 = \|Fx\|_{\ell^1}$ for some $F \in \mathscr{F}_0$. Note that $L_F = \{\min L_F\} \cup (L_F \backslash \{\min L_F\})$. Clearly $L_F \backslash \{\min L_F\} \in \mathscr{F}_0$ as

$\mathscr{F}_0$ is spreading. Then

$$\begin{aligned}||Fx||_{\ell^1} &= ||L_F y||_{\ell^1} = ||\{\min L_F\}y||_{\ell^1} + ||(L_F\backslash\{\min L_F\})y||_{\ell^1} \\ &\leqslant ||\{\min L_F\}y||_0 + ||(L_F\backslash\{\min L_F\})y||_0\end{aligned}$$

as $\{\min L_F\}$ and $L_F\backslash\{\min L_F\}$ are both in $\mathscr{F}_0$.

Suppose that the proposition is true for some m. If $||x||_{m+1} = ||x||_m$, the conclusion follows from induction as $||\cdot||_m \leqslant ||\cdot||_{m+1}$. Otherwise,

$$||x||_{m+1} = \theta_n \sum_{i=1}^{r} ||E_i x||_m$$

for some $n \in \mathbb{N}$ and some $\mathscr{F}_n$-admissible sequence $(E_i)_{i=1}^r$. Applying the inductive hypothesis, there exist $F_1^i < F_2^i < F_3^i$ such that

$$||E_i x||_m \leqslant ||F_1^i y||_m + ||F_2^i y||_m + ||F_3^i y||_m$$

for each $1 \leqslant i \leqslant r$. We may assume that $F_1^i \cup F_2^i \cup F_3^i \subseteq L_{E_i}$. Observe that

$$\min E_i \leqslant \min F_2^i < \min F_3^i < \min F_1^{i+1} \wedge \min L_{E_{i+1}} < \min E_{i+1}$$

for all $1 \leqslant i \leqslant r$. (Ignore the undefined terms F_1^{r+1}, $L_{E_{r+1}}$ and E_{r+1} here and subsequently in the proof.) Hence

$$\bigcup_{i=1}^{r} \{\min F_2^i, \min F_3^i, \min F_1^{i+1} \wedge \min L_{E_{i+1}}\} \in \mathscr{F}_n[\mathscr{A}_3] \subseteq (\mathscr{F}_n)^2.$$

Clearly, $\{\min F_1^1\} \in \mathscr{S}_0 \subseteq \mathscr{F}_n$ as well. Thus

$$\{\min F_1^1\} \cup \bigcup_{i=1}^{r} \{\min F_2^i, \min F_3^i, \min F_1^{i+1} \wedge \min L_{E_{i+1}}\} \in (\mathscr{F}_n)^3.$$

Since $\bigcup_{i=1}^r \{\min F_1^i, \min F_2^i, \min F_3^i\}$ is a spreading of this set, it belongs to $(\mathscr{F}_n)^3$. Choose $G_1 < G_2 < G_3$ so that $\bigcup_{i=1}^r \bigcup_{j=1}^3 F_j^i \subseteq \bigcup_{k=1}^3 G_k$ and that $\{F_j^i : F_j^i \subseteq G_k\}$ is $\mathscr{F}_n$-admissible, $k = 1, 2, 3$. Then

$$\begin{aligned}||x||_{m+1} &= \theta_n \sum_{i=1}^{r} ||E_i x||_m \\ &\leqslant \theta_n \sum_{i=1}^{r} (||F_1^i y||_m + ||F_2^i y||_m + ||F_3^i y||_m) \\ &\leqslant ||G_1 y||_{m+1} + ||G_2 y||_{m+1} + ||G_3 y||_{m+1}. \qquad \square\end{aligned}$$

Remark. *The sets* $\mathscr{S}_\alpha, 0<\alpha<\omega_1$, *satisfy the "moreover" condition in Corollary* 8, *i.e., if* $0<\alpha<\omega_1$, *then* $\mathscr{S}_\alpha[\mathscr{A}_3]\subseteq(\mathscr{S}_\alpha)^2$.

Proof. The proof is by induction on α. Suppose that $J\in\mathscr{S}_1[\mathscr{A}_3]$. Then $J=\bigcup_{i=1}^k J_i, J_1<J_2<\cdots<J_k, J_i\in\mathscr{A}_3$, and $\{J_1,\dots,J_k\}$ is $\mathscr{S}_1$-admissible. Then $|J|\leqslant 3k\leqslant 3\min J$. Thus we can write $J=H_1\cup H_2$, $H_1<H_2$ so that $|H_1|\leqslant\min J\leqslant\min H_1$ and $|H_2|\leqslant 2\min J\leqslant\min H_2$. Hence $J=H_1\cup H_2\in(\mathscr{S}_1)^2$.

Suppose that the remark is true for some α. Let $J\in\mathscr{S}_{\alpha+1}[\mathscr{A}_3]$. Then $J=\bigcup_{i=1}^k J_i$, $J_1<J_2<\cdots<J_k, J_i\in\mathscr{A}_3$, and $\{\min J_1,\dots,\min J_k\}\in\mathscr{S}_{\alpha+1}$. We can write $\{\min J_1,\dots,\min J_k\}$ as $\bigcup_{j=1}^p H_j$ so that $H_1<\cdots<H_p$, $H_j\in\mathscr{S}_\alpha$ and $\{H_1,\dots,H_p\}$ is $\mathscr{S}_1$-admissible. For each $j, L_j=\bigcup_{\min J_i\in H_j} J_i\in\mathscr{S}_\alpha[\mathscr{A}_3]\subseteq(\mathscr{S}_\alpha)^2$ by induction. Hence we can write $L_j=L_j^1\cup L_j^2$, where $L_j^1<L_j^2$, $L_j^1,L_j^2\in\mathscr{S}_\alpha$. Now $\{\min L_1^1,\min L_1^2,\dots,\min L_p^1,\min L_p^2\}\in\mathscr{S}_1[\mathscr{A}_3]\subseteq(\mathscr{S}_1)^2$ by the above. Hence $J=\bigcup_{j=1}^p L_j=\bigcup_{j=1}^p(L_j^1\cup L_j^2)\in(\mathscr{S}_1)^2[\mathscr{S}_\alpha]=(\mathscr{S}_1[\mathscr{S}_\alpha])^2=(\mathscr{S}_{\alpha+1})^2$. This proves the remark for the case $\alpha+1$.

Suppose that the remark is true for all $0<\beta<\alpha$, where α is a countable limit ordinal. Let $J\in\mathscr{S}_\alpha[\mathscr{A}_3]$. Then $J\in\mathscr{S}_{\alpha_n}[\mathscr{A}_3]$ for some $n\leqslant\min J$, where (α_n) is the sequence of ordinals used to define $\mathscr{S}_\alpha$. This implies that $J\in(\mathscr{S}_{\alpha_n})^2$ for some $n\leqslant\min J$ by induction. Hence $J\in(\mathscr{S}_\alpha)^2$. $\square$

3. Bounds on the ℓ^1-index

Let us recall the relevant terminology concerning trees. A *tree* on a set S is a subset T of $\bigcup_{n=1}^\infty S^n$ such that $(x_1,\dots,x_n)\in T$ whenever $n\in\mathbb{N}$ and $(x_1,\dots,x_{n+1})\in T$. If $(x_1,\dots,x_n)\in T$ and $1\leqslant m<n$, the sequence $(x_1,\dots,x_m)$ is said to be an ancestor of $(x_1,\dots,x_n)$. A tree T is *well-founded* if there is no infinite sequence (x_n) in S such that $(x_1,\dots,x_n)\in T$ for all n. Given a well-founded tree T, we define the *derived tree* $D(T)$ to be the set of all $(x_1,\dots,x_n)\in T$ such that $(x_1,\dots,x_n,x)\in T$ for some $x\in S$. Inductively, we let $D^0(T)=T$, $D^{\alpha+1}(T)=D(D^\alpha(T))$, and $D^\alpha(T)=\bigcap_{\beta<\alpha}D^\beta(T)$ if α is a limit ordinal. The *order* of a well-founded tree T is the smallest ordinal $o(T)$ such that $D^{o(T)}(T)=\emptyset$. If E is a Banach space and $1\leqslant K<\infty$, an ℓ^1-K-tree on E is a tree T on $S(E)=\{x\in E:\|x\|=1\}$ such that $\|\sum_{i=1}^n a_ix_i\|\geqslant K^{-1}\sum_{i=1}^n|a_i|$ whenever $(x_1,\dots,x_n)\in T$ and $(a_i)\subseteq\mathbb{R}$. If E has a basis (e_i), a *block tree* on E is a tree T on E so that every $(x_1,\dots,x_n)\in T$ is a finite block basis of (e_i). An ℓ^1-K-block tree on E is a block tree that is also an ℓ^1-K-tree. The index $I(E,K)$ is defined to be $\sup\{o(T): T \text{ is an } \ell^1\text{-}K\text{-tree on } E\}$. If E has a basis (e_i), the index $I_b(E,K)$ is defined similarly, with the supremum taken over all ℓ^1-K-block trees. The Bourgain ℓ^1-*index* of E is the ordinal $I(E)=\sup\{I(E,K):1\leqslant K<\infty\}$. The index $I_b(E)$ is defined similarly. Bourgain proved that if E is a separable Banach space not containing a copy of ℓ^1, then $I(E)<\omega_1$ [8]. Judd and Odell [12] showed that $I(E)$ and $I_b(E)$ are closely related for a Banach space E with a basis. Precisely, if $I_b(E) = \omega^n$ for some $n < \omega$, then $I(E) = \omega^n$ or ω^{n+1}, while $I_b(E) = I(E)$ if $I_b(E) \geqslant \omega^\omega$. We refer the reader to [2,12] for in depth discussions of these and related indices.

Our concern for the rest of the paper is the calculation of the index $I_b(X)$, where X is the mixed Tsirelson space $T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^\infty)$. We begin with an easy lower bound on $I_b(X)$.

Proposition 10. $I_b(X) \geqslant \alpha_0 \cdot \sup_{n \in \mathbb{N}} \alpha_n^\omega$.

Proof. For all $m, n \in \mathbb{N}$, denote the family $[\overbrace{\mathscr{F}_n, \ldots, \mathscr{F}_n}^{m \text{ times}}, \mathscr{F}_0]$ by $\mathscr{B}_{mn}$. Observe that $\imath(\mathscr{B}_{mn}) = \alpha_0 \cdot \alpha_n^m$ for all $m, n \in \mathbb{N}$ by [13, Proposition 10]. For any $(a_k) \in c_{00}$, $\|\sum a_k e_k\| \geqslant \theta_n^m \|\sum a_k e_k\|_{\mathscr{B}_{mn}}$. Thus $I_b(X, \frac{1}{\theta_n^m}) \geqslant \imath(\mathscr{B}_{mn}) = \alpha_0 \cdot \alpha_n^m$ for all $m, n \in \mathbb{N}$. Therefore,

$$I_b(X) \geqslant \sup_{m,n \in \mathbb{N}} \alpha_0 \cdot \alpha_n^m = \alpha_0 \cdot \sup_{n \in \mathbb{N}} \alpha_n^\omega. \qquad \square$$

In the remainder of this section, we apply Proposition 3 to obtain an upper bound on the ℓ^1-index of X. For each $n \in \mathbb{N}$, let

$$\mathscr{C}(n) = \{(0, n_1, \ldots, n_s) : n_1, \ldots, n_s, s \in \mathbb{N},\ n_1 + n_2 + \cdots + n_s \leqslant n\}$$

and

$$\pi_n = \sup\{\theta_{n_1} \ldots \theta_{n_s} : n_1 + \cdots + n_s > n\}.$$

Obviously $\mathscr{C}(n)$ is a finite set. Denote its cardinality by $p(n)$. It is clear that $\lim_{n \to \infty} \pi_n = 0$.

Lemma 11. *Suppose that $\mathscr{H}$ is a regular family containing $\mathscr{S}_0$ and that ρ is the norm on the space $T(\mathscr{H}, (\theta_n, \mathscr{F}_n)_{n=1}^\infty)$. For all $x \in c_{00}$ and all $n \in \mathbb{N}$, we have*

$$\rho(x) \leqslant \pi_n \|x\|_{\ell^1} + p(n) \|x\|_{\mathscr{M}[\mathscr{H}]},$$

where $\mathscr{M} = \bigcup_{(0, n_1, \ldots, n_s) \in \mathscr{C}(n)} [\mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}]$.

Proof. There exists an $(\mathscr{F}_n)$-admissible tree $\mathscr{T}$ such that

$$\rho(x) = \sum_{E \in \mathscr{L}} t(E) \|Ex\|_{\mathscr{H}},$$

where $\mathscr{L}$ is the set of all leaves of $\mathscr{T}$. Let $\mathscr{L}_{(n_1,\dots,n_s)}$ be the set of all $E\in\mathscr{L}$ such that $h(E)=(0,n_1,\dots,n_s)$. Then

$$\rho(x)=\left(\sum_{(0,n_1,\dots,n_s)\in\mathscr{C}(n)}+\sum_{(0,n_1,\dots,n_s)\notin\mathscr{C}(n)}\right)\sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}}t(E)\|Ex\|_{\mathscr{H}}.$$

If $(0,n_1,\dots,n_s)\in\mathscr{C}(n)$, then $\mathscr{L}_{(n_1,\dots,n_s)}$ is $[\mathscr{F}_{n_1},\dots,\mathscr{F}_{n_s}]$-admissible and thus $\mathscr{M}$-admissible. Since $t(E)\leqslant 1$ for all E,

$$\sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}}t(E)\|Ex\|_{\mathscr{H}}\leqslant\sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}}\|Ex\|_{\mathscr{H}}\leqslant\|x\|_{\mathscr{M}[\mathscr{H}]}.$$

Therefore,

$$\sum_{(0,n_1,\dots,n_s)\in\mathscr{C}(n)}\sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}}t(E)\|Ex\|_{\mathscr{H}}\leqslant p(n)\|x\|_{\mathscr{M}[\mathscr{H}]}.$$

On the other hand, since $\theta_{n_1}\dots\theta_{n_s}\leqslant\pi_n$ if $(0,n_1,\dots,n_s)\notin\mathscr{C}(n)$,

$$\begin{aligned}&\sum_{(0,n_1,\dots,n_s)\notin\mathscr{C}(n)}\sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}}t(E)\|Ex\|_{\mathscr{H}}\\&\quad\leqslant\sum_{(0,n_1,\dots,n_s)\notin\mathscr{C}(n)}\sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}}\pi_n\|Ex\|_{\mathscr{H}}\leqslant\pi_n\|x\|_{\ell^1}.\qquad\square\end{aligned}$$

Lemma 12. *Let $\mathscr{M}$ be as defined in Lemma* 11, *then $\iota(\mathscr{M})\leqslant\alpha_n^n$.*

Proof. The lemma follows immediately from the fact that

$$\iota(\mathscr{H}[\mathscr{N}])\leqslant\iota(\mathscr{N})\cdot\iota(\mathscr{H})$$

if $\mathscr{H}$ and $\mathscr{N}$ are regular families of finite subsets of $\mathbb{N}$ (cf. [13, Proposition 10]). $\square$

Proposition 13 (Leung and Tang [13, Proposition 12]). *Let T be a well-founded block tree on some basis (e_i). Define*

$$\mathscr{H}(T)=\{\{\max\operatorname{supp}x_i:i=1,\dots,n\}:(x_1,x_2,\dots,x_n)\in T\}$$

and

$$\mathscr{G}(T)=\{G: G\ \textit{is a spreading of a subset of some}\ F\in\mathscr{H}(T)\}.$$

If $\mathscr{G}(T)$ is compact, then $\iota(\mathscr{G}(T))\geqslant o(T)$.

Given a countable ordinal η, define the order (or the logarithm) $\ell(\eta)$ of the ordinal η to be γ_1, where $\eta = \omega^{\gamma_1} \cdot k_1 + \cdots + \omega^{\gamma_p} \cdot k_p$ in Cantor normal form. Clearly, $\ell(\eta_1 \cdot \eta_2) = \ell(\eta_1) + \ell(\eta_2)$. Therefore $\ell(\eta^n) = \ell(\eta) \cdot n$ and $\ell(\eta^\omega) = \ell(\eta) \cdot \omega$. Obviously, if $\ell(\eta) = \gamma$, then $\omega^\gamma \leqslant \eta < \omega^{\gamma+1}$. Observe that in the notation of Proposition 3, if we take ρ to be the norm on the space $T(\mathscr{F}_0 \cup \mathscr{G}, (\theta_n, \mathscr{F}_n)_{n=1}^\infty)$, then $\|\cdot\| \leqslant \rho$ and $\rho_2 \leqslant \rho$. Thus inequality (2) implies

$$\left\|\sum_{k=1}^p a_k x_k\right\| \leqslant (K+2)\rho\left(\sum_{k=1}^p a_k e_{i_k}\right) + 4\epsilon \sum_{k=1}^p |a_k|.$$

If $(x_k)_{k=1}^n$ and $(y_k)_{k=1}^n$ are sequences in possibly different normed spaces, and $0 < K < \infty$, we write $(x_k)_{k=1}^n \overset{K}{\succeq} (y_k)_{k=1}^n$ to mean $K\|\sum_{k=1}^n a_k x_k\| \geqslant \|\sum_{k=1}^n a_k y_k\|$ for all $(a_k) \in c_{00}$.

Proposition 14. *Suppose for all* $\epsilon > 0$, *there exist a regular family* $\mathscr{G}_\epsilon$ *and* $m_0 \in \mathbb{N}$ *such that for all* $m \geqslant m_0$, *there exist* $n_1, \ldots, n_s \in \mathbb{N}$ *satisfying* $\theta_m < \epsilon\theta_{n_1} \ldots \theta_{n_s}$ *and* $\mathscr{F}_m \subseteq [\mathscr{G}_\epsilon, \mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}]$. *Then*

$$I_b(X) \leqslant \sup_{\epsilon > 0} \sup_{n \in \mathbb{N}} [(\alpha_0 \vee \imath(\mathscr{G}_\epsilon)) \cdot \alpha_n^\omega].$$

Proof. Suppose otherwise. There exists $H > 1$ and an ℓ^1-H-block tree T on X such that

$$o(T) > \sup_{\epsilon > 0} \sup_{n \in \mathbb{N}} [(\alpha_0 \vee \imath(\mathscr{G}_\epsilon)) \cdot \alpha_n^\omega].$$

Pick $\epsilon_0 < \frac{1}{8H}$. According to Proposition 3 and the remark above, there exists a constant K such that for all $(a_k) \in c_{00}$ and all normalized block basic sequences $(x_k)_{k=1}^n$,

$$\left\|\sum_{k=1}^n a_k x_k\right\| \leqslant K\rho\left(\sum_{k=1}^n a_k e_{i_k}\right) + 4\epsilon_0 \sum_{k=1}^n |a_k|,$$

where ρ is the norm on $T(\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}, (\theta_n, \mathscr{F}_n)_{n=1}^\infty)$. Let $\ell(\alpha_n) = \gamma_n$ for all $n \in \mathbb{N}$ and $\ell(\alpha_0 \vee \imath(\mathscr{G}_{\epsilon_0})) = \gamma$. Then

$$\begin{aligned}\ell\left((\alpha_0 \vee \imath(\mathscr{G}_{\epsilon_0})) \cdot \sup_{n \in \mathbb{N}} \alpha_n^\omega\right) &= \ell(\alpha_0 \vee \imath(\mathscr{G}_{\epsilon_0})) + \ell\left(\sup_{n \in \mathbb{N}} \alpha_n^\omega\right) \\ &\geqslant \ell(\alpha_0 \vee \imath(\mathscr{G}_{\epsilon_0})) + \ell(\alpha_n^\omega) = \gamma + \gamma_n \cdot \omega\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $o(T) > \omega^{\gamma + \gamma_n \cdot \omega}$ for all $n \in \mathbb{N}$. Given $F \in \mathscr{H}(T)$, there exists $(x_1, x_2, \ldots, x_n) \in T$ such that $F = \{i_k\}_{k=1}^n = \{\max \operatorname{supp} x_k\}_{k=1}^n$. Since

$(x_1, x_2, \ldots, x_n) \in T$, $(x_1, x_2, \ldots, x_n) \overset{H}{\succeq} \ell^1(|F|)$-basis. Thus

$$K\rho\left(\sum_{k=1}^{n} a_k e_{i_k}\right) + 4\epsilon_0 \sum_{k=1}^{n} |a_k| \geqslant \frac{1}{H} \sum_{k=1}^{n} |a_k|.$$

Hence

$$\rho\left(\sum_{k=1}^{n} a_k e_{i_k}\right) \geqslant \frac{1}{2KH} \sum_{k=1}^{n} |a_k|.$$

Since it is clear that $(e_k)_{k\in G} \overset{1}{\succeq} (e_k)_{k\in F}$ whenever G is a spreading of F, it follows that

$$\rho\left(\sum_{k\in G} a_k e_k\right) \geqslant \frac{1}{2KH} \sum_{k\in G} |a_k| \tag{6}$$

for all $G \in \mathscr{G}(T)$. Assume that $\gamma_n \neq 0$ for some n. Then $\gamma_n \neq 0$ for all sufficiently large n. Choose $m \in \mathbb{N}$ such that $\pi_m < 1/(4KH)$ and $\gamma_m \neq 0$. If $\mathscr{G}(T)$ is compact, then $\iota(\mathscr{G}(T)) > \omega^{\gamma+\gamma_m\cdot\omega}$ by Proposition 13. Since $\mathscr{G}(T)$ is regular, the same holds for $\mathscr{G}(T) \cap [L]^{<\infty}$ for any $L \in [\mathbb{N}]$. Thus by [11, Corollary 1.2], there exists $L \in [\mathbb{N}]$ such that $\mathscr{S}_{\gamma+\gamma_m\cdot\omega} \cap [L]^{<\infty} \subseteq \mathscr{G}(T)$. The same conclusion clearly holds if $\mathscr{G}(T)$ is not compact. Hence inequality (6) holds for all $(a_k) \in c_{00}$ and all $G \in \mathscr{S}_{\gamma+\gamma_m\cdot\omega} \cap [L]^{<\infty}$. Now, defining $\mathscr{M}$ to be as in Lemma 11 corresponding to m,

$$\iota(\mathscr{M}[\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}]) \leqslant \iota(\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}) \cdot \iota(\mathscr{M}) = (\alpha_0 \vee \iota(\mathscr{G}_{\epsilon_0})) \cdot \alpha_m^m < \omega^{\gamma+\gamma_m\cdot m+1}.$$

Using [11, Corollary 1.2] again, we obtain $M \in [L]$ such that $\mathscr{M}[\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}] \cap [M]^{<\infty} \subseteq \mathscr{S}_{\gamma+\gamma_m\cdot m+1}$. It follows from [16, Proposition 3.6] that there are $F \in \mathscr{S}_{\gamma+\gamma_m\cdot\omega}(M)$ and $(a_j)_{j\in F} \subseteq \mathbb{R}^+$ such that $\sum_{j\in F} a_j = 1$ and if $G \subseteq F$ with $G \in \mathscr{S}_{\gamma+\gamma_m\cdot m+1}$, then $\sum_{j\in G} a_j < \frac{1}{4p(m)KH}$. Note that $F \in \mathscr{S}_{\gamma+\gamma_m\cdot\omega} \cap [M]^{<\infty} \subseteq \mathscr{G}(T)$. Consider $x = \sum_{j\in F} a_j e_j$. By Lemma 11,

$$\begin{aligned}\rho(x) &\leqslant \pi_m \|x\|_{\ell^1} + p(m)\|x\|_{\mathscr{M}[\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}]}\\ &\leqslant \pi_m + p(m)\|x\|_{\mathscr{S}_{\gamma+\gamma_m\cdot m+1}}\\ &< \frac{1}{4KH} + \frac{1}{4KH} = \frac{1}{2KH}\end{aligned}$$

contrary to (6). This proves the proposition in case $\gamma_n \neq 0$ for some n.

If $\gamma_n = 0$ for all n, then $\alpha_n^{\omega} = \omega$ for all n. (Recall that we assume $\alpha_n > 1$ for all $n \in \mathbb{N}$.) Write $\alpha_0 \vee \iota(\mathscr{G}_{\epsilon_0}) = \omega^{\lambda_1} \cdot m_1 + \cdots + \omega^{\lambda_k} \cdot m_k$ in Cantor normal form. Then

$$\iota(\mathscr{G}(T)) \geqslant o(T) > [\alpha_0 \vee \iota(\mathscr{G}_{\epsilon_0})] \cdot \omega = \omega^{\lambda_1 + 1}.$$

By [11, Corollary 1.2], there exists $L \in [\mathbb{N}]$ such that $\mathscr{S}_{\lambda_1+1} \cap [L]^{<\infty} \subseteq \mathscr{G}(T)$. Hence, for all $(a_k) \in c_{00}$ and all $G \in \mathscr{S}_{\lambda_1+1} \cap [L]^{<\infty}$, inequality (6) holds. Choose $m \in \mathbb{N}$ such that $\pi_m < 1/(4KH)$ and define $\mathscr{M}$ as in Lemma 11 corresponding to m. Then

$$\iota(\mathscr{M}[\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}]) = [\alpha_0 \vee \iota(\mathscr{G}_{\epsilon_0})] \cdot r < \omega^{\lambda_1} \cdot (m_1 + 1) r$$

for some $r \in \mathbb{N}$. Applying [11, Theorem 1.1], there exists $M \in [L]$ such that $\mathscr{M}[\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}] \cap [M]^{<\infty} \subseteq (\mathscr{S}_{\lambda_1})^{(m_1+1)r}$. By [16, Proposition 3.6], there exist $F \in \mathscr{S}_{\lambda_1+1}(M) \subseteq \mathscr{S}_{\lambda_1+1} \cap [M]^{<\infty} \subseteq \mathscr{S}_{\lambda_1+1} \cap [L]^{<\infty}$ and $(a_j)_{j \in F} \subseteq \mathbb{R}^+$ such that $\sum_{j \in F} a_j = 1$ and if $G \subseteq F$ with $G \in \mathscr{S}_{\lambda_1}$, then $\sum_{j \in G} a_j < \frac{1}{4p(m)KH(m_1+1)r}$. Consider $x = \sum_{j \in F} a_j e_j$. By Lemma 11,

$$\begin{aligned} \rho(x) &\leqslant \pi_m \|x\|_{\ell^1} + p(m)\|x\|_{\mathscr{M}[\mathscr{F}_0 \cup \mathscr{G}_{\epsilon_0}]} \\ &\leqslant \pi_m + p(m)\|x\|_{(\mathscr{S}_{\lambda_1})^{(m_1+1)r}} \\ &\leqslant \pi_m + p(m)(m_1+1)r\|x\|_{\mathscr{S}_{\lambda_1}} \\ &< \frac{1}{4KH} + \frac{1}{4KH} = \frac{1}{2KH} \end{aligned}$$

contradicting (6). □

Theorem 15. (1) $\alpha_0 \cdot \sup_{n \in \mathbb{N}} \alpha_n^{\omega} \leqslant I_b(X) \leqslant (\alpha_0 \vee \alpha) \cdot \sup_{n \in \mathbb{N}} \alpha_n^{\omega}$.

(2) *If* $\alpha_0 \geqslant \alpha$, *then* $I_b(X) = \alpha_0 \cdot \sup_{n \in \mathbb{N}} \alpha_n^{\omega}$.

(3) *If* $\alpha_0 < \alpha$ *and* $\alpha = \alpha_n$ *for some* $n \in \mathbb{N}$, *then* $I_b(X) = \alpha^{\omega}$.

(4) *If* $\alpha_n < \alpha$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* α *is not of the form* $\alpha = \omega^{\omega^{\xi}}, \xi < \omega_1$, *then* $I_b(X) = \alpha^{\omega}$.

Proof. 1. The first inequality follows from Proposition 10. Since $\mathscr{S}_0 \subseteq \mathscr{F}_n$ for all n, $\mathscr{F}_m \subseteq \mathscr{F} \subseteq [\mathscr{F}, \mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}]$ for all $m, n_1, \ldots, n_s \in \mathbb{N}$. The second inequality follows from Proposition 14 upon taking $\mathscr{G}_{\epsilon} = \mathscr{F}$ and m_0 to be a number such that $\theta_{m_0} < \epsilon\theta_1$.

2. and 3. are clear.

4. If $\sup_{n \in \mathbb{N}} \alpha_n^{\omega} = \alpha^{\omega}$, then $\alpha \cdot \sup_{n \in \mathbb{N}} \alpha_n^{\omega} = \alpha \cdot \alpha^{\omega} = \alpha^{1+\omega} = \alpha^{\omega}$ and the result follows by part 1. So it suffices to show that $\sup_{n \in \mathbb{N}} \alpha_n^{\omega} = \alpha^{\omega}$. Since $\alpha_n^{\omega} \leqslant \alpha^{\omega}$ for all $n \in \mathbb{N}$, $\sup_{n \in \mathbb{N}} \alpha_n^{\omega} \leqslant \alpha^{\omega}$. Suppose that $\alpha = \omega^{\gamma_1} \cdot k_1 + \cdots + \omega^{\gamma_p} \cdot k_p$ in Cantor normal form and either $k_1 > 1$ or $p > 1$. Then there exists $n \in \mathbb{N}$ such that $\alpha_n \geqslant \omega^{\gamma_1}$. Hence $\sup_{n \in \mathbb{N}} \alpha_n^{\omega} \geqslant \omega^{\gamma_1 \cdot \omega} = \alpha^{\omega}$.

Now suppose that $\alpha = \omega^{\gamma_1}$, and $\gamma_1 \neq \omega^{\xi}$ for any $\xi < \omega_1$. Then there exists $n \in \mathbb{N}$ such that $\alpha_n \geqslant \omega^{\omega^{\ell(\gamma_1)}}$. Thus

$$\sup_{n\in\mathbb{N}} \alpha_n^{\omega} \geqslant [\omega^{\omega^{\ell(\gamma_1)}}]^{\omega} = \omega^{\omega^{\ell(\gamma_1)}\cdot\omega} = \omega^{\omega^{\ell(\gamma_1)+1}} = \omega^{\gamma_1\cdot\omega} = \alpha^{\omega}. \qquad \square$$

The following corollary answers Question 1 in [12].

Corollary 16. *If* η *is a countable ordinal not of the form* ω^{ξ} *for some limit ordinal* $\xi < \omega_1$, *then there exists a Banach space* Y *such that* $I(Y) = \omega^{\eta}$.

Proof. Write $\eta = \omega^{\gamma_1}\cdot m_1 + \cdots + \omega^{\gamma_k}\cdot m_k$ in Cantor normal form. If γ_k is 0 or a successor ordinal, then the result follows immediately from [13, Corollary 14]. If γ_k is a limit ordinal, let (β_n) be a sequence of ordinals increasing to γ_k. Choose regular families $(\mathscr{F}_n)_{n=0}^{\infty}$ such that $\alpha_n = \iota(\mathscr{F}_n) = \omega^{\omega^{\beta_n}}$, $n \in \mathbb{N}$, and $\alpha_0 = \iota(\mathscr{F}_0) = \omega^{\omega^{\gamma_1}\cdot m_1 + \cdots + \omega^{\gamma_k}\cdot(m_k-1)}$. Then $\alpha = \sup_{n\in\mathbb{N}} \alpha_n = \sup_{n\in\mathbb{N}} \omega^{\omega^{\beta_n}} = \omega^{\omega^{\gamma_k}} \leqslant \alpha_0$ as $k > 1$ or $m_k > 1$. Let $Y = T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\infty})$. By 2. in Theorem 15, $I_b(Y) = \alpha_0 \cdot \sup_{n\in\mathbb{N}} \alpha_n^{\omega} = \omega^{\omega^{\gamma_1}\cdot m_1 + \cdots + \omega^{\gamma_k}\cdot m_k} = \omega^{\eta}$. Finally, since $I_b(Y) \geqslant \omega^{\omega}, I(Y) = I_b(Y) = \omega^{\eta}$ by [12, Corollary 5.13]. $\square$

4. Attaining the upper bound

Henceforth, we shall consider only the case where $\alpha_n < \alpha$ for all $n \in \mathbb{N} \cup \{0\}$ and α is of the form $\omega^{\omega^{\xi}}$. Under these conditions, Theorem 15 yields the estimate

$$\omega^{\omega^{\xi}} \leqslant I_b(X) \leqslant \omega^{\omega^{\xi}\cdot 2}.$$

The next theorem gives a sufficient condition for the upper estimate to be attained. Given $m \in \mathbb{N}$ and $\epsilon > 0$, define

$$\begin{aligned} \gamma = \gamma(\epsilon, m) = \ell(\alpha_0) + \max\{\ell(\alpha_{n_s} \ldots \alpha_{n_1}) :& \\ \epsilon\theta_{n_1}\theta_{n_2}\ldots\theta_{n_s} > \theta_m\}& \quad (\max\emptyset = 0). \end{aligned}$$

Theorem 17. *Assume* $\xi \neq 0$. *If there exists* $\epsilon > 0$ *such that for all* $\beta < \omega^{\xi}$, *there exists* $m \in \mathbb{N}$ *satisfying* $\gamma(\epsilon, m) + 2 + \beta < \ell(\alpha_m)$, *then* $I_b(X) = \omega^{\omega^{\xi}\cdot 2}$.

Before giving the proof of Theorem 17, let us observe an interesting corollary.

Corollary 18. *If* ξ *is a limit ordinal, then* $I_b(X) = \omega^{\omega^{\xi}\cdot 2}$.

Proof. Since ξ is a limit ordinal, the sequence $(\ell(\ell(\alpha_n)))$ converges to ξ. Hence for all $\beta < \xi$, there exists $m \in \mathbb{N}$ such that $\ell(\ell(\alpha_m)) > \beta \vee \ell(\ell(\alpha_{m-1})) \vee \ell(\ell(\alpha_0))$. Suppose $\theta_{n_1} \ldots \theta_{n_s} > \theta_m$ for some $n_1, \ldots, n_s \in \mathbb{N}$. Then $n_1, \ldots, n_s < m$. Hence $\ell(\alpha_0)$,

$\ell(\alpha_{n_i}) < \omega^{\ell(\ell(\alpha_m))}$ for all $1 \leqslant i \leqslant s$. Now for all $1 \leqslant i \leqslant s$, $\alpha_{n_i} \leqslant \omega^{\ell(\alpha_{n_i})+1}$. Thus

$$\alpha_0 \cdot \alpha_{n_s} \cdots \alpha_{n_1} \leqslant \omega^{\ell(\alpha_0)+1+\ell(\alpha_{n_s})+1+\cdots+\ell(\alpha_{n_1})+1}.$$

Therefore,

$$\begin{aligned} &\ell(\alpha_0 \cdot \alpha_{n_s} \dots \alpha_{n_1}) + 2 + \omega^{\beta} \\ &\quad \leqslant \ell(\alpha_0) + 1 + \ell(\alpha_{n_s}) + 1 + \cdots + \ell(\alpha_{n_1}) + 1 + 2 + \omega^{\beta} \\ &\quad < \omega^{\ell(\ell(\alpha_m))}. \end{aligned}$$

The last inequality follows from the fact that if $\eta_1, \dots, \eta_k < \omega^{\tau}$, then $\eta_1 + \cdots + \eta_k < \omega^{\tau}$. Hence we have $\ell(\alpha_0 \cdot \alpha_{n_s} \dots \alpha_{n_1}) + 2 + \omega^{\beta} < \ell(\alpha_m)$. Applying Theorem 17 with $\epsilon = 1$ yields the required result. $\square$

Lemma 19. *Let* $m \in \mathbb{N}$ *and* $\epsilon > 0$ *be given. Then for all* $M \in [\mathbb{N}]$, *there exists* $x \in c_{00}$ *satisfying* $\|x\| \leqslant 1 + \frac{1}{\epsilon}$, $\|x\|_{\ell^1} = \frac{1}{\theta_m}$, *and* $\operatorname{supp} x \in \mathscr{S}_{\gamma+2} \cap [M]^{<\infty}$, *where* $\gamma = \gamma(\epsilon, m)$ *is as defined above.*

Proof. Let $\mathscr{N} = \{(n_1, \dots, n_s) : \epsilon\theta_{n_1} \dots \theta_{n_s} > \theta_m\}$. Clearly $\mathscr{N}$ is a finite set. Denote its cardinality by c. By assumption, there exists $L \in [M]$ such that $[\mathscr{F}_{n_1}, \dots, \mathscr{F}_{n_s}, \mathscr{F}_0] \cap [L]^{<\infty} \subseteq \mathscr{S}_{\gamma+1}$ for all $(n_1, \dots, n_s) \in \mathscr{N}$ (cf. [11]). By Odell et al. [16, Proposition 3.6], there exists $y \in c_{00}$, $\|y\|_{\ell^1} = 1$ such that $\operatorname{supp} y \in \mathscr{S}_{\gamma+2} \cap [L]^{<\infty}$ and $\|y\|_{\mathscr{S}_{\gamma+1}} \leqslant \theta_m / c$. Let $x = y/\theta_m$. Then $\|x\|_{\ell^1} = \frac{1}{\theta_m}$ and $\operatorname{supp} x \in \mathscr{S}_{\gamma+2} \cap [M]^{<\infty}$. Choose a complete $(\mathscr{F}_n)$-admissible tree $\mathscr{T}$ such that $\|x\| = \mathscr{T}x$. Denote by $\mathscr{L}(\mathscr{T})$ the set of all leaves of $\mathscr{T}$. For a fixed $(n_1, \dots, n_s) \in \mathscr{N}$, the set $\{E \in \mathscr{L}(\mathscr{T}) : h(E) = (0, n_1, \dots, n_s)\}$ is $[\mathscr{F}_{n_1}, \dots, \mathscr{F}_{n_s}]$-admissible. Since $\operatorname{supp} x \in [L]^{<\infty}$, we conclude by the choice of L that

$$\sum_{\substack{E \in \mathscr{L}(\mathscr{T}) \\ h(E) = (0, n_1, \dots, n_s)}} \|Ex\|_{\mathscr{F}_0} \leqslant \|x\|_{[\mathscr{F}_{n_1}, \dots, \mathscr{F}_{n_s}, \mathscr{F}_0]} \leqslant \|x\|_{\mathscr{S}_{\gamma+1}}.$$

Therefore,

$$\begin{aligned} \|x\| &\leqslant \sum_{\substack{E \in \mathscr{L}(\mathscr{T}) \\ \epsilon t(E) \leqslant \theta_m}} t(E) \|Ex\|_{\mathscr{F}_0} + \sum_{\substack{E \in \mathscr{L}(\mathscr{T}) \\ \epsilon t(E) > \theta_m}} t(E) \|Ex\|_{\mathscr{F}_0} \\ &\leqslant \frac{\theta_m}{\epsilon} \|x\|_{\ell^1} + \sum_{(n_1, \dots, n_s) \in \mathscr{N}} \theta_{n_1} \theta_{n_2} \dots \theta_{n_j} \sum_{\substack{E \in \mathscr{L}(\mathscr{T}) \\ h(E) = (0, n_1, \dots, n_s)}} \|Ex\|_{\mathscr{F}_0} \\ &\leqslant \frac{1}{\epsilon} + \sum_{(n_1, \dots, n_s) \in \mathscr{N}} \|x\|_{\mathscr{S}_{\gamma+1}} \leqslant \frac{1}{\epsilon} + \frac{c}{\theta_m} \|y\|_{\mathscr{S}_{\gamma+1}} \leqslant 1 + \frac{1}{\epsilon}. \qquad \square \end{aligned}$$

Lemma 20. *Under the assumptions of Theorem* 17, *there exists a strictly increasing sequence* $(q_k) \subseteq \mathbb{N}$ *such that for all* $F \in \mathcal{S}_{\omega^\xi}$, *there are normalized vectors* $(x_k)_{k \in F}$ *with* $\operatorname{supp} x_k \subseteq [q_k, q_{k+1})$ *for all* $k \in F$ *and*

$$\left\| \sum_{k \in F} a_k x_k \right\| \geqslant \frac{\epsilon}{1+\epsilon} \sum_{k \in F} |a_k|$$

for all $(a_k) \in c_{00}$.

Proof. Since $\xi \neq 0$, ω^ξ is a limit ordinal. Suppose that $\mathcal{S}_{\omega^\xi}$ is defined by the sequence (β_k) increasing to ω^ξ. For each k, apply the hypothesis of Theorem 17 to choose $m_k \in \mathbb{N}$ such that $\gamma(\epsilon, m_k) + 2 + \beta_k < \ell(\alpha_{m_k})$. Write $\gamma_k = \gamma(\epsilon, m_k)$. Using Lemma 19, obtain a strictly increasing sequence $(q_k)_{k=1}^{\infty}$ in $\mathbb{N}$ and $(x_k^i)_{k=1}^{i}{}_{i=1}^{\infty} \subseteq c_{00}$ such that $\|x_k^i\|_{\ell^1} = \frac{1}{\theta_{m_k}}$, $\|x_k^i\| \leqslant 1 + \frac{1}{\epsilon}$, $\operatorname{supp} x_k^i \subseteq [q_i, q_{i+1})$, and $\operatorname{supp} x_k^i \in \mathcal{S}_{\gamma_k+2} \cap [M_i]^{<\infty}$, where $M_i \in [\mathbb{N}]$ is chosen so that $M_{i+1} \subseteq M_i \cap [q_{i+1}, \infty)$ and

$$\bigcup_{j=1}^{i} \mathcal{S}_{\beta_j}[\mathcal{S}_{\gamma_i+2}] \cap [M_i]^{<\infty} \subseteq \mathcal{F}_{m_i}.$$

Note that this choice is possible by Gasparis [11] since

$$\iota\left(\bigcup_{j=1}^{i} \mathcal{S}_{\beta_j}[\mathcal{S}_{\gamma_i+2}]\right) = \omega^{\gamma_i+2+\beta_i} < \omega^{\ell(\alpha_{m_i})} \leqslant \alpha_{m_i} = \iota(\mathcal{F}_{m_i}).$$

If $F = \{i_1, \dots, i_r\} \in \mathcal{S}_{\omega^\xi}$, $i_1 < \cdots < i_r$, then $F \in \mathcal{S}_{\beta_k}$ for some $k \leqslant i_1$. Consider the block basic sequence $(x_{i_1}^{i_1}, x_{i_1}^{i_2}, \dots, x_{i_1}^{i_r})$. By choice, $\operatorname{supp} x_{i_1}^{i_j} \in \mathcal{S}_{\gamma_{i_1}+2} \cap [M_{i_j}]^{<\infty}$ and $\operatorname{supp} x_{i_1}^{i_j} \subseteq [q_{i_j}, q_{i_j+1})$, $1 \leqslant j \leqslant r$. Moreover, the set $\{q_{i_1}, \dots, q_{i_r}\}$ is a spreading of $\{i_1, \dots, i_r\} = F$ and hence belongs to $\mathcal{S}_{\beta_k}$. Thus

$$\bigcup_{j=1}^{r} \operatorname{supp} x_{i_1}^{i_j} \in \mathcal{S}_{\beta_k}[\mathcal{S}_{\gamma_{i_1}+2}] \cap [M_{i_1}]^{<\infty} \subseteq \mathcal{F}_{m_{i_1}}.$$

Therefore, given any $(a_j) \in c_{00}$,

$$\begin{aligned}
\left\| \sum_{j=1}^{r} a_j x_{i_1}^{i_j} \right\| &\geqslant \theta_{m_{i_1}} \left\| \sum_{j=1}^{r} a_j x_{i_1}^{i_j} \right\|_{\ell^1} \\
&= \theta_{m_{i_1}} \sum_{j=1}^{r} |a_j| \, \|x_{i_1}^{i_j}\|_{\ell^1} \\
&= \theta_{m_{i_1}} \sum_{j=1}^{r} |a_j| \frac{1}{\theta_{m_{i_1}}} = \sum_{j=1}^{r} |a_j|.
\end{aligned}$$

Normalizing the sequence $(x_{i_1}^{i_1}, x_{i_1}^{i_2}, \dots, x_{i_1}^{i_r})$ yields the desired result. □

Lemma 21. *Suppose the assumptions of Theorem* 17 *hold. Then there exists a strictly increasing sequence* $(q_k)\subseteq\mathbb{N}$ *such that whenever* $F\in\mathscr{F}_n[\mathscr{S}_{\omega^\xi}]$ *for some* $n\in\mathbb{N}$, *there are normalized vectors* $(x_k)_{k\in F}$, $\operatorname{supp} x_k\subseteq[q_k,q_{k+1})$, *satisfying*

$$\left\|\sum_{k\in F} a_k x_k\right\| \geqslant \frac{\epsilon\theta_n}{1+\epsilon}\sum_{k\in F}|a_k|$$

for all $(a_k)\in c_{00}$.

Proof. Choose (q_k) using Lemma 20. If $F\in\mathscr{F}_n[\mathscr{S}_{\omega^\xi}]$ for some $n\in\mathbb{N}$, write $F=\bigcup_{j=1}^s F_j$, with $F_1<\cdots<F_s$, $F_j\in\mathscr{S}_{\omega^\xi}$, $1\leqslant j\leqslant s$, and $\{\min F_j\}_{j=1}^s\in\mathscr{F}_n$. For all $1\leqslant j\leqslant s$, there exist normalized vectors $(x_k)_{k\in F_j}$ such that $\operatorname{supp} x_k\subseteq[q_k,q_{k+1})$ for all $k\in F_j$ and $\|\sum_{k\in F_j} a_k x_k\|\geqslant\frac{\epsilon}{1+\epsilon}\sum_{k\in F_j}|a_k|$ for any $(a_k)\in c_{00}$. Therefore,

$$\begin{aligned}\left\|\sum_{k\in F} a_k x_k\right\| &= \left\|\sum_{j=1}^s\left(\sum_{k\in F_j} a_k x_k\right)\right\|\\ &\geqslant \theta_n\sum_{j=1}^s\left\|E_j\sum_{j=1}^s\left(\sum_{k\in F_j} a_k x_k\right)\right\|, \quad \text{where } E_j=\bigcup_{k\in F_j}\operatorname{supp} x_k\\ &= \theta_n\sum_{j=1}^s\left\|\sum_{k\in F_j} a_k x_k\right\|\\ &\geqslant \frac{\epsilon\theta_n}{1+\epsilon}\sum_{k\in F}|a_k|\end{aligned}$$

for any $(a_k)\in c_{00}$. □

To complete the proof of Theorem 17, we apply a compactness argument to condense the block basic sequences obtained in Lemma 21 into a tree. Let Y be a set and let $(A_n)_{n=1}^\infty$ be a sequence of pairwise disjoint finite subsets of Y. Suppose that a given set

$$\mathscr{X}\subseteq\bigcup_{\emptyset\neq F\in[\mathbb{N}]^{<\infty}}\left(\prod_{n\in F} A_n\right)$$

is hereditary in the sense that $(x_n)_{n\in G}\in\mathscr{X}$ whenever $(x_n)_{n\in F}\in\mathscr{X}$ and $\emptyset\neq G\subseteq F$.

Proposition 22. *Let* $\mathscr{H}\subseteq[\mathbb{N}]^{<\infty}$ *be a regular family with* $\omega_1>\iota(\mathscr{H})\geqslant\alpha\geqslant 1$. *Suppose for all nonempty* $F\in\mathscr{H}$, *there exists* $(x_n)_{n\in F}\in\mathscr{X}$. *Then there exists a tree* T *on* Y *such that* $T\subseteq\mathscr{X}$ *and* $o(T)\geqslant\alpha$.

Proof. Assume that $\mathscr{H}$ is regular and nonempty. There exists $n_0 \in \mathbb{N}$ such that $\{n\} \in \mathscr{H}$ for all $n \geqslant n_0$. By hypothesis, there exists $(x_n) \in \mathscr{X}$ for all $n \geqslant n_0$. Let $T = \{(x_n) : n \geqslant n_0\}$. Then $T \subseteq \mathscr{X}$ and $o(T) \geqslant 1$.

Suppose the proposition is true for some $\alpha \geqslant 1$. Let $\mathscr{H} \subseteq [\mathbb{N}]^{<\infty}$ be a regular family satisfying the hypothesis such that $\omega_1 > \imath(\mathscr{H}) \geqslant \alpha + 1$. Pick a singleton set $\{n_0\} \in \mathscr{H}^{(\alpha)}$ and let

$$\mathscr{G} = \{G \in [\mathbb{N}]^{<\infty} : n_0 < G, \{n_0\} \cup G \in \mathscr{H}\}.$$

Then $\mathscr{G}$ is regular and $\imath(\mathscr{G}) \geqslant \alpha \geqslant 1$. Correspondingly, let

$$\mathscr{Y} = \{(x_n)_{n \in G} : \emptyset \neq G \in \mathscr{G},\ \text{there exists } (x_{n_0}) \\ \text{such that } (x_n)_{n \in \{n_0\} \cup G} \in \mathscr{X}\}.$$

Since $\mathscr{X}$ is hereditary, so is $\mathscr{Y}$. Let a nonempty set $G \in \mathscr{G}$ be given. Then there exists $(x_n)_{n \in \{n_0\} \cup G} \in \mathscr{X}$ such that $(x_n)_{n \in G} \in \mathscr{Y}$. By the inductive hypothesis, there exists a tree T_0 on Y such that $T_0 \subseteq \mathscr{Y}$ and $o(T_0) \geqslant \alpha$. List the elements in A_{n_0} as $z^1_{n_0}, \ldots, z^p_{n_0}$. Let M be the collection of maximal nodes of T_0. If $(x_n)_{n \in G} \in M$, there exists i, $1 \leqslant i \leqslant p$, such that $(z^i_{n_0}) \cup (x_n)_{n \in G} \in \mathscr{X}$. Partition M into a disjoint union $\bigcup_{i=1}^{p} M_i$ so that $(x_n)_{n \in G} \in M_i$ implies $(z^i_{n_0}) \cup (x_n)_{n \in G} \in \mathscr{X}$. Now let T_i be the subtree of T_0 consisting of all nodes in M_i and their ancestors. By [12, Lemma 5.10], there exists i such that $o(T_i) \geqslant \alpha$. Define

$$T = \{(z^i_{n_0}) \cup (x_n)_{n \in H} : (x_n)_{n \in H} \in T_i\}.$$

Then T is a tree on Y such that $T \subseteq \mathscr{X}$ and $o(T) \geqslant \alpha + 1$.

Suppose α is a countable limit ordinal and the result holds for all $1 \leqslant \beta < \alpha$. Let $\mathscr{H} \subseteq [\mathbb{N}]^{<\infty}$ be a regular family of finite subsets of $\mathbb{N}$ satisfying the hypothesis such that $\imath(\mathscr{H}) \geqslant \alpha$. If $1 \leqslant \beta < \alpha$, then $\imath(\mathscr{H}) \geqslant \beta \geqslant 1$. Hence there exists a tree T_β on Y such that $T_\beta \subseteq \mathscr{X}$ and $o(T_\beta) \geqslant \beta$. Clearly the tree $T = \bigcup_{\beta < \alpha} T_\beta$ satisfies the requirements of the proposition. $\square$

Proof of Theorem 17. In view of (1) in Theorem 15, it suffices to show that $I_b(X) \geqslant \omega^{\omega^\xi} \cdot \alpha_n$ for all $n \in \mathbb{N}$. In order to set up to apply Proposition 22, let $Y = X$. Choose a sequence (q_k) as in Lemma 21 and fix $n \in \mathbb{N}$. Let A_k be a finite $\frac{\epsilon\theta_n}{2(1+\epsilon)}$-net of the unit sphere of $[e_j]_{j=q_k}^{q_{k+1}-1}$ for each $k \in \mathbb{N}$. Define $c_n = \frac{\epsilon\theta_n}{2(1+\epsilon)}$ and set

$$\mathscr{X} = \{(y_k)_F : \emptyset \neq F \in \mathscr{F}_n[\mathscr{S}_{\omega^\xi}],\ y_k \in A_k, (y_k) \overset{c_n}{\succeq} \ell^1(|F|)\text{-basis}\}.$$

Clearly $\mathscr{X}$ is hereditary. According to Lemma 21, whenever $F \in \mathscr{F}_n[\mathscr{S}_{\omega^\xi}]$, there exist normalized vectors $(x_k)_{k \in F}$, supp $x_k \subseteq [q_k, q_{k+1})$, such that

$$\left\| \sum_{k \in F} a_k x_k \right\| \geqslant \frac{\epsilon \theta_n}{1+\epsilon} \sum_{k \in F} |a_k|$$

for all $(a_k) \in c_{00}$. Choose $(y_k)_{k \in F}$ such that $y_k \in A_k$ and $\|x_k - y_k\| \leqslant \frac{\epsilon \theta_n}{2(1+\epsilon)}$ for all $k \in F$. For all $(a_k) \in c_{00}$,

$$\begin{aligned}
\left\| \sum_{k \in F} a_k y_k \right\| &\geqslant \left\| \sum_{k \in F} a_k x_k \right\| - \left\| \sum_{k \in F} a_k (x_k - y_k) \right\| \\
&\geqslant \frac{\epsilon \theta_n}{1+\epsilon} \sum_{k \in F} |a_k| - \sum_{k \in F} |a_k| \|x_k - y_k\| \\
&\geqslant \frac{\epsilon \theta_n}{1+\epsilon} \sum_{k \in F} |a_k| - \frac{\epsilon \theta_n}{2(1+\epsilon)} \sum_{k \in F} |a_k| \\
&= \frac{\epsilon \theta_n}{2(1+\epsilon)} \sum_{k \in F} |a_k|.
\end{aligned}$$

Thus $(y_k)_{k \in F} \in \mathscr{X}$. By Proposition 22, there exists a tree T on X such that $T \subseteq \mathscr{X}$ and $o(T) \geqslant \iota(\mathscr{F}_n[\mathscr{S}_{\omega^\xi}]) = \omega^{\omega^\xi} \cdot \alpha_n$. Since $T \subseteq \mathscr{X}$, it is an ℓ^1-c_n-block tree. Thus $I_b(X) \geqslant \omega^{\omega^\xi} \cdot \alpha_n$. ☐

In general, the converse of Theorem 17 is far from true, as the following theorem shows.

Theorem 23. *Suppose that* $0 < \xi < \omega_1$, $(\alpha_n)_{n=0}^\infty$ *is a sequence of ordinals such that* $\sup_{n \in \mathbb{N} \cup \{0\}} \alpha_n = \omega^{\omega^\xi}$ *nontrivially (i.e.,* $\alpha_n < \omega^{\omega^\xi}$ *for all* n*) and* $(\theta_n)_{n=1}^\infty$ *is a nonincreasing null sequence in* $(0, 1)$. *Then there exists a sequence* $(\mathscr{F}_n)_{n=0}^\infty$ *of regular families of finite subsets of* $\mathbb{N}$ *such that* $\iota(\mathscr{F}_n) = \alpha_n$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $I_b(T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^\infty)) = \omega^{\omega^\xi \cdot 2}$.

Proof. The proof is similar to that of Theorem 17 once we have obtained Proposition 25 below. ☐

Lemma 24. *Suppose that* $\omega \leqslant \beta < \omega_1$, *where* $\beta = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_m} \cdot k_m$ *in Cantor normal form, and* $g : \mathbb{N} \to \mathbb{N}$ *is a function increasing to* ∞. *There exist regular families* $\mathscr{G}$ *and* $\mathscr{H}$ *such that* $\omega \cdot \iota(\mathscr{G}) = \omega^{\beta_1} \cdot k_1$, $\mathscr{S}_0 \subseteq \mathscr{G}$ *and* $\iota(\mathscr{H}) = \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_m} \cdot k_m$. *In particular,* $\iota((\mathscr{H}, \mathscr{G}[\mathscr{S}_1^g])) = \beta$. *(If* $m = 1$, *take* $\mathscr{H} = \emptyset$.*)*

Proof. Note that $\beta_1 > 0$ since $\beta \geqslant \omega$. Define

$$\mathcal{G} = \begin{cases} (\mathcal{S}_{\beta_1 - 1})^{k_1} & \text{if } 0 < \beta_1 < \omega, \\ (\mathcal{S}_{\beta_1})^{k_1} & \text{if } \omega \leqslant \beta_1 < \omega_1 \end{cases}$$

and $\mathcal{H} = ((\mathcal{S}_{\beta_m})^{k_m}, \dots, (\mathcal{S}_{\beta_2})^{k_2})$. Clearly $\iota(\mathcal{H}) = \omega^{\beta_2} \cdot k_2 + \cdots + \omega^{\beta_m} \cdot k_m$ and

$$\omega \cdot \iota(\mathcal{G}) = \begin{cases} \omega \cdot \omega^{\beta_1 - 1} \cdot k_1 & \text{if } 0 < \beta_1 < \omega \\ \omega \cdot \omega^{\beta_1} \cdot k_1 & \text{if } \omega \leqslant \beta_1 < \omega_1 \end{cases} = \omega^{\beta_1} \cdot k_1. \qquad \square$$

If β is a nonzero countable ordinal whose Cantor normal form is $\omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_m} \cdot k_m$, write $\mathcal{R}_\beta$ for the family $((\mathcal{S}_{\beta_m})^{k_m}, \dots, (\mathcal{S}_{\beta_1})^{k_1})$.

Proposition 25. *Under the hypotheses of Theorem* 23, *there exist regular families* $(\mathcal{F}_n)_{n=0}^{\infty}$ *and* $\mathcal{G}$ *with* $\iota(\mathcal{F}_n) = \alpha_n$, $\iota(\mathcal{G}) = \omega^{\omega^{\xi}}$, *and* $(q_m) \subseteq \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *and all* $F \in \mathcal{F}_n[\mathcal{G}]$, *there is a normalized sequence* $(x_m)_{m \in F}$ *such that* $\operatorname{supp} x_m \subseteq [q_m, q_{m+1})$ *and*

$$\left\| \sum_{m \in F} a_m x_m \right\| \geqslant \frac{\theta_n}{2} \sum_{m \in F} |a_m|$$

for all $(a_m) \in c_{00}$. *Here the norm* $\|\cdot\|$ *is taken in the space* $T(\mathcal{F}_0, (\theta_n, \mathcal{F}_n)_{n=1}^{\infty})$.

Proof. We define the families $(\mathcal{F}_n)_{n=0}^{\infty}$ and a corresponding sequence of functions $(g_n)_{n=1}^{\infty}$ inductively. Let $\mathcal{F}_0 = \mathcal{R}_{\alpha_0}$, $\mathcal{F}_1 = \mathcal{R}_{\alpha_1}$ and $g_1(k) = k$ for all $k \in \mathbb{N}$. Suppose that g_n and $\mathcal{F}_n$ have been defined. If $\alpha_{n+1} < \omega$, let $\mathcal{F}_{n+1} = \mathcal{R}_{\alpha_{n+1}}$ and $g_{n+1} = g_n$. If $\alpha_{n+1} \geqslant \omega$, pick $x(k,n) \in c_{00}$ for each $k \in \mathbb{N}$ such that

1. $\min \operatorname{supp} x(k,n) \geqslant k$,
2. $\|x(k,n)\|_{\ell^1} = 1/\theta_{n+1}$, and
3. $\|x(k,n)\|_{[\mathcal{F}_{n_1}, \dots, \mathcal{F}_{n_s}, \mathcal{F}_0]} \leqslant \frac{1}{|A|}$ whenever $(n_1, \dots, n_s) \in A$,

where $A = \{(n_1, \dots, n_s) : \theta_{n_1} \dots \theta_{n_s} > \theta_{n+1}\}$. Choose an increasing function $g_{n+1} : \mathbb{N} \to \mathbb{N}$ such that $g_{n+1} \geqslant g_n$ and $\operatorname{supp} x(k,p) \subseteq [k, g_{n+1}(k))$ for all $1 \leqslant p \leqslant n$, $k \in \mathbb{N}$. Then choose families $\mathcal{G}_{n+1}$ and $\mathcal{H}_{n+1}$ corresponding to α_{n+1} and g_{n+1} using Lemma 24. Finally, define $\mathcal{F}_{n+1} = (\mathcal{H}_{n+1}, \mathcal{G}_{n+1}[\mathcal{S}_1^{g_{n+1}}])$. Note that $\iota(\mathcal{F}_n) = \alpha_n$ for all n. This completes the inductive definition of the families $(\mathcal{F}_n)_{n=0}^{\infty}$.

Claim. *If* $\alpha_{n+1} \geqslant \omega$, *then* $\|x(k,n)\| \leqslant 2$ *for all* $k \in \mathbb{N}$.

Let $x = x(k,n)$ and suppose $\|x\| = \sum_{E \in \mathcal{E}} t(E) \|Ex\|_{\mathcal{F}_0}$, where $\mathcal{E}$ is the set of all leaves of an $(\mathcal{F}_n)$-admissible tree. Take

$$\mathcal{E}' = \{E \in \mathcal{E} : h(E) = (0, n_1, \dots, n_s), (n_1, \dots, n_s) \in A\}$$

and $\mathscr{E}'' = \mathscr{E}\backslash\mathscr{E}'$. Now $E \in \mathscr{E}''$ only if $t(E) \leqslant \theta_{n+1}$. Therefore,

$$\sum_{E\in\mathscr{E}''} t(E)\|Ex\|_{\mathscr{F}_0} \leqslant \theta_{n+1} \sum_{E\in\mathscr{E}''} \|Ex\|_{\mathscr{F}_0} \leqslant \theta_{n+1}\|x\|_{\ell^1} = 1.$$

If $(n_1, \dots, n_s) \in A$, let $\mathscr{L}_{(n_1,\dots,n_s)} = \{E \in \mathscr{E}' : h(E) = (0, n_1, \dots, n_s)\}$. Now

$$\sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}} t(E)\|Ex\|_{\mathscr{F}_0} \leqslant \sum_{E\in\mathscr{L}_{(n_1,\dots,n_s)}} \|Ex\|_{\mathscr{F}_0} \leqslant \|x\|_{[\mathscr{F}_{n_1},\dots,\mathscr{F}_{n_s},\mathscr{F}_0]} \leqslant \frac{1}{|A|}$$

by condition (3). Hence

$$\sum_{E\in\mathscr{E}'} t(E)\|Ex\|_{\mathscr{F}_0} \leqslant \sum_{(n_1,\dots,n_s)\in A} \frac{1}{|A|} = 1.$$

Thus

$$\|x\| = \sum_{E\in\mathscr{E}} t(E)\|Ex\| = \sum_{E\in\mathscr{E}'} t(E)\|Ex\| + \sum_{E\in\mathscr{E}''} t(E)\|Ex\| \leqslant 2.$$

This proves the claim.

Since $\alpha_n < \sup_m \alpha_m = \omega^{\omega^\xi}$ for all $n \in \mathbb{N}$, there exist $n_1 < n_2 < n_3 < \cdots$ such that $\sup_s \alpha_{n_s+1} = \omega^{\omega^\xi}$ and $\alpha_{n_s+1} \geqslant \omega$ for all $s \in \mathbb{N}$. Note that this implies by choice that $\sup_s \iota(\mathscr{G}_{n_s+1}) = \omega^{\omega^\xi}$. Now choose $q_1 < q_2 < q_3 < \cdots$ such that $q_{s+1} > \max \operatorname{supp} x(q_s, n_r), 1 \leqslant r \leqslant s$. Let $L = \{q_1, q_2, q_3, \dots\} \in [\mathbb{N}]$ and $q(F) = \{q_m : m \in F\}$ for all $F \in [\mathbb{N}]^{<\infty}$. Define

$$\mathscr{G} = \{F : s \leqslant F \text{ and } q(F) \in \mathscr{G}_{n_s+1} \text{ for some } s \in \mathbb{N}\}.$$

Then $\iota(\mathscr{G}) = \omega^{\omega^\xi}$. For $s \leqslant m$, supp $x(q_m, n_s) \subseteq [q_m, g_{n_s+1}(q_m)) \in \mathscr{S}_1^{g_{n_s+1}}$. Hence if $s \leqslant F$, $q(F) \in \mathscr{G}_{n_s+1}$ for some $s \in \mathbb{N}$, and $x_m = \frac{x(q_m,n_s)}{\|x(q_m,n_s)\|}$ for all $m \in F$, then

$$\bigcup_{m\in F} \operatorname{supp} x_m \in \mathscr{G}_{n_s+1}[\mathscr{S}_1^{g_{n_s+1}}] \subseteq \mathscr{F}_{n_s+1}.$$

Thus, for all $(a_m) \in c_{00}$,

$$\begin{aligned}
\left\|\sum_{m\in F} a_m x_m\right\| &\geqslant \theta_{n_s+1}\left\|\sum_{m\in F} a_m x_m\right\|_{\mathscr{F}_{n_s+1}} \\
&= \theta_{n_s+1}\left\|\sum_{m\in F} a_m x_m\right\|_{\ell^1} \\
&\geqslant \frac{\theta_{n_s+1}}{2}\sum_{m\in F} |a_m|\, \|x(q_m, n_s)\|_{\ell^1} \text{ by the claim,} \\
&= \frac{1}{2}\sum_{m\in F} |a_m| \quad \text{by condition (2).}
\end{aligned}$$

Finally, if $F \in \mathscr{F}_n[\mathscr{G}]$ for some $n \in \mathbb{N}$, write $F = \bigcup_{s=1}^{k} F_s$ where $F_1 < \cdots < F_k$, $F_s \in \mathscr{G}$, $1 \leqslant s \leqslant k$, and $\{\min F_1, \ldots, \min F_k\} \in \mathscr{F}_n$. For $1 \leqslant s \leqslant k$, choose a normalized sequence $(x_m)_{m \in F_s}$ as above. Now for all $(a_m) \in c_{00}$,

$$\begin{aligned}\left\|\sum_{m\in F} a_m x_m\right\| &= \left\|\sum_{j=1}^{k}\left(\sum_{m\in F_s} a_m x_m\right)\right\| \\ &\geqslant \theta_n \sum_{j=1}^{k}\left\|\sum_{m\in F_s} a_m x_m\right\| \\ &\geqslant \frac{\theta_n}{2}\sum_{m\in F}|a_m|. \qquad \square\end{aligned}$$

5. Standard Schreier families

For all limit ordinals $\alpha < \omega_1$, fix a sequence of ordinals strictly increasing to α. If $\beta = \omega^{\beta_1}\cdot m_1 + \cdots + \omega^{\beta_k}\cdot m_k$ is a limit ordinal, determine $\mathscr{S}_\beta$ using the sequence

$$\hat{\beta}_n = \begin{cases} \omega^{\beta_1}\cdot m_1 + \cdots + \omega^{\beta_k}\cdot (m_k - 1) + \omega^{\beta_k - 1}\cdot n & \text{if } \beta_k \text{ is a successor,} \\ \omega^{\beta_1}\cdot m_1 + \cdots + \omega^{\beta_k}\cdot (m_k - 1) + \omega^{\zeta_n} & \text{if } \beta_k \text{ is a limit,} \end{cases}$$

where (ζ_n) is the chosen sequence of ordinals increasing to β_k. It is clear that if α is a countable limit ordinal such that $\ell(\alpha) \leqslant \eta$ for some $\eta < \omega_1$, then $(\omega^{\eta}\cdot\widehat{m+\alpha})_n = \omega^{\eta}\cdot m + \hat{\alpha}_n$ for all $m, n \in \mathbb{N}$. Throughout this section, we assume that the Schreier families $\mathscr{S}_\alpha$ are defined using these choices. For such "standard" Schreier families, the converse of Theorem 17 holds. The crucial set theoretic property of the standard Schreier families is encapsulated in the next lemma.

Lemma 26. *If α and η are countable ordinals such that $\ell(\alpha) \leqslant \eta$ and $m \in \mathbb{N}$, then $\mathscr{S}_\alpha[\mathscr{S}_{\omega^\eta\cdot m}] = \mathscr{S}_{\omega^\eta\cdot m+\alpha}$.*

Proof. The proof is by induction on α. The case $\alpha = 0$ is clear. The result holds for $\alpha = 1$ by definition of $\mathscr{S}_{\omega^\eta\cdot m+1}$. Suppose the lemma is true for some α. Then

$$\begin{aligned}\mathscr{S}_{\alpha+1}[\mathscr{S}_{\omega^\eta\cdot m}] &= (\mathscr{S}_1[\mathscr{S}_\alpha])[\mathscr{S}_{\omega^\eta\cdot m}] = \mathscr{S}_1[\mathscr{S}_\alpha[\mathscr{S}_{\omega^\eta\cdot m}]] \\ &= \mathscr{S}_1[\mathscr{S}_{\omega^\eta\cdot m+\alpha}] = \mathscr{S}_{\omega^\eta\cdot m+\alpha+1}.\end{aligned}$$

Suppose α is a limit ordinal and the lemma holds for all $\gamma<\alpha$. By the remark above, $\omega^\eta\cdot m+\hat{\alpha}_n=(\widehat{\omega^\eta\cdot m+\alpha})_n$ for all $m,n\in\mathbb{N}$. Now

$$\begin{aligned} &F\in\mathcal{S}_\alpha[\mathcal{S}_{\omega^\eta\cdot m}]\\ &\quad\Leftrightarrow F\in\mathcal{S}_{\hat{\alpha}_n}[\mathcal{S}_{\omega^\eta\cdot m}] \text{ for some } n\leqslant\min F,\\ &\quad\Leftrightarrow F\in\mathcal{S}_{\omega^\eta\cdot m+\hat{\alpha}_n} \text{ for some } n\leqslant\min F \text{ by induction},\\ &\quad\Leftrightarrow F\in\mathcal{S}_{\omega^\eta\cdot m+\alpha}.\qquad\square \end{aligned}$$

For the next theorem, fix a countable successor ordinal ξ and a nondecreasing sequence of ordinals $(\beta_n)_{n=1}^\infty$ such that $\sup_{n\in\mathbb{N}}\beta_n=\omega^\xi$ nontrivially. Also let $\mathcal{F}_0$ be a regular family containing $\mathcal{S}_0$ such that $\iota(\mathcal{F}_0)=\alpha_0<\omega^{\omega^\xi}$, and let $(\theta_n)_{n=1}^\infty$ be a nonincreasing null sequence in $(0,1)$. In the present context, the ordinal $\gamma(\epsilon,m)$ defined at the beginning of Section 4 becomes

$$\gamma=\gamma(\epsilon,m)=\ell(\alpha_0)+\max\{\beta_{n_s}+\cdots+\beta_{n_1}:\epsilon\theta_{n_1}\theta_{n_2}\ldots\theta_{n_s}>\theta_m\}$$

for all $m\in\mathbb{N}$ and $\epsilon>0$ ($\max\emptyset=0$). Denote the immediate predecessor of ξ by $\xi-1$.

Theorem 27. *Follow the notation above and apply the standard choices to define Schreier families. If there exists* $\epsilon>0$ *such that for all* $\beta<\omega^\xi$, *there exists* $m\in\mathbb{N}$ *satisfying* $\gamma(\epsilon,m)+2+\beta<\beta_m$, *then* $I_b(T(\mathcal{F}_0,(\theta_n,\mathcal{S}_{\beta_n})_{n=1}^\infty))=\omega^{\omega^{\xi}\cdot 2}$. *Otherwise,* $I_b(T(\mathcal{F}_0,(\theta_n,\mathcal{S}_{\beta_n})_{n=1}^\infty))=\omega^{\omega^\xi}$.

Proof. If there exists $\epsilon>0$ with the above properties, then Theorem 17 yields that $I_b(T(\mathcal{F}_0,(\theta_n,\mathcal{S}_{\beta_n})_{n=1}^\infty))=\omega^{\omega^{\xi}\cdot 2}$. Now assume that such ϵ does not exist. Given $\epsilon>0$, there exists $r=r(\epsilon)\in\mathbb{N}$ such that for all $m\in\mathbb{N}$, $\gamma(\epsilon,m)+2+\omega^{\xi-1}\cdot r\geqslant\beta_m$. Let $m_0\in\mathbb{N}$ be such that $\beta_{m_0}>\ell(\alpha_0)+2+\omega^{\xi-1}\cdot r$. Fix $m\geqslant m_0$. In particular, $\gamma(\epsilon,m)\neq(\alpha_0)$. Hence there exist $n_1,\ldots,n_s\in\mathbb{N}$ such that $\epsilon\theta_{n_1}\ldots\theta_{n_s}>\theta_m$ and $\ell(\alpha_0)+\beta_{n_s}+\cdots+\beta_{n_1}+2+\omega^{\xi-1}\cdot r\geqslant\beta_m$. Choose $r_0\in\mathbb{N}$ such that $\ell(\alpha_0)+2\leqslant\omega^{\xi-1}\cdot r_0$ and write $\beta_n=\omega^{\xi-1}\cdot r_n+\gamma_n$ for all $n\in\mathbb{N}$, where $r_n\in\mathbb{N}\cup\{0\}$ and $\gamma_n<\omega^{\xi-1}$. Then $r_0+r_{n_1}+\cdots+r_{n_s}+r\geqslant r_m$. If $r_n>0$,

$$\begin{aligned}\mathcal{S}_{\beta_n}&=\mathcal{S}_{\omega^{\xi-1}\cdot r_n+\gamma_n}=\mathcal{S}_{\gamma_n}[\mathcal{S}_{\omega^{\xi-1}\cdot r_n}]\quad\text{by Lemma 26}\\ &\supseteq\mathcal{S}_{\omega^{\xi-1}\cdot r_n}.\end{aligned}$$

The inclusion is obvious if $r_n = 0$. Therefore, using Lemma 26 again,

$$\begin{aligned}\left[\mathscr{S}_{\omega^{\xi-1}\cdot(r_0+r+1)}, \mathscr{S}_{\beta_{n_1}}, \ldots, \mathscr{S}_{\beta_{n_s}}\right] &\supseteq \left[\mathscr{S}_{\omega^{\xi-1}\cdot(r_0+r+1)}, \mathscr{S}_{\omega^{\xi-1}\cdot r_{n_1}}, \ldots, \mathscr{S}_{\omega^{\xi-1}\cdot r_{n_s}}\right] \\ &= \mathscr{S}_{\omega^{\xi-1}\cdot(r_{n_s}+\cdots+r_{n_1}+r_0+r+1)}.\end{aligned}$$

Since $\beta_m \leqslant \omega^{\xi-1}\cdot(r_0 + r_{n_1} + \cdots + r_{n_s} + r + 1)$, it follows from [16, Proposition 3.2(a)] that there exists $j_m \in \mathbb{N}$, such that

$$\begin{aligned}\mathscr{S}_{\beta_m} \cap [\mathbb{N}_{j_m}]^{<\infty} &\subseteq \mathscr{S}_{\omega^{\xi-1}\cdot(r_{n_s}+\cdots+r_{n_1}+r_0+r+1)} \\ &\subseteq [\mathscr{S}_{\omega^{\xi-1}\cdot(r_0+r+1)}, \mathscr{S}_{\beta_{n_1}}, \ldots, \mathscr{S}_{\beta_{n_s}}],\end{aligned}$$

where $\mathbb{N}_j$ is the integer interval $[j, \infty)$ for all $j \in \mathbb{N}$. By Proposition 1, there exists a sequence $(\ell_m) \subseteq \mathbb{N}$ converging to ∞ such that, defining $\mathscr{F}_n$ to be $(\mathscr{S}_{\beta_n} \cap [\mathbb{N}_{\ell_n}]^{<\infty}) \cup \mathscr{S}_0$ for all $n \in \mathbb{N}$, $T(\mathscr{F}_0, (\theta_n, \mathscr{S}_{\beta_n})_{n=1}^{\infty})$ is isomorphic to $T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\infty})$. Let $k_m = \max\{j_m, \ell_{n_1}, \ldots, \ell_{n_s}\}$,

$$\mathscr{B}_m = \{B \in [\mathbb{N}]^{<\infty} : \ell_m \leqslant B \text{ and } |B| \leqslant k_m\}$$

and define $\mathscr{H} = (\bigcup_{m=m_0}^{\infty} \mathscr{B}_m) \cup \mathscr{S}_{\omega^{\xi-1}\cdot(r_0+r+1)}$. If $m \geqslant m_0$, then $\mathscr{F}_m \subseteq [(\mathscr{H})^2, \mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}]$. Indeed, if $F \in \mathscr{F}_m$, then $F \in \mathscr{S}_0$ or $F \in \mathscr{S}_{\beta_m} \cap [\mathbb{N}_{\ell_m}]^{<\infty}$. In the former case it is clear that $F \in [(\mathscr{H})^2, \mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}]$. Suppose $F \in \mathscr{S}_{\beta_m} \cap [\mathbb{N}_{\ell_m}]^{<\infty}$. Then $F = F_1 \cup F_2$, where $F_1 = F \cap [\ell_m, k_m)$ and $F_2 = F \backslash F_1$. Clearly $F_1 \in \mathscr{B}_m \subseteq [\mathscr{H}, \mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}]$ and

$$\begin{aligned}F_2 &\in \mathscr{S}_{\beta_m} \cap [\mathbb{N}_{k_m}]^{<\infty} \\ &\subseteq [\mathscr{S}_{\omega^{\xi-1}\cdot(r_0+r+1)}, \mathscr{S}_{\beta_{n_1}} \cap [\mathbb{N}_{k_m}]^{<\infty}, \ldots, \mathscr{S}_{\beta_{n_s}} \cap [\mathbb{N}_{k_m}]^{<\infty}] \\ &\subseteq [\mathscr{H}, \mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}].\end{aligned}$$

Hence $\mathscr{F}_m \subseteq [(\mathscr{H})^2, \mathscr{F}_{n_1}, \ldots, \mathscr{F}_{n_s}]$. This proves that the family $\mathscr{G}_\varepsilon = (\mathscr{H})^2$ satisfies the hypothesis of Proposition 14. Note that $\iota((\mathscr{H})^2) = \iota(\mathscr{H}) \cdot 2 = \omega^{\omega^{\xi-1}\cdot(r_0+r+1)} \cdot 2$. Applying Proposition 14, we obtain

$$I_b(T(\mathscr{F}_0, (\theta_n, \mathscr{F}_n)_{n=1}^{\infty})) \leqslant \sup_{\epsilon>0} \sup_{n \in \mathbb{N}} [\omega^{\omega^{\xi-1}\cdot(r_0+r(\epsilon)+1)} \cdot 2 \cdot \omega^{\beta_n \cdot \omega}] = \omega^{\omega^{\xi}}.$$

Since the reverse inequality holds by Theorem 15, the proof is complete. □

It is worthwhile to record the statement of Theorem 27 for finite β_n's. Of course, no choices need to be made in defining the Schreier families $\mathscr{S}_n$, $n \in \mathbb{N} \cup \{0\}$.

Corollary 28. *Suppose that* $\mathscr{F}_0$ *is a regular family containing* $\mathscr{S}_0$ *such that* $\iota(\mathscr{F}_0) < \omega^\omega$, *and that* (θ_n) *is a nonincreasing null sequence in* $(0,1)$ *such that* $\theta_{n+m} \geqslant \theta_n\theta_m$ *for all* $n, m \in \mathbb{N}$. *Let* $X = T(\mathscr{F}_0, (\theta_n, \mathscr{S}_n)_{n=1}^\infty)$. *If* $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$, *then* $I(X) = \omega^{\omega\cdot 2}$. *Otherwise*, $I(X) = \omega^\omega$.

Proof. Observe that with the assumption on the sequence (θ_n),

$$\gamma(\epsilon, m) = \ell(\alpha_0) + \max\{n : \epsilon\theta_n > \theta_m\}$$

for all $\epsilon > 0$ and all $m \in \mathbb{N}$. Assume that $\lim_m \limsup_n \dfrac{\theta_{m+n}}{\theta_n} > 0$. Then there exists $\epsilon > 0$ such that $\limsup_n \dfrac{\theta_{m+n}}{\theta_n} > \epsilon$ for all $m \in \mathbb{N}$. Given $\beta < \omega$, let $m = 2 + \beta + \ell(\alpha_0)$ and choose k such that $\dfrac{\theta_{m+k}}{\theta_k} > \epsilon$. If $\epsilon\theta_n > \theta_{m+k}$, then $\epsilon\theta_n > \epsilon\theta_k$, which implies that $n < k$. Therefore $\gamma(\epsilon, m+k) < \ell(\alpha_0) + k$. Hence

$$\gamma(\epsilon, m+k) + 2 + \beta < \ell(\alpha_0) + 2 + \beta + k = m + k.$$

Consequently, by Theorem 27, $I_b(T(\mathscr{F}_0, (\theta_n, \mathscr{S}_n)_{n=1}^\infty)) = \omega^{\omega\cdot 2}$.

Conversely, suppose that $I_b(T(\mathscr{F}_0, (\theta_n, \mathscr{S}_n)_{n=1}^\infty)) = \omega^{\omega\cdot 2}$. By Theorem 27, there exists $\epsilon > 0$ such that for all $m \in \mathbb{N}$, there exists j_m such that $\gamma(\epsilon, j_m) + 2 + m < j_m$. In particular, $\lim_m j_m = \infty$. For all $m \in \mathbb{N}$, $j_m - m > \gamma(\epsilon, j_m)$ implies that $\epsilon\theta_{j_m - m} \leqslant \theta_{j_m}$. If $n \geqslant m$, $\theta_{j_n} \geqslant \epsilon\theta_{j_n - n} \geqslant \epsilon\theta_{j_n - m}$. Thus $\lim_n \sup \dfrac{\theta_{m+n}}{\theta_n} \geqslant \epsilon$. $\square$

We conclude by looking at a case of the result when $\xi = 0$.

Proposition 29. *Suppose that* $\mathscr{F}_0$ *is a regular family containing* $\mathscr{S}_0$ *and* $\iota(\mathscr{F}_0) < \omega$. *Let* (θ_n) *be a nonincreasing null sequence in* $(0,1)$. *Denote the space* $T(\mathscr{F}_0, (\theta_n, \mathscr{A}_n)_{n=1}^\infty)$ *by* Y. *Assume that every term* $(\theta_n, \mathscr{A}_n)$ *is essential in the sense that there exists a nonzero* $x \in Y$ *such that* $\|x\| = \theta_n \sum_{j=1}^n \|E_j x\|$ *for some* $E_1 < \cdots < E_n$. *Then* $I_b(Y) = \omega^2$ *if*

$$\inf_{r\in\mathbb{N}} \sup_n \frac{\theta_{rn}}{\theta_n} > 0.$$

Otherwise, $I_b(Y) = \omega$.

Proof. Clearly $I_b(Y) \geqslant \omega$. Suppose $\inf_{r\in\mathbb{N}} \sup_n \dfrac{\theta_{rn}}{\theta_n} = 0$. Given $\epsilon > 0$, choose $r \in \mathbb{N}$ such that $\sup_n \dfrac{\theta_{rn}}{\theta_n} < \epsilon$. Let $\mathscr{G}_\epsilon = \mathscr{A}_{2r}$. Take any $m \geqslant r$, and let $n = \lfloor \frac{m}{r} \rfloor$. Then $n \in \mathbb{N}$ and $m \geqslant rn$. Hence $\theta_m \leqslant \theta_{rn} < \epsilon\theta_n$. Also $\frac{m}{r} < n + 1$ implies that $m < 2rn$. Hence

$\mathscr{A}_m \subseteq \mathscr{A}_{2r}[\mathscr{A}_n] = \mathscr{G}_\epsilon[\mathscr{A}_n]$. By Proposition 14,

$$I_b(Y) \leqslant \sup_{\epsilon > 0} \sup_{n \in \mathbb{N}} [(\alpha_0 \vee \iota(\mathscr{G}_\epsilon)) \cdot \alpha_n^\omega] = \omega.$$

On the other hand, assume that $\inf_{r \in \mathbb{N}} \sup_n \dfrac{\theta_{rn}}{\theta_n} = \epsilon > 0$. Let $r \in \mathbb{N}$ be given. There exists n such that $\dfrac{\theta_{rn}}{\theta_n} > \dfrac{\epsilon}{2}$. Choose a normalized vector x such that $1 = \theta_n \sum_{i=1}^n ||E_i x||$ for some $E_1 < \cdots < E_n$. Define a sequence (y_j) so that each y_j is a right shift of x so that $\operatorname{supp} y_j < \operatorname{supp} y_{j+1}$. Clearly $||y_j|| \geqslant ||x|| = 1$ for all j. However, there exists $s \in \mathbb{N}$ such that $|F| \leqslant s$ for all $F \in \mathscr{F}_0$. Thus $||| \cdot ||| \leqslant || \cdot || \leqslant s ||| \cdot |||$, where $||| \cdot |||$ is the norm of the space $T(\mathscr{S}_0, (\theta_n, \mathscr{A}_n)_{n=1}^\infty)$. Note that the norm $||| \cdot |||$ is shift invariant. Hence

$$||y_j|| \leqslant s|||y_j||| = s|||x||| \leqslant s||x|| = s \text{ for all } j.$$

Moreover, for an appropriate shift $E_1^j < \cdots < E_n^j$ of $E_1 < \cdots < E_j$, we have

$$\theta_n \sum_{i=1}^n ||E_i^j y_j|| \geqslant \theta_n \sum_{i=1}^n ||E_i x|| = 1.$$

Therefore, for any $(a_j) \in c_{00}$,

$$\begin{aligned} \left|\left| \sum_{j=1}^r a_j y_j \right|\right| &\geqslant \theta_{rn} \sum_{j=1}^r \sum_{i=1}^n |a_j| ||E_i^j y_j|| \\ &\geqslant \frac{\epsilon}{2} \theta_n \sum_{j=1}^r \sum_{i=1}^n |a_j| ||E_i^j y_j|| \\ &\geqslant \frac{\epsilon}{2} \sum_{j=1}^r |a_j|. \end{aligned}$$

Thus $\left(\dfrac{y_j}{||y_j||}\right)_{j=1}^r$ is a normalized ℓ^1-$\dfrac{\epsilon}{2s}$-block basis. Since $r \in \mathbb{N}$ is arbitrary, we obtain an ℓ^1-$\dfrac{\epsilon}{2s}$-block tree of order ω. Hence $I_b(Y, \frac{2s}{\epsilon}) \geqslant \omega$. By [12, Theorem 5.6 and Lemma 5.7], $I_b(Y) \geqslant \omega^2$. By (1) of Theorem 15, $I_b(Y) \leqslant \omega^2$. Therefore $I_b(Y) = \omega^2$. $\square$

Remark. Let (θ_n) be a nonincreasing null sequence in $(0, 1)$ such that $\theta_{m+n} \geqslant \theta_m \theta_n$ for all $m, n \in \mathbb{N}$ and set $X = T(\mathscr{S}_0, (\theta_n, \mathscr{S}_n)_{n=1}^\infty)$. Argyros, Deliyanni and Manoussakis [6, Proposition 3.1] showed that if $\lim_{n \to \infty} \theta_n^{1/n} = 1$, then X contains ℓ_ω^1-spreading models hereditarily. In particular, this implies that $I(X) \geqslant \omega^{\omega \cdot 2}$. The authors [14] have shown that X contains ℓ^1_ω-spreading models (without the "hereditary") under the strictly weaker assumption $\lim_m \limsup_n \frac{\theta_{m+n}}{\theta_n} > 0$. More generally, it has been proven that the assumptions of Theorem 17 ensure that $T(\mathscr{S}_0, (\theta_n, \mathscr{F}_n)_{n=1}^\infty)$ contains an $\ell^1_{\omega^\xi}$-spreading model [14].

References

[1] D.E. Alspach, S. Argyros, Complexity of weakly null sequences, Diss. Math. 321 (1992) 1–44.

[2] D.E. Alspach, R. Judd, E. Odell, The Szlenk index and local ℓ_1-indices, preprint.

[3] G. Androulakis, E. Odell, Distorting mixed Tsirelson spaces, Israel J. Math. 109 (1999) 125–149.

[4] S.A. Argyros, I. Deliyanni, Examples of asymptotic ℓ^1 Banach spaces, Trans. Amer. Math. Soc. 349 (1997) 973–995.

[5] S.A. Argyros, I. Deliyanni, D.N. Kutzarova, A. Manoussakis, Modified mixed Tsirelson spaces, J. Funct. Anal. 159 (1998) 43–109.

[6] S.A. Argyros, I. Deliyanni, A. Manoussakis, Distortion and spreading models in modified mixed Tsirelson spaces, preprint.

[7] S.F. Bellenot, Tsirelson superspaces and ℓ_p, J. Funct. Anal. 69 (1986) 207–228.

[8] J. Bourgain, On convergent sequences of continuous functions, Bull. Soc. Math. Bel. 32 (1980) 235–249.

[9] P.G. Casazza, W.B. Johnson, L. Tzafriri, On Tsirelson's space, Israel J. Math. 47 (1984) 81–98.

[10] T. Figiel, W.B. Johnson, A uniformly convex Banach space which contains no ℓ_p, Compositio Math. 29 (1974) 179–190.

[11] I. Gasparis, A dichotomy theorem for subsets of the power set of the natural numbers, Proc. Amer. Math. Soc. 129 (2001) 759–764.

[12] R. Judd, E. Odell, Concerning the Bourgain ℓ_1 index of a Banach space, Israel J. Math. 108 (1998) 145–171.

[13] D. Leung, W.-K. Tang, The ℓ^1-indices of Tsirelson type spaces, Proc. Amer. Math. Soc. 131 (2003) 511–521.

[14] D. Leung, W.-K. Tang, ℓ^1-spreading models in mixed Tsirelson spaces, in preparation.

[15] Edward Odell, Nicole Tomczak-Jaegermann, On certain norms on Tsirelson's space, Illinois J. Math. 44 (2000) 51–71.

[16] Edward Odell, Nicole Tomczak-Jaegermann, Roy Wagner, Proximity to ℓ_1 and distortion in asymptotic ℓ_1 spaces, J. Funct. Anal. 150 (1997) 101–145.