Dedicated to the memory of Lior Tzafriri

MINIMALITY PROPERTIES OF TSIRELSON TYPE SPACES

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Abstract. In this paper, we study minimality properties of partly modified mixed Tsirelson spaces. A Banach space with a normalized basis \((e_k)\) is said to be subsequentially minimal if for every normalized block basis \((x_k)\) of \((e_k)\), there is a further block basis \((y_k)\) of \((x_k)\) such that \((y_k)\) is equivalent to a subsequence of \((e_k)\). Sufficient conditions are given for a partly modified mixed Tsirelson space to be subsequentially minimal and connections with Bourgain’s \(\ell^1\)-index are established. It is also shown that a large class of mixed Tsirelson spaces fails to be subsequentially minimal in a strong sense.

The class of mixed Tsirelson spaces plays an important role in the structure theory of Banach spaces and has been well investigated (e.g., [2, 3, 5, 17, 20, 21]). In this paper, we will study aspects of the subspace structure of mixed Tsirelson spaces and (partly) modified mixed Tsirelson spaces (see definitions below). We are particularly interested in properties connected with minimality. A infinite-dimensional Banach space \(X\) is minimal if every infinite-dimensional subspace has a further subspace isomorphic to \(X\). The work of Gowers [15] had motivated some recent studies on minimality (e.g., [11], [12], [22]).

A Banach space \(X\) with a normalized basis \((e_k)\) is said to be subsequentially minimal if for every normalized block basis \((x_k)\) of \((e_k)\), there is a further block \((y_k)\) of \((x_k)\) such that \((y_k)\) is equivalent to a subsequence of \((e_k)\). It is well known that the Tsirelson space \(T[(S_1, 1/2)]\) has the property that every normalized block basis of its standard basis is equivalent to a subsequence of \((e_k)\) [8]. In particular, it is subsequentially minimal. In [18, Theorem 9], it was shown that if a nonincreasing null sequence \((\theta_n)\) in \((0, 1)\) is regular \((\theta_{m+n} \geq \theta_m \theta_n)\) and satisfies

\[
\lim \limsup_m \frac{\theta_{m+n}}{\theta_n} > 0,
\]

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Then the space $T[\{(S_n, \theta_n)\}_{n=1}^\infty]$ is subsequentially minimal if and only if every block subspace of $T[\{(S_n, \theta_n)\}_{n=1}^\infty]$ admits an $\ell^1$-$S_\omega$-spreading model, if and only if every block subspace of $T[\{(S_n, \theta_n)\}_{n=1}^\infty]$ has Bourgain $\ell^1$-index greater than $\omega^\omega$. In particular, if $\sup_n \theta_n^{1/n} = 1$, then the mixed Tsirelson space $T[\{(S_n, \theta_n)\}_{n=1}^\infty]$ is subsequentially minimal [20].

This paper is divided into two parts. In the first part, we investigate the analogs of the results quoted above in the context of partly modified mixed Tsirelson spaces. In this connection, it is worthwhile to point out that a subsequentially minimal partly modified mixed Tsirelson space is quasi-minimal in the sense of Gowers [15]. Since these spaces are strongly asymptotic $\ell^1$, by [10] they do not contain minimal subspaces and therefore, they are strictly quasi-minimal. The only typical known example of a strictly quasi-minimal space was the Tsirelson space. While the unit vector basis of Tsirelson space satisfies the block property (every normalized block basis is equivalent to a subsequence of the unit vector basis) [8], among our examples of strictly quasi-minimal spaces there are cases which do not have this property. The subsequentially minimal mixed Tsirelson spaces, mentioned above, are also quasi-minimal, however it is not known if they are strictly quasi-minimal (see the remarks in [10]). In the second part of the paper, we give a general sufficient condition for a (unmodified) mixed Tsirelson space to fail to be subsequentially minimal in a strong sense.

In a recent article [13], Ferenczi and Rosendal undertook a deep analysis of minimality and proved several dichotomy results. Generally speaking, their results show that in every (infinite dimensional) Banach space, one can find a further (infinite dimensional closed) subspace that either has some form of minimality or a related form of “tightness”. They also refined Gower’s classification of Banach spaces. The subsequentially minimal partly modified mixed Tsirelson spaces considered in §2 all belong to Class 5c in the Ferenczi-Rosendal classification. (The fact that they are tight with constants follow from Proposition 4.2 of [13].) This puts them in the same class as Tsirelson’s space $T$. However, every normalized block basis in $T$ satisfies the block property. On the other hand, if $(\theta_n)$ is a regular sequence with $\sup_n \theta_n^{1/n} = 1$, then our results show that $T[\{(S_n, \sigma_n, \theta_n)\}_{n=1}^\infty]$ is saturated with block sequences that satisfy the block property and also saturated with block sequences that fail the block property. (The same can be said for the space $X_{M(1),u}$ constructed in [3].) Thus it is unclear if there is a further dichotomy within Class 5c that can distinguish $T$ from these other spaces. Let us also mention that a dichotomy between subsequential minimality on the one hand and a property called “tight by range” on the other hand was established [13, Theorem 1.3]. The property of being “tight by range” is stronger than what we term “strongly non-subsequentially minimal” in §4 below.
1. Preliminaries

Denote by \( \mathbb{N} \) the set of natural numbers. For any infinite subset \( M \) of \( \mathbb{N} \), let \( [M] \), respectively \( [M]^{< \infty} \), be the set of all infinite and finite subsets of \( M \) respectively. These are subspaces of the power set of \( \mathbb{N} \), which is identified with \( 2^\mathbb{N} \) and endowed with the topology of pointwise convergence. A subset \( \mathcal{F} \) of \( [\mathbb{N}]^{< \infty} \) is said to be hereditary if \( G \in \mathcal{F} \) whenever \( G \subseteq F \) and \( F \in \mathcal{F} \). It is spreading if for all strictly increasing sequences \( (m_i)_{i=1}^k \) and \( (n_i)_{i=1}^k \in \mathcal{F} \) if \( (m_i)_{i=1}^k \in \mathcal{F} \) and \( m_i \leq n_i \) for all \( i \). We also call \( (n_i)_{i=1}^k \) a spreading of \( (m_i)_{i=1}^k \). A regular family is a subset of \( [\mathbb{N}]^{< \infty} \) that is hereditary, spreading and compact (as a subspace of \( 2^\mathbb{N} \)). If \( I \) and \( J \) are nonempty finite subsets of \( \mathbb{N} \), we write \( I < J \) to mean \( \max I < \min J \). We also allow that \( \emptyset < I \) and \( I < \emptyset \). For a singleton \( \{n\} \), \( \{n\} < J \) is abbreviated to \( n < J \). If \( \mathcal{F}, \mathcal{G} \subseteq [\mathbb{N}]^{< \infty} \), let

\[
\mathcal{F}[\mathcal{G}]=\{\cup_{i=1}^k G_i : G_i \in \mathcal{G}, G_1 < \cdots < G_k, (\min G_i)_{i=1}^k \in \mathcal{F}\}
\]

and

\[
(\mathcal{F}, \mathcal{G}) = \{F \cup G : F < G, F \in \mathcal{F}, G \in \mathcal{G}\}.
\]

Inductively, set \( (\mathcal{F})^1 = \mathcal{F} \) and \( (\mathcal{F})^{n+1} = (\mathcal{F}, (\mathcal{F})^n) \) for all \( n \in \mathbb{N} \). It is clear that \( \mathcal{F}[\mathcal{G}] \) and \( (\mathcal{F}, \mathcal{G}) \) are regular if both \( \mathcal{F} \) and \( \mathcal{G} \) are. A class of regular families that has played a central role is the class of generalized Schreier families \([1]\).

Let \( S_0 \) consist of all singleton subsets of \( \mathbb{N} \) together with the empty set. Then define \( S_1 \) to be the collection of all \( A \in [\mathbb{N}]^{< \infty} \) such that \( |A| \leq \min A \) together with the empty set, where \( |A| \) denotes the cardinality of the set \( A \). If \( S_\alpha \) has been defined for some countable ordinal \( \alpha \), set \( S_{\alpha+1} = S_1[S_\alpha] \). For a countable limit ordinal \( \alpha \), specify a sequence \( (\alpha_n) \) that strictly increases to \( \alpha \). Then define

\[
S_\alpha = \{F : F \in S_{\alpha_n} \text{ for some } n \leq \min F\} \cup \{\emptyset\}.
\]

Given a nonempty compact family \( \mathcal{F} \subseteq [\mathbb{N}]^{< \infty} \), let \( \mathcal{F}^{(0)} = \mathcal{F} \) and \( \mathcal{F}^{(1)} \) be the set of all limit points of \( \mathcal{F} \). Continue inductively to derive \( \mathcal{F}^{(\alpha+1)} = (\mathcal{F}^{(\alpha)})^{(1)} \) for all ordinals \( \alpha \) and \( \mathcal{F}^{(\alpha)} = \cap_{\beta < \alpha} \mathcal{F}^{(\beta)} \) for all limit ordinals \( \alpha \). The index \( \iota(\mathcal{F}) \) is taken to be the smallest \( \alpha \) such that \( \mathcal{F}^{(\alpha+1)} = \emptyset \). Since \( [\mathbb{N}]^{< \infty} \) is countable, \( \iota(\mathcal{F}) < \omega_1 \) for any compact family \( \mathcal{F} \subseteq [\mathbb{N}]^{< \infty} \). It is well known that \( \iota(S_\alpha) = \omega^\alpha \) for all \( \alpha < \omega_1 \) \([1, \text{Proposition 4.10}]\).

A sequence \( (x_n) \) in a normed space said to dominate a sequence \( (y_n) \) in a possibly different space if there is a finite constant \( K \) such that \( \| \sum a_n x_n \| \leq K \| \sum a_n y_n \| \) for all \( (a_n) \in c_{00} \). If two sequences dominate each other, then they are equivalent and we write \( (x_n) \sim (y_n) \). If \( (e_n) \) is a basic sequence and \( F \subseteq \mathbb{N} \), \( \langle (e_n)_{n \in F} \rangle \) denotes the closed linear space of \( \{e_n : n \in F\} \). If \( (e_n) \) is a normalized basis of \( X \), then by \( (x_n) \prec (e_n) \) or \( (x_n) \prec X \) we shall mean that \( (x_n) \) is a normalized block basis of \( (e_n) \). We say that \( Y \) is a block subspace of \( X \), \( Y \preceq X \), if \( X \) has a basis \( (x_n) \) and \( Y = \langle (y_n)_{n \in \mathbb{N}} \rangle \) for some
A normalized sequence \((x_n)\) is said to be an \(\ell^1\)-spreading model with constant \(K\) if \(\| \sum_{n \in F} a_n x_n \| \geq K^{-1} \sum_{n \in F} |a_n|\) whenever \(F \in S_\beta\).

**Partly modified mixed Tsirelson spaces**

Let \((\theta_n)\) be a null sequence in the interval \((0, 1)\) and \(\sigma_n \in \{U, M\}\) for every \(n\). We say that a family \((E_i)_{i=1}^k\) of subsets of \(\mathbb{N}\) is \((S_n, \sigma_n)\)-adapted if \((\min E_i)_{i=1}^k \in S_n\) and

\[
\begin{cases}
E_i \cap E_j = \emptyset, & 1 \leq i \neq j \leq k \quad \text{if } \sigma_n = M, \\
E_i < E_{i+1}, & 1 \leq i < k \quad \text{if } \sigma_n = U.
\end{cases}
\]

An \((S_n, \sigma_n)\)-adapted family \((E_i)_{i=1}^k\) is said to be \(S_n\)-admissible (respectively \(\mathcal{S}_n\)-allowable) if \(\sigma_n = U\) (respectively \(\sigma_n = M\)). Define the space \(X = T[(S_n, \sigma_n, \theta_n)^\infty_{n=1}]\) to be the completion of \(c_{00}\) under the implicitly defined norm

\[
\|x\| = \max \left\{ \|x\|_{c_0}, \sup_n \theta_n \sup_{i} \sum \|E_i x\| \right\},
\]

where the last supremum is taken over all \((S_n, \sigma_n)\)-adapted families \((E_i)\). If \(\sigma_n = U\) for all \(n\) (respectively \(\sigma_n = M\) for all \(n\)), then \(X\) is a mixed Tsirelson space (respectively modified mixed Tsirelson space). If \(\sigma_0 = M\) for at least one \(p_0\), we call \(X\) a partly modified mixed Tsirelson space.

**Norming Trees**

Equation (1) can be viewed as an iterative prescription for computing the norm. The procedure may be summarized in terms of norming trees, from which the existence and uniqueness of a norm satisfying equation (1) also follows. An \(((S_n, \sigma_n)n)\)-adapted tree \(T\) is a finite collection of elements \((E_i^n), 0 \leq m \leq r, 1 \leq i \leq k(m)\), in \([N]^{<\infty}\) with the following properties.

1. \(k(0) = 1\),
2. Every \(E_i^{m+1}\) is a subset of some \(E_j^m\),
3. For each \(j\) and \(m\), the collection \(\{E_i^{m+1} : E_i^{m+1} \subseteq E_j^m\}\) is \((S_k, \sigma_k)\)-adapted for some \(k\).

The set \(E_1^0\) is called the root of the adapted tree. The elements \(E_i^n\) are called nodes of the tree. If \(E_i^n \subseteq E_j^m\) and \(n > m\), we say that \(E_i^n\) is a descendant of \(E_j^m\) and \(E_j^m\) is an ancestor of \(E_i^n\). If, in the above notation, \(n = m + 1\), then \(E_i^n\) is said to be an immediate successor of \(E_j^m\), and \(E_j^m\) the immediate predecessor or parent of \(E_i^n\). Nodes with no descendants are called terminal nodes or leaves of the tree. The collection of all leaves of \(T\) is denoted by \(L(T)\). Assign tags to the individual nodes inductively as follows. Let \(t(E_i^0) = 1\). If \(t(E_i^m)\) has been defined and the collection \((E_j^{m+1})\) of all immediate successors of \(E_i^m\) forms an \((S_k, \sigma_k)\)-adapted collection, then define \(t(E_i^{m+1}) = \theta_k t(E_i^m)\) for all immediate successors \(E_j^{m+1}\) of \(E_i^m\). If \(x \in c_{00}\) and \(T\) is an adapted tree, let \(Tx = \sum t(E)\|Ex\|_{c_0}\) where the sum is taken over all leaves in \(T\). It follows from the implicit description (equation (1))
of the norm in $X$ that $\|x\| = \max T x$, with the maximum taken over the set of all adapted trees. Let us also point out that if $E$ is a collection of pairwise disjoint nodes of an adapted tree $T$ so that $E \subseteq \cup \mathcal{E}$ for every leaf $E$ of $T$ and $x \in c_{00}$, then $T x \leq \sum_{F \in \mathcal{E}} t(F)\|Fx\|$. Given a node $E \in T$ with tag $t(E) = \prod_{i=1}^{m} \theta_{n_{i}}$, define $\text{ord}_{T}(E) = \sum_{i=1}^{m} n_{i}$. When there is no confusion, we write $\text{ord}(E)$ instead of $\text{ord}_{T}(E)$.

Let $T$ be an adapted tree. A node $E \in T$ is said to be a \textit{sibling} of $F \in T$ if they have the same parent. If $(z_{i})$ is a block sequence, we say that $E$ \textit{begins at} $z_{k}$ if $E \cap \text{supp} z_{k} \neq \emptyset$ and $E \cap \text{supp} z_{j} = \emptyset$ for all $j < k$. To say that $E$ \textit{begins before} $z_{k}$ means that $E$ begins at $z_{j}$ for some $j < k$ and we denote this condition by $E < z_{k}$.

\textbf{Definition.} Given a finite constant $K \geq 1$, an $\ell^{1}$-$K$-tree in a Banach space $B$ is a tree in $B$ so that every node $(x_{1}, \ldots, x_{n})$ is a normalized sequence such that $\|\sum a_{k}x_{k}\| \geq K^{-1} \sum |a_{k}|$ for all $(a_{k})$. If $B$ has a basis, an $\ell^{1}$-$K$-block tree is a block tree that is also an $\ell^{1}$-$K$-tree. Suppose that $B$ does not contain $\ell^{1}$, let $I(B, K) = \sup o(T)$, where the sup is taken over the set of all $\ell^{1}$-$K$-trees in $X$. The \textit{Bourgain $\ell^{1}$-index} of $B$ is defined to be $I(B) = \sup_{K<\infty} I(B, K)$. The \textit{block $\ell^{1}$-index} $I_{b}(B)$ is defined analogously using block trees if $B$ has a basis. In [16, Lemmas 5.7 and 5.11], it was shown that $I_{b}(B) \neq I_{b}(B, K)$ and $I(B) \neq I(B, K)$ for every $K$. In particular, $I_{b}(B), I(B)$ are limit ordinals. It was also shown that [16, Corollary 5.13] $I(B) = I_{b}(B)$ when both are defined and either one has value $\geq \omega^{\omega}$.

\section{Sufficient conditions for subsequential minimality}

The purpose of the present section is to give sufficient conditions for a partly modified mixed Tsirelson space to be subsequentially minimal. Prior experience with mixed Tsirelson spaces [18] informs us that there may be some connection with the Bourgain $\ell^{1}$-index. This indeed turns out to be the case but the proof requires a different approach.

The main result of the section is the following theorem. The smallest integer greater than or equal to $a \in \mathbb{R}$ is denoted by $\lceil a \rceil$. For the rest of the section, $X$ will denote a partly modified mixed Tsirelson space.
Theorem 1. Let $X$ be a partly modified mixed Tsirelson space. If $Y < X$ and $I(Y) > \omega^{\omega}$, then there exists $(x_n) < Y$ such that $(x_n) \sim (e_{p_n})$, where $p_n = \min \text{supp } x_n$. Consequently, $X$ is subsequentially minimal if $I(Y) > \omega^{\omega}$ for all $Y < X$.

Let us remark that by Theorem 1 and Proposition 14, any partly modified mixed Tsirelson space $X$ such that $I(Y) > \omega^{\omega}$ for all block subspaces $Y$ of $X$ is saturated with subspaces with subsequentially minimal bases. Thus $X$ is *sequentially minimal* in the terminology of [13]. Before proceeding with the proof of the theorem, let us draw the following corollary.

**Corollary 2.** Suppose that there exists $\varepsilon > 0$ such that

$$\sup \left\{ \frac{n}{m} : \theta_n \geq \varepsilon^m \right\} = \infty.$$  

Then $X$ is subsequentially minimal. This holds in particular if $\sup \theta_n^{1/n} = 1$.

**Proof.** Clearly, for any $n \in \mathbb{N}$ and any $Y < X$, every normalized block sequence in $Y$ is an $\ell^1$-$S_n$-spreading model with constant $\theta_n^{-1}$. By [16], if $Y$ contains an $\ell^1$-$S_{2n}$-spreading model with constant $K$, then it contains an $\ell^1$-$S_n$-spreading model with constant $\sqrt{K}$. With the assumption of the corollary, for any $k \in \mathbb{N}$, there are $m, n$ so that $n/m \geq 2k$ and $\theta_n \geq \varepsilon^m$. Choose $i$ and $j$ so that $2^i \leq m < 2^{i+1}$ and $2^j \leq n < 2^{j+1}$. Then any $Y < X$ contains an $\ell^1$-$S_{2j}$-spreading model with constant $\theta_n^{-1}$, and hence, by the remark above, an $\ell^1$-$S_{2^{j-i}}$-spreading model with constant $\theta_n^{-1/2^i}$. Since $\theta_n^{-1/2^i} \leq \varepsilon^{-2}$ and $2^{j-i} \geq k$, $Y$ has an $\ell^1$-$S_k$-spreading model with constant $\varepsilon^{-2}$ for all $k$. Hence there is an $\ell^1$-$\varepsilon^{-2}$-tree on $Y$ of order $\omega^{\omega}$. Thus $I_\varepsilon(Y, \varepsilon^{-2}) \geq \omega^{\omega}$ and so $I(Y) = I_\varepsilon(Y) > I_\varepsilon(Y, \varepsilon^{-2}) \geq \omega^{\omega}$. The desired result now follows from Theorem 1.

Finally, assume that $\sup \theta_n^{1/n} = 1$. Given $0 < \varepsilon < 1$ and $k \in \mathbb{N}$, there exists $n > k$ such that $\theta_n^{1/n} > \varepsilon^{1/k}$. Set $m = \lceil n/k \rceil \geq 2$. Then $\theta_n \geq \varepsilon^m$ and $n/m \geq k(1 - 1/m) \geq k/2$. \hfill $\square$

The proof of Theorem 1 occurs in two stages. First we show that from any block subspace of $X$ with a high $\ell^1$-index a “slow-growing” block sequence may be extracted (see property $(\ast)$ defined below). In the second part, we show that this block sequence is equivalent to a subsequence of the unit vector basis $(e_k)$.

**Definition.** Let $Y = [(y_k)]$ be a block subspace of $X$, we say that $Y$ has property $(\ast)$ if there exists a constant $C < \infty$ such that for all $n \in \mathbb{N}$, there exists a normalized vector $x \in Y_n = \text{span}\{(y_k)_{k=n}^{\infty}\}$ such that $\sum \|E_i x\| \leq C$ whenever $(E_i)$ is $S_n$-allowable.

First we recall a needed lemma.
Lemma 3 ([17, Proposition 14]). Let $T$ be a well-founded block tree in a Banach space $B$ with a basis. Define

$$\mathcal{H} = \{(\max \text{supp } x_j)_{j=1}^r : (x_j)_{j=1}^r \in T\}$$

and

$$\mathcal{G} = \{G : G \text{ is a spreading of a subset of some } H \in \mathcal{H}\}.$$  

Then $\mathcal{G}$ is hereditary and spreading. If $\mathcal{G}$ is compact, then $\iota(\mathcal{G}) \geq o(T)$.

Lemma 4. If $I(Y) > \omega^\omega$ then $Y$ has property (*).

Proof. There exists $K < \infty$ such that $I_b(Y, K) \geq \omega^\omega$. Let $T$ be an $\ell^1$-$K$-block tree in $Y$ such that $o(T) \geq \omega^\omega$. Given $n \geq p_0$, consider the tree $\hat{T}$ consisting of all nodes of the form $(x_j)_{j=n}^r$ for some $(x_j)_{j=1}^r \in T$, $r \geq n$.

Then $\hat{T}$ is an $\ell^1$-$K$-block tree in $Y_n$ such that $o(\hat{T}) \geq \omega^\omega$. Define

$$\mathcal{H} = \{(\max \text{supp } x_j)_{j=n}^r : (x_j)_{j=n}^r \in \hat{T}\}$$

and

$$\mathcal{G} = \{G : G \text{ is a spreading of a subset of some } H \in \mathcal{H}\}.$$  

By Lemma 3, $\mathcal{G}$ is hereditary and spreading, and either $\mathcal{G}$ is noncompact or it is compact with $\iota(\mathcal{G}) \geq o(\hat{T}) \geq \omega^\omega > \omega^{n+1}$. By [14, Theorem 1.1], there exists $M \in \mathbb{N}$ such that

$$\mathcal{S}_{n+1} \cap [M]^{<\infty} \subseteq \mathcal{G}.$$  

Now [21, Proposition 3.6] gives a finite set $G \in \mathcal{S}_{n+1} \cap [M]^{<\infty}$ and a sequence of positive numbers $(a_p)_{p \in G}$ such that $\sum a_p = 1$ and $\sum_{p \in F} a_p < (\theta_{p_0})^P$, where $P = \left\lceil \frac{n}{p_0} \right\rceil$, whenever $F \subseteq G$ and $F \in \mathcal{S}_n$. By definition, there exist a node $(x_j)_{j=n}^r \in \hat{T}$ and a subset $J$ of the integer interval $[n, r]$ such that $G$ is a spreading of $(\max \text{supp } x_j)_{j \in J}$. Denote the unique order preserving bijection from $J$ onto $G$ by $u$ and consider the vector $y = \sum_{j \in J} a_{u(j)} x_j$. Since $(x_j)_{j=n}^r$ is a normalized $\ell^1$-$K$-block sequence in $Y_n$ and $\sum_{j \in J} a_{u(j)} = 1$, $y \in Y_n$ and $\|y\| \geq 1/K$.

Let $(E_i)$ be $\mathcal{S}_n$-allowable. Let $J_1 = \{j \in J : \text{some } E_i \text{ begins at } x_j\}$ and $J_2 = J \setminus J_1$. Note that $\mathcal{S}_n \subseteq \mathcal{S}_{p_0} P = [\mathcal{S}_{p_0}]^P$. Thus for each $j$, $(\theta_{p_0})^P \sum_i \|E_i x_j\| \leq \|x_j\| = 1$. Also, since $\{u(j) : j \in J_1\} \in \mathcal{S}_n$, $\sum_{j \in J_1} a_{u(j)} < (\theta_{p_0})^P$. Hence

$$\sum_{i} \left\| E_i \sum_{j \in J_1} a_{u(j)} x_j \right\| \leq \sum_{j \in J_1} a_{u(j)} \sum_i \|E_i x_j\| \leq \sum_{j \in J_1} a_{u(j)} \left(\frac{1}{\theta_{p_0}}\right)^P < 1.$$  

(2)
On the other hand, the collection \( \{ E_i \cap \text{supp } x_j : E_i \prec x_j \} \) of pairwise disjoint sets is \( S_1 \)-allowable and thus \( S_{p_0} \)-allowable. Therefore,

\[
(3) \quad \sum_i \left\| E_i \sum_{j \in J_2} a_u(j) x_j \right\| \leq \sum_i a_u(j) \sum_i \left\| E_i x_j \right\|
= \sum_i a_u(j) \sum_{E_i \prec x_j} \left\| E_i x_j \right\|
= \sum_i a_u(j) \sum_{E_i \prec x_j} \left\| (E_i \cap \text{supp } x_j) \cdot x_j \right\|
\leq \sum_{j \in J_2} a_u(j) \frac{1}{b_{p_0}}.
\]

Combining inequalities (2) and (3) gives

\[
\sum \left\| E_i y \right\| = \sum \left\| E_i \sum_{j \in J} a_u(j) x_j \right\|
\leq \sum \left\| E_i \sum_{j \in J_1} a_u(j) x_j \right\| + \sum \left\| E_i \sum_{j \in J_2} a_u(j) x_j \right\|
\leq 1 + \frac{1}{b_{p_0}}.
\]

It is clear that the normalized element \( x = y/\|y\| \) satisfies the statement of the lemma with the constant \( C = \left( 1 + \frac{1}{b_{p_0}} \right) K \). \( \square \)

We record the quantitative statement of Lemma 4 for future reference.

**Lemma 5.** Let \( T \) be an \( \ell^1 \)-\( K \)-block tree on a block subspace \( Y \) of \( X \) of order \( \alpha(T) \geq \omega^\omega \). Then for all \( n \in \mathbb{N} \), there is a normalized vector \( x \) in the span of a node of \( T \) such that \( \sum \left\| E_i x \right\| \leq K(1 + \theta_{p_0}^{-1}) \) whenever \( (E_i) \) is \( S_n \)-allowable.

For each \( n \in \mathbb{N} \), define

\[ \xi_n = \sup \{ \theta_{m_1} \cdots \theta_{m_j} : m_1 + \cdots + m_j > n \} \].

Then \( (\xi_n) \) is a null sequence. Assume that \( Y \) has property (*). Taking \( n_0 = 0 \), choose a normalized, finitely supported vector \( x_1 \) so that \( \sum \left\| E_s x_1 \right\| \leq C \) whenever \( (E_s) \) is \( S_{n_0} \)-allowable. Since \( (\xi_n) \) is a null sequence, there exists \( n_1 > n_0 \) so that \( \xi_{n_1} \left\| x_1 \right\|_{\ell_1} \leq 1/2 \). Let \( q_1 = \max \text{supp } x_1 \). We can choose a normalized vector \( x_2 \in \text{span} \{ (e_k)_{k=2q_1} \} \) so that \( \sum \left\| E_s x_1 \right\| \leq C \) whenever \( (E_s) \) is \( S_{n_1} \)-allowable. Continuing inductively, we obtain \( (x_k) \prec Y \) and a strictly increasing sequence \( (n_k) \), \( n_0 = 1 \), so that for each \( k \),

\[ (a) \quad \sum \left\| E_s x_k \right\| \leq C \] whenever \( (E_s) \) is \( S_{n_k-1} \)-allowable,
\[ (b) \quad \xi_{n_k} \left\| x_k \right\|_{\ell_1} \leq \frac{1}{2^k}, \]
\[ (c) \quad 2q_k \leq p_{k+1} \text{ for all } k, \text{ where } p_k = \min \text{supp } x_k \text{ and } q_k = \max \text{supp } x_k. \]

Let \( (b_k)^N_{k=1} \in c_0^+ \) and set \( x = \sum_{k=1}^N b_k x_k. \)
Lemma 6. Let $T$ be an adapted tree. If $E$ is a collection of pairwise disjoint nodes of $T$ such that $\text{ord}(E) \leq m$ for all $E \in E$, then $E$ is $S_m$-allowable.

Proof. Note that if $T$ is an adapted tree, then it is an allowable tree with nodes of the same orders. The conclusion follows from [19, Lemma 3]. □

Lemma 7. Given any adapted tree $T$, there exists an adapted tree $T'$ such that

(a) if $E \in T'$ and $E \cap \text{supp} x_k \neq \emptyset$, then $\text{ord}(E) \leq n_k$,

(b) $T x \leq T' x + \sum \frac{b_k}{2^k}$.

Proof. Given an adapted tree $T$ and $F \subseteq \mathbb{N}$, define

$T_F = \{ E \cap F : E \in T, E \cap F \neq \emptyset \}$.

Clearly $T_F$ is an adapted tree. For all $k = 2, \ldots, N$, define a set $F_k$ by

$F_k = \bigcup \{ E \cap \text{supp} x_k : E \in T, \text{ord}(E) > n_k \}$.

Then

$T x_k = \sum_{E \in L(T)} t(E) \| E x_k \|
= \sum_{E \in L(T), \text{ord}(E) \leq n_k} t(E) \| E x_k \|
+ \sum_{E \in L(T), \text{ord}(E) > n_k} t(E) \| E x_k \|
\leq T_{F_k} x_k + \xi_{n_k} \| x_k \|_1 \leq T_{F_k} x_k + \frac{1}{2^k}$.

Let $T' = T_{(F_2 \cap F_3 \cap \ldots \cap F_N)}$. Note that $T'$ satisfies (a) and $T' x_k = T_{F_k} x_k$ if $2 \leq k \leq N$. Hence

$T x = \sum b_k T x_k
\leq b_1 T x_1 + \sum_{k=2}^{N} b_k \left( T_{F_k} x_k + \frac{1}{2^k} \right)
= \sum b_k T' x_k + \sum \frac{b_k}{2^k}$

= $T' x + \sum \frac{b_k}{2^k}$.

Define $E_k$ to be the set

$\{ E \in T' : E \text{ begins at } x_k \text{ and has a sibling that begins before } x_k \}$.

Lemma 8. $\sum_{E \in E_k} \| E x_k \| \leq C$ for all $k = 2, \ldots, N$.

Proof. Note that if $E \in E_k$, $E$ has a sibling $E'$ that begins before $x_k$. Hence $\text{ord}(E) = \text{ord}(E') \leq n_{k-1}$ by property (a) of Lemma 7. By Lemma 6, $E_k$ is $S_{n_{k-1}}$-allowable. The conclusion follows from condition (α). □
Proof of Theorem 1. As \((e_k)\) is a 1-unconditional basis of \(X\), it is enough to consider nonnegative coefficients. As above, consider \((b_k)_{k=1}^N \in c_0^+\) and set \(x = \sum_{k=1}^N b_k x_k, y = \sum_{k=1}^N b_k e_{p_k}\). It is easy to see that \(||y|| \leq ||x||\). We will show that \(||x|| \leq (2 + C)||y||\), where \(C\) is the constant in condition \((\alpha)\). Given an adapted tree \(T\), we obtain an adapted tree \(T'\) as in Lemma 7. We may further assume that every node \(E \in T'\backslash L(T')\) is the union of its immediate successors, that \(E \subseteq \cup_k \text{supp} x_k\) for every \(E \in T'\) and that, relabeling if necessary, the root of \(T'\) begins at \(x_1\). With these assumptions, every node \(E \in L(T')\) that intersects \(\text{supp} x_k, \ k \geq 2\), is a descendant of some node in \(E_k\). For each \(k \geq 2\), choose \(E_k \in E_k\) such that \(t(E_k) = \max\{t(E) : E \in E_k\}\). By Lemma 8, for \(k \geq 2\),

\[
T'x_k = \sum_{E \in E_k} t(E) ||Ex_k|| \leq t(E_k) \sum_{E \in E_k} ||Ex|| \leq t(E_k) C.
\]

Therefore,

\[
Tx \leq T(b_1 x_1) + \sum_{k=2}^N b_k T'x_k + \sum_{k=2}^N \frac{b_k}{2^k} \leq b_1 + C \sum_{k=2}^N b_k t(E_k) + \sum_{k=2}^N \frac{b_k}{2^k} \leq C \sum_{k=2}^N t(E_k) b_k + 2 ||(b_k)||_{c_0} \leq C \sum_{k=2}^N t(E_k) b_k + 2 ||y||.
\]

To complete the proof, it suffices to appeal to Proposition 9 below to see that \(\sum_{k=2}^N t(E_k) b_k \leq ||y||\).

\[\square\]

Remark. The proof above shows that if \((x_k)\) is a (possibly finite) normalized block sequence in \(X\) satisfying conditions \((\alpha)\), \((\beta)\) and \((\gamma)\) for some \((n_k)\), then \((x_k)\) is \((2 + C)\)-equivalent to \((e_{p_k})\).

Proposition 9. There is an \((S_n, \sigma_n)_{n=1}^\infty\)-adapted tree \(T''\) so that

\[
T''y \geq \sum_{k=1}^N b_k t(E_k).
\]

In particular, \(\sum_{k=1}^N b_k t(E_k) \leq ||y||\).

The tree \(T''\) is constructed by substituting each node \(E \in T'\) with one or two nodes, which we now proceed to describe. For each \(E \in T'\), define \(G_E = \{p_j : E_j \subseteq E\}\). If \(E \in T'\) and \(E \neq E_k\) for any \(k\), substitute \(G_E\) for \(E\). If \(E = E_k\) for some \(k\), substitute two nodes, namely \(\{p_k\}\) and \(G_E\), in place of \(E\). The resulting collection of nodes after the substitutions we denote by...
\[ T^{\prime\prime} \]. Note that since the root of \( T' \) begins at \( x_1 \), it cannot be equal to \( E_k \) for any \( k \). Thus the root of \( T' \) is substituted with a single node. If \( E \in T' \) has immediate successors \( (F_i)_{i=1}^s \) which form an \((S_n, \sigma_n)\)-adapted family, then in the process, the \( F_i's \) are substituted with sets from the collection \((G_F)_{i=1}^s \cup P\), where \( P = \{ \{p_k\} : F_i = E_k \text{ for some } i \} \), whereas \( G_E \) is one of the (perhaps the only) substitutes for \( E \). Claim 1 below shows that \( T'' \) remains a tree; Claims 2 and 3 show that \((G_F)_{i=1}^s \cup P\) is an \((S_n, \sigma_n)\)-adapted family if \((F_i)_{i=1}^s \) is \((S_n, \sigma_n)\)-adapted.

**Claim 1.** \((G_F)_{i=1}^s \cup P\) is a family of pairwise disjoint subsets of \( G_E \).

By definition, \( \{p_k\} \subseteq G_E \) for any \( \{p_k\} \in P \). Let us show that \( G_{F_i} \subseteq G_E \).

Indeed, if \( p_j \in G_{F_i} \), then \( E_j \subseteq F_i \subseteq E \). Thus \( p_j \in G_E \).

Now if \( i \neq i' \), then \( F_i \cap F_{i'} = \emptyset \). By definition, \( G_{F_i} \) is disjoint from \( G_{F_{i'}} \).

If \( F_i = E_k \) for some \( i \) and \( k \), then for any \( i' \) (including \( i \) itself), \( E_k \subseteq F_{i'} \) cannot hold. Therefore, \( \{p_k\} \) and \( G_{F_{i'}} \) are disjoint for all \( i' \). Since obviously any two sets in \( P \) are disjoint, the claim is established.

**Claim 2.** If \((F_i)_{i=1}^s \) consists of successive sets, then so does \((G_F)_{i=1}^s \cup P\).

First we show that if \( F_i < F_{i'} \), then \( G_{F_i} < G_{F_{i'}} \). Let \( p_j \in G_{F_i} \) and \( p_{j'} \in G_{F_{i'}} \). Then \( E_j \not\subseteq F_i \) and \( E_{j'} \not\subseteq F_{i'} \). Since \( E_j \) begins at \( x_j \) and \( E_{j'} \) begins at \( x_{j'} \), it follows that \( j < j' \) and hence \( p_j < p_{j'} \). This shows that \( G_{F_i} < G_{F_{i'}} \).

Next, if \( F_i < F_{i'} = E_k \) for some \( i, i' \) and \( k \), then we claim that \( G_{F_i} < \{p_k\} \). To see the first inequality, pick a point \( p_j \in G_{F_i} \). Then \( E_j \not\subseteq F_i \). In particular, \( E_j < F_{i'} = E_k \). Since \( E_j \) begins at \( x_j \) and \( E_k \) begins at \( x_k \), we deduce that \( j < k \) and thus \( p_j < p_k \). Hence \( G_{F_i} < \{p_k\} \). Similarly, if \( p_j \in G_{F_{i'}} \), then \( E_j \not\subseteq F_{i'} = E_k \). Since \( E_j \) begins at \( x_j \) and \( E_k \) begins at \( x_k \), we deduce that \( k < j \). This shows that \( \{p_k\} < G_{F_{i'}} \).

Let \( \hat{P} = \{p_k : \{p_k\} \in P\} \).

**Claim 3.** \((\min G_{F_i})_{i=1}^s \cup \hat{P} \in S_n\)

The proof of this claim requires several short lemmas.

**Lemma 10.** For any \( E \in T' \), \( \min G_E \geq 2 \min E \).

**Proof.** Suppose that \( p_j \in G_E \). Then \( E_j \not\subseteq E \). Since \( E_j \) has a sibling that begins before \( x_j \), \( E \) begins before \( x_j \). This implies that

\[
2 \min E \leq 2q_{j-1} \leq p_j \quad \text{by (g)}.
\]

**Lemma 11.** \( \hat{P} \) is a spreading of a subset of \((\min F_i)_{i=1}^s \) and \( p_k \geq 2 \min F_i \) for all \( p_k \in \hat{P} \).

**Proof.** We may assume that \( \min F_1 < \cdots < \min F_s \). For each \( k \), let \( H_k = \{\min F_i : \min F_i \in \supp x_k\} \). List the \( k\)'s such that \( H_k \neq \emptyset \) in increasing
order as \( k_1 < \cdots < k_r \). Since every \( F_i \) begins at or after \( x_{k_i} \), \( E_{k_1} \neq F_i \) for any \( i \). Therefore, \( \hat{P} \subseteq (p_{k_i})_{i=2}^{r} \). For each \( 2 \leq \ell \leq r \), choose \( i_{\ell-1} \) such that \( \min F_{i_{\ell-1}} \subseteq H_{k_{\ell-1}} \). Then \( (p_{k_i})_{i=2}^{r} \) is a spreading of \( (\min F_{i_{\ell-1}})_{\ell=2}^{r} \). Also note that \( |p_k| \leq p_{k_2} \geq 2q_{k_1} \geq 2 \min F_1 \) for all \( p_k \in \hat{P} \).

It follows from Lemmas 10 and 11 that \( (\min G_{F_i})_{i=1}^{a} \cup \hat{P} \) can be written as \( \cup_{j \in B} A_j \), where \( B = \{2 \min F_1 \} \cup (\min F_i)_{i=2}^{a} \), \( \min A_j \geq j \) and \( |A_j| \leq 2 \) for all \( j \in B \).

**Lemma 12.** Suppose that \( n \in \mathbb{N} \), \( L \in \mathcal{S}_n \) and \( B \) is a spreading of \( L \) such that \( \min B \geq 2 \min L \). If \( |A_j| \leq 2 \) and \( \min A_j \geq j \) for all \( j \in B \), then \( \cup_{j \in B} A_j \in \mathcal{S}_n \).

**Proof.** It is easy to see that we may assume \( A_j < A_{j'} \) if \( j < j' \). Write \( L = \cup_{k=1}^{p} L_k \), where \( L_1 < \cdots < L_p \) are in \( \mathcal{S}_{n-1} \) and \( p \leq \min L_1 \). Then \( B = \cup_{k=1}^{p} B_k \), where each \( B_k \) is a spreading of \( L_k \) and \( B_1 < \cdots < B_p \). Denoting by \( \mathcal{A}_2 \) the collection of subsets of \( \mathbb{N} \) having at most two elements, we appeal to [17, Remark on p.312] to deduce that

\[
\cup_{j \in B} A_j \in \mathcal{S}_{n-1} [\mathcal{A}_2] \subseteq (\mathcal{S}_{n-1})^2 .
\]

Hence \( \cup_{j \in B} A_j = \cup_{k=1}^{2p} C_k \), where \( C_1 < \cdots < C_{2p} \) are in \( \mathcal{S}_{n-1} \). Since \( 2p \leq 2 \min L_1 \leq \min B \leq \min C_1 \), the conclusion of the lemma follows. \( \square \)

**Completion of proof of Proposition 9.** It follows from the claims and lemmas above that the nodes of \( T'' \) form an \( (\mathcal{S}_n, \sigma_m)_{m=1}^{\infty} \)-adapted tree, where the tag of any node in \( T'' \) is the same as the tag of the node in \( T' \) for which it is a substitute. Moreover, it follows from Claim 1 that all nodes in \( P \) are terminal. Therefore,

\[
T''y \geq \sum_{(p_k) \in P} t(\{p_k\})b_k = \sum_{k=2}^{N} t(E_k)b_k .
\]

\( \square \)

Recall that a Banach space \( Z \) is said to be *minimal* if every infinite dimensional subspace of \( Z \) has a further subspace isomorphic to \( Z \). This definition is due to Rosenthal. In [15], Gowers introduced the more general notion of *quasi-minimal* spaces. Two Banach spaces are said to be *totally incomparable* if they do not have isomorphic infinite dimensional subspaces.

A Banach space is said to be quasi-minimal if it does not contain a pair of totally incomparable infinite dimensional closed subspaces. Using Theorem 1, Corollary 2 and Proposition 14 below, we obtain

**Corollary 13.** Let \( X = T[(\mathcal{S}_n, \sigma_n, \theta_n)]_{n=1}^{\infty} \) be a partly modified mixed Tsirelson space so that \( I(Y) > \omega^\omega \) for every block subspace \( Y \) of \( X \). Then \( X \) is quasi-minimal. This holds if there exists \( \varepsilon > 0 \) such that \( \sup\{n/m : \theta_n \geq \varepsilon^m \} = \infty \), and, in particular, if \( \sup \theta_{n}^{1/n} = 1 \).
Proposition 14. Let \((p_k)\) and \((q_k)\) be subsequences of \(\mathbb{N}\) so that \(p_k \leq q_k < 2q_k \leq p_{k+1}\) for all \(k\). Then the sequences \((e_{p_k})\) and \((e_{q_k})\) are 2-equivalent in any partly modified mixed Tsirelson space \(X = T[(S_n, \sigma_n, \theta_n)_{n=1}^{\infty}]\).

**Proof.** Define a sequence of norms on \(X\) follows. Let \(\|x\|_0 = \|x\|_{\infty}\) and
\[
\|x\|_{i+1} = \max\{\|x\|_0, \sup_n \sum_m \theta_n \|E_m x\|_i\},
\]
where the final supremum is taken over all \((S_n, \sigma_n)\)-adapted families \((E_m)\).

It is clear that \(\|x\| = \lim \|x\|_i\) for all \(x \in X\). For any finite subset \(E\) of \((q_k)\), let the shift of \(E\) be the set \(s(E) = \{p_k : q_k \in E\}\). We claim that for any \(i\), any \((a_k) \in c_{00}\) and any \(E \subseteq (q_k)\), there exist \(p_j \in s(E)\) and \(F \subseteq s(E)\) such that \(p_j < F\) and
\[
\|E \sum a_k e_{q_k}\|_{i+1} \leq \|a_j\| + \|F \sum a_k e_{p_k}\|_{i}.
\]
Once the claim is proved, it follows easily that \(\|\sum a_k e_{q_k}\| \leq 2\|\sum a_k e_{p_k}\|\).

Since each \(S_n\) is spreading, we clearly have \(\|\sum a_k e_{p_k}\| \leq \|\sum a_k e_{q_k}\|\), and the proof of proposition would be complete. We now prove the claim (4) by induction on \(i\). The case \(i = 0\) is trivial. Suppose that the claim holds for some \(i\). We may assume that
\[
\|E \sum a_k e_{q_k}\|_{i+1} = \theta_n \sum_{m=1}^d \|E_m \sum a_k e_{q_k}\|_i,
\]
where \((E_m)_{m=1}^d\) is an \((S_n, \sigma_n)\)-adapted family of subsets of \(E\), arranged so that \((\min E_m)_{m=1}^d\) is an increasing sequence. By induction, for each \(m\), there are \(p_{jm} \in s(E_m)\) and \(F_m \subseteq s(E_m)\) such that \(p_{jm} < F_m\) and
\[
\|E_m \sum a_k e_{q_k}\|_{i} \leq \|a_{jm}\| + \|F_m \sum a_k e_{p_k}\|_{i}.
\]
Observe that for every \(m\), \(2\min E_m \leq 2q_{jm} < p_{jm+1} \leq \min F_m\). Also, for \(m \geq 2\), \(2\min E_{m-1} \leq \min s(E_m) \leq p_{jm}\). Let \(m_0\) be such that \(p_{jm_0}\) is the minimum of the sequence \((p_{jm})_{m=1}^d\). Then \((p_{jm})_{m \neq m_0} \cup (\min F_m)_{m=1}^d\) may be written as \(\cup_{j \in B_0} A_j\), where \(B\) is a spreading of \((\min E_m)_{m=1}^d\) such that \(\min B \geq 2\min E_1\), \(|A_j| \leq 2\) and \(A_j \geq j\) for all \(j \in B\). By Lemma 12, \((p_{jm})_{m \neq m_0} \cup (\min F_m)_{m=1}^d\) is \((S_n, \sigma_n)\)-adapted. We may then conclude that
\[
\|E \sum a_k e_{q_k}\|_{i+1} = \theta_n \sum_{m=1}^d \|E_m \sum a_k e_{q_k}\|_i \leq \theta_n \|a_{jm_0}\| + \theta_n (\sum_{m \neq m_0} \|a_{jm}\| + \sum_{m=1}^d \|F_m \sum a_k e_{p_k}\|_i)
\]
\[
\leq \|a_{jm_0}\| + \|F \sum a_k e_{p_k}\|_{i+1},
\]
where \(F = \{p_{jm} : m \neq m_0\} \cup \cup_{m=1}^d F_m \subseteq s(E)\) and \(F > p_{jm_0} \in s(E)\). \(\square\)
If $X = T[(S_n, \sigma_n, \theta_n)_{n=1}^\infty]$ is a partly modified mixed Tsirelson space where $\sigma_{p_0} = M$, then it is clear that every disjointly supported sequence $(x_k)_{k=1}^n$ in $[(e_k)_{k=n}^\infty]$ is $\theta_{p_0}^{-1}$-equivalent to the unit vector basis of $\ell^1(n)$. Such spaces are called strongly asymptotic $\ell^1$ spaces. In [10], it was proved that every minimal, strongly asymptotic Banach space with a basis is isomorphic to a subspace of $\ell^1$. Since partly modified spaces are reflexive (this may be proved using the arguments of [3]; alternatively, it follows from the computation of the $\ell^1$-index below (Theorem 17)), we get that no partly modified mixed Tsirelson space contains a minimal subspace. Hence the class of partly modified mixed Tsirelson spaces $X$ such that $I(Y) > \omega^\omega$ for every subspace $Y$ of $X$ provides examples of strictly quasi-minimal Banach spaces in the sense of Gowers [15]. By [9], Tsirelson’s space $T$ does not contain a minimal subspace. Also $T$ has the block property [8]: every normalized block basis is equivalent to a subsequence of the unit vector basis. It follows easily that $T$ is strictly quasi-minimal. However, among the strictly quasi-minimal partly modified mixed Tsirelson spaces are spaces that fail the block property. Indeed, it can be deduced from the arguments in §2 of [3] that the space $X_{M(1),u}$ constructed there is one such example. We may also obtained further examples using the arguments in the present paper.

Proposition 15. Let $(\theta_n)$ be a regular sequence so that $\sup \theta_{1/n} = 1$. Then $X = T[(S_n, \sigma_n, \theta_n)_{n=1}^\infty]$ is saturated with block subspaces that fail the block property.

Proof. If $\lim \theta_{1/n} = 1$, then the hypothesis of Theorem 23 is fulfilled. In fact, assume that there exists $m$ such that

$$\limsup_n \inf_{n_1 + \cdots + n_s \geq n} \frac{\theta_{n+n}}{\theta_{n_1} \cdots \theta_{n_s}} < \frac{1}{2}. $$

Since $(\theta_n)$ is regular, $\theta_{n_1} \cdots \theta_{n_s} \leq \theta_n$ if $n_1 + \cdots + n_s \geq n$. So we have

$$\limsup_n \frac{\theta_{m+n}}{\theta_n} < \frac{1}{2}. $$

Pick $n_0$ so that $\frac{\theta_{m+n}}{\theta_n} < \frac{1}{2}$ for $n \geq n_0$. For all $k \in \mathbb{N}$,

$$\theta_{km+n_0} \leq \frac{1}{2} \theta_{(k-1)m+n_0} \leq \cdots \leq \left(\frac{1}{2}\right)^k \theta_{m+n_0}. $$

Thus

$$\limsup_k \frac{\theta_{km+n_0}}{\theta_{km+n+0}} \leq \frac{1}{2^m}, $$

a contradiction. By Theorem 23, $X = T[(S_n, \sigma_n, \theta_n)_{n=1}^\infty]$ contains $\ell^1$-spreading models with uniform constant, say $K$. 

Claim. For any \( \varepsilon > 0 \), there exists \( n_0 \) so that
\[
\|x\| \leq \varepsilon \|x\|_{\ell^1} + \|x\|_{S_{n_0}}
\]
for all \( x \in X \), where \( \|x\|_{S_{n_0}} = \sup_{E \in S_{n_0}} \|Ex\|_{\ell^1} \).

Let \( T \) be an adapted tree so that \( \|x\| = \sum t(E)\|Ex\|_{c_0} \), where the sum is taken over all leaves of \( T \). It is clear that there exists \( n_0 \) so that any leaf \( E \in T \) with \( t(E) > \varepsilon \) satisfies \( \text{ord}(E) \leq n_0 \). By Lemma 6, the set of all leaves \( E \) with \( t(E) > \varepsilon \) is \( S_{n_0} \)-allowable. Thus
\[
\sum' t(E)\|Ex\|_{c_0} \leq \|x\|_{S_{n_0}},
\]
where \( \sum' \) is taken over all leaves in \( T \) with \( t(E) > \varepsilon \). Let \( \sum'' \) be the sum over the remaining leaves. Then
\[
\|x\| \leq \sum'' t(E)\|Ex\|_{c_0} + \sum' t(E)\|Ex\|_{c_0} \leq \varepsilon \|x\|_{\ell^1} + \|x\|_{S_{n_0}},
\]
as desired.

Next, we show that there is a normalized block basis \((y_n)\) in \( X \) and a subsequence \((e_{p_n})\) of the unit vector basis of \( X \) with \( y_n < e_{p_n} < y_{n+1} \) for all \( n \) so that \((y_n)\) is not equivalent to \((e_{p_n})\). Indeed, if this were not true, there would be a uniform constant \( M \) so that any two such sequences are \( M \)-equivalent. Let \( K \) be the constant chosen before the Claim and let \( n_0 \) be obtained from the Claim corresponding to \( \varepsilon = \frac{1}{2KM} \). Take any \( m > n_0 \). There is a normalized block basis \((y_n)\) in \( X \) that is an \( \ell^1 \)-\( S_m \)-spreading model with constant \( K \) and a subsequence \((e_{p_n})\) of the unit vector basis of \( X \) so that \( y_n < e_{p_n} < y_{n+1} \) for all \( n \). For any finite sequence \((a_n)\) supported on a set in \( S_m \),
\[
\sum |a_n| \leq K\sum a_n y_n \\
\leq KM\sum a_n e_{p_n} \\
\leq \frac{1}{2} \sum |a_n| + KM\sum a_n e_{p_n} \|S_{n_0}\|.
\]
Hence \( \|a_n\|_{S_m} \leq 2KM\sum a_n e_{p_n} \|S_{n_0}\| \) for any sequence \((a_n)\). It is easy to see that this does not hold since \( m > n_0 \).

If \( X \) has the block property, then \((y_n)\) chosen above is equivalent to some subsequence \((e_{p_n})\) of the unit vector basis. Because of Proposition 14, we obtain a subsequence \((y_{n_k})\) and \( r_1 < r_2 < \cdots \) so that \( y_{n_k} < e_{r_k} < y_{n_{k+1}} \). Since \((y_{n_k})\) is still an \( \ell^1 \)-\( S_m \)-spreading model with constant \( K \), the same contradiction ensues.

The argument above may be carried out in any subspace \([e_{q_n}]\) of \( X \), with \((e_{q_n})\) a subsequence of the unit vector basis. Since \( X \) is subsequentially minimal by Corollary 2, it follows that \( X \) is saturated with block subspaces that fail the block property and also saturated with normalized block basic sequences \((z_n)\) so that \((z_{2n-1})\) is not equivalent to \((z_{2n})\). \( \square \)
3. The Bourgain $\ell^1$-index

In this section, we develop the techniques in §2 further to investigate the Bourgain $\ell^1$-index of partly modified Tsirelson spaces. In the first part of the section, we show that $I(X)$ does not exceed $\omega^\omega-2$. In the second part, we pinpoint the value of $I(X)$ in certain cases in terms of the sequence of coefficients $(\theta_n)$.

In the following proposition, we will require the concepts of block subtrees, minimal trees $T_\alpha$ and replacement trees $T'(\alpha, \beta)$ defined, constructed and developed in [16]. We refer the reader to that paper for details. The execution of the following proof is comparable to that of [16, Lemma 4.2]. When two trees $T$ and $T'$ are isomorphic, we write $T \simeq T'$. Given two finite sequences $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_n)$, let $\vec{x} \sqcup \vec{y} = (x_1, \ldots, x_m, y_1, \ldots, y_n)$. We say that a normalized vector $x$ satisfies property $(\ast)$ for the couple $(n, C) \in \mathbb{N} \times \mathbb{R}^+$ if $\sum |E_i x| \leq C$ whenever $(E_i)$ is $S_n$-allowable.

**Proposition 16.** If $T$ is an $\ell^1$-$K$-block tree of order $o(T) \geq \omega^\omega \cdot \alpha$, then for any $n_0 \in \mathbb{N}$ and any positive sequence $(\varepsilon_i)$, there exists a block subtree $T'$ of $T$, isomorphic to $T_\alpha$, such that every node $(x_1, \ldots, x_d) \in T'$ satisfies

1. There exist $n_1 < \cdots < n_{d-1}$, with $n_1 > n_0$, such that each $x_i$ satisfies property $(\ast)$ for the couple $(n_{i-1}, C)$, where $C = (1 + \theta_{n_0}^{-1}) K$,
2. $\xi_{n_i} \|x_i\|_\ell^1 \leq \varepsilon_i$ for $1 \leq i < d$, and
3. $2 \max \supp x_i \leq \min \supp x_{i+1}$ if $1 \leq i < d$.

**Proof.** The proof is by induction on $\alpha$. The case $\alpha = 1$ follows from Lemma 5. Suppose that $T$ is an $\ell^1$-$K$-block tree of order $o(T) \geq \omega^\omega \cdot (\alpha + 1)$. According to [16, Lemma 3.7] and replacing $T$ by a subtree if necessary, we may assume that $T$ is isomorphic to the “replacement tree” $T'(\alpha + 1, \omega^\omega)$.

From the definition of $T'(\alpha + 1, \omega^\omega)$, we see that $(T'(\alpha + 1, \omega^\omega))^{(\omega^\omega \cdot \alpha)}$ is the minimal tree $T_{\omega^\omega}$. Applying the case $\alpha = 1$ to $T^{(\omega^\omega \cdot \alpha)} \simeq T_{\omega^\omega}$, we obtain a normalized block $y$ of a node $\vec{x} = (x_1, \ldots, x_m)$ in $T^{(\omega^\omega \cdot \alpha)}$ such that $y$ satisfies $(\ast)$ for the couple $(n_0, C)$. Choose $n_1 > n_0$ such that $\xi_{n_1} \|y\|_\ell^1 \leq \varepsilon_1$.

Without loss of generality, we may assume that $\vec{x}$ is a terminal node in $T^{(\omega^\omega \cdot \alpha)}$. By the construction of $T'(\alpha + 1, \omega^\omega)$, the subtree $T_{\vec{x}}$ of $T'$ consisting of all nodes $\vec{z} > \vec{x}$ is isomorphic to $T'(\omega^\omega \cdot \alpha)$ and hence has order $\omega^\omega \cdot \alpha$. Consider the “restricted subtree” $R(T_{\vec{x}})$ [16, Definition 4.1] consisting of all $(w_j, \ldots, w_k)$, where $\vec{x} \sqcup (w_1, \ldots, w_k) \in T_{\vec{z}}$ and $j$ is the smallest integer such that $\min \supp w_j \geq 2 \max \supp x_m$. Then $R(T_{\vec{x}})$ is an $\ell^1$-$K$-block tree of order $\omega^\omega \cdot \alpha$. Apply the inductive hypothesis to $R(T_{\vec{x}})$ with the parameters $n_1$ and $(\varepsilon_i)_{i+1}$ to obtain a block subtree $T'' \in R(T_{\vec{x}})$. Define $T' = \{(y) \sqcup \vec{w} : \vec{w} \in T''\}$. It is easy to check that $T'$ satisfies the desired conclusion (for the ordinal $\alpha + 1$).

Suppose that $T$ is an $\ell^1$-$K$-block tree of order $o(T) \geq \omega^\omega \cdot \alpha$, where $\alpha$ is a limit ordinal. Let $(\alpha_n)$ be a sequence of ordinals strictly increasing to $\alpha$. Then $T$ contains pairwise disjoint subtrees $T_n$ with $o(T_n) \geq \omega^\omega \cdot \alpha_n$ for all $n$. For each $n$, apply the inductive hypothesis to obtain a block subtree
Theorem 1, we see that every node \((x, \cdots, x_d) \in T'\) is \((2 + C)\)-equivalent to \((e_{p_i})\), where \(p_i = \min \text{ supp } x_i\). For \(y \in c_{00}\), let \(\|y\|_{S_p} = \sup_{E \in S_p} \|Ey\|_{\ell^1}\).

**Theorem 17.** The Bourgain \(\ell^1\)-index of \(X = T[(S_n, \sigma_n, \theta_n)]\) is \(I(X) \leq \omega^{\omega^2}\).

**Proof.** If \(I(X) > \omega^{\omega^2}\), then by [16, Corollary 5.13], there exists an \(\ell^1\)-block tree \(T\) with \(o(T) \geq \omega^{\omega^2}\) for some \(K > 0\). Let \(n\) be chosen so that \(\xi_n < \frac{1}{2K(2 + C)}\). By Proposition 16, we obtain an \(\ell^1\)-block tree \(T'\) of \(T\) with \(o(T') = \omega^{n+1}\) such that every node \((x_1, \cdots, x_d) \in T'\) is \((2 + C)\)-equivalent to \((e_{p_i})\). Define 
\[\mathcal{H} = \{(p_j)_{j=n}^r : (x_j)_{j=n}^r \in T'\}\]
and 
\[\mathcal{G} = \{G : G \text{ is a spreading of a subset of some } H \in \mathcal{H}\} \]
By Lemma 3, \(\mathcal{G}\) is hereditary and spreading, and either \(\mathcal{G}\) is noncompact or it is compact with \(\iota(\mathcal{G}) \geq o(T') \geq \omega^{n+1} > \omega^n\). By [14, Theorem 1.1], there exists \(M \in [N]\) such that \(S_n \cap [M]^\infty \subseteq \mathcal{G}\). As in the proof of Lemma 4, we obtain a node \((x_j)_{j=n}^r \in T'\), \(J \subseteq [n, r]\), an order preserving map \(u\) from \(J\) onto a spreading of \((p_j)_{j \in J}\) and a sequence of positive numbers \((a_{u(j)})_{j \in J}\) such that \(\sum_{j \in J} a_{u(j)} = 1\) and \(\sum_{j \in A} a_{u(j)} < \xi_n\) whenever \(\{u(j) : j \in A\} \in S_{n-1}\). Let \(y = \sum_{j \in J} a_{u(j)} x_j\). Since \((x_j)_{j}^r\) is a normalized \(\ell^1\)-block sequence, \(\|y\| \geq 1/K\). On the other hand,
\[\|y\| = \|\sum_{j \in J} a_{u(j)} x_j\| \leq (2 + C) \|\sum_{j \in J} a_{u(j)} e_{p_j}\| \]
\[\leq (2 + C) (\sum_{j \in J} a_{u(j)} e_j \|S_{n-1} + \xi_n\|a_{u(j)}\|_{\ell^1}) \leq 2(2 + C) \xi_n,\]
contradicting the choice of \(n\).

In the second half of the section, we obtain an estimate on the norms of vectors spanned by normalized block sequences in \(X\) (Proposition 21), from which the value of the Bourgain \(\ell^1\)-index \(I(X)\) may be deduced. For the remainder of the section, assume that \((x_k)\) is a normalized block sequence in \(X = T[S_n, \sigma_n, \theta_n]\), \((a_k) \in c_{00}\) and \(q_k = \max \text{ supp } x_k\). Set \(x = \sum a_k x_k\). Recall the assumption that \(\sigma_{p_0} = M\) for some \(p_0\). Given a node \(E\) in an adapted tree \(T\), we say that it is a long node (with respect to \(x\)) if \(E \cap \text{ supp } x_k \neq \emptyset\) for more than one \(k\). Otherwise, we term the node short.

**Lemma 18.** For any \(N\), there exists an adapted tree \(T\) such that all long nodes \(E \in T\) satisfy \(t(E) > \theta_N\) and 
\[\|x\| \leq Tx + \frac{\theta_N}{\theta_{p_0}} \|a_k\|_{\ell^1}.\]
Proof. Choose an adapted tree $T'$ such that $\|x\| = T'x$. Let $E$ be the collection of minimal elements in the set of long nodes $E$ with $t(E) \leq \theta_N$. For each $E \in E$, let $k_E$ be the smallest $k$ such that $\text{supp} x_k \cap E \neq \emptyset$ and let $F_E = \text{supp} x_{kE} \cap E$. For each $k$, the nonempty sets in the collection $\{(E \setminus F_E) \cap \text{supp} x_k\}$ is $S_1$-allowable and hence $S_{\rho_0}$-allowable. Thus,

$$
\sum_{E \in E} t(E) \|(E \setminus F_E) x_k\| \leq \theta_N \sum_{E \in E} \|(E \setminus F_E) x_k\| \leq \frac{\theta_N}{\theta_{\rho_0}}.
$$

Then

$$
\sum_{E \in E} t(E) \|(E \setminus F_E) x\| \leq \frac{\theta_N}{\theta_{\rho_0}} \|(a_k)\|_{\ell^1}.
$$

Let $T$ be the tree obtained from $T'$ by changing all nodes $G \in T'$, $G \subseteq E$ for some $E \in E$ to $G \cap F_E$. Then $T$ is an adapted tree such that all long nodes $H$ in $T$ satisfies $t(H) > \theta_N$. Moreover,

$$
\|x\| = T'x \leq Tx + \sum_{E \in E} t(E) \|(E \setminus F_E) x\|
$$

$$
\leq Tx + \frac{\theta_N}{\theta_{\rho_0}} \|(a_k)\|_{\ell^1}.
$$

Let $N$ and let $T$ be the tree given by Lemma 18. For any $\varepsilon > 0$, let $k(\varepsilon) = \max \{n_1 + \cdots + n_j : \theta_{n_1} \cdots \theta_{n_j} > \varepsilon\}$. Let $E$ denote the set of all minimal short nodes in $T$.

**Lemma 19.** If $E_1 = \{E \in E : E$ has a long sibling$\}$, then

$$
\sum_{E \in E_1} t(E) \|Ex\| \leq \left\| \sum a_k e_{q_k} \right\|_{S_{k(\theta_N)}}.
$$

**Proof.** If $E \in E_1$, then $t(E) > \theta_N$ and hence ord$(E) \leq k(\theta_N)$. Hence by Lemma 6, $E_1$ is $S_{k(\theta_N)}$-allowable. Since each $E \in E_1$ is a short node, it follows that the set $Q_0 = \{q_k : \text{supp} x_k \cap E \neq \emptyset$ for some $E \in E_1\} \in S_{k(\theta_N)}$. Thus

$$
\sum_{E \in E_1} t(E) \|Ex\| \leq \sum_{q_k \in Q_0} |a_k| \leq \left\| \sum a_k e_{q_k} \right\|_{S_{k(\theta_N)}}.
$$

For $m, n \in \mathbb{N}$, define $\eta_{m,n} = \inf \frac{\theta_{m+n}}{\theta_{n_1} \cdots \theta_{n_s}}$, where the infimum is taken over all $n_1, \ldots, n_s$ such that $n_1 + \cdots + n_s \geq n$, with the additional requirement that $\sigma_{n_1} = \sigma_{n_2} = \cdots = \sigma_{n_s} = M$ if $\sigma_{m+n} = M$. Let us make the obvious remark that the quantity $\eta_{m,n}$ majorizes the quantity

$$
\inf_{n_1 + \cdots + n_s \geq n} \frac{\theta_{m+n}}{\theta_{n_1} \cdots \theta_{n_s}};
$$

which occurs in Theorem 23 below.
Lemma 20. Suppose that \( \inf \limsup_{m,n} \eta_{m,n} = 0 \). For any \( \varepsilon > 0 \), there exist \( m \) and \( n_0 \) such that

\[
\sum_{E \in \mathcal{E} \setminus \mathcal{E}_1} t(E) \|Ex\| \leq \varepsilon \|(a_k)\|_1 + 2 \sum a_k e_{q_k} \left\| S_{k(\theta_N)} + n_0 + m \right\|.
\]

Proof. Choose \( m \) and \( n_0 \) such that \( \eta_{m,n} < \varepsilon \) if \( n \geq n_0 \). Let \( \mathcal{D} = \{D_i\} \) be the set of all parents of nodes in \( \mathcal{E} \setminus \mathcal{E}_1 \). In particular, each \( D_i \) is a long node and hence \( t(D_i) > \theta_N \). It follows that \( \text{ord}(D_i) \leq k(\theta_N) \). Also, the nodes in \( \mathcal{D} \) are pairwise disjoint since no \( E \in \mathcal{E} \setminus \mathcal{E}_1 \) can have a long sibling. For each \( i \), there exists some \( n_i \) such that \( \mathcal{F}_i = \{E \in \mathcal{E} \setminus \mathcal{E}_1 : E \subseteq D_i\} \) is \( (S_{n_i}, \sigma_{n_i}) \)-adapted. Let \( I = \{i : n_i \leq n_0 + m\} \). Then \( \text{ord}(E) = \text{ord}(D_i) + n_i \leq k(\theta_N) + n_0 + m \) for all \( E \in \bigcup_{i \in I} \mathcal{F}_i \). By Lemma 6, \( \bigcup_{i \in I} \mathcal{F}_i \) is an \( S_{k(\theta_N)} + n_0 + m \)-allowable collection of short nodes. It follows that

\[
Q_0 = \left\{ q_k : \text{supp} \ x_k \cap E \neq \emptyset \text{ for some } E \in \bigcup_{i \in I} \mathcal{F}_i \right\} \in S_{k(\theta_N)} + n_0 + m.
\]

Therefore,

\[
(5) \quad \sum_{E \in \bigcup_{i \in I} \mathcal{F}_i} t(E) \|Ex\| \leq \sum_{q_k \in Q_0} |a_k| \leq \sum a_k e_{q_k} \left\| S_{k(\theta_N)} + n_0 + m \right\|.
\]

Now consider those \( i \notin I \). Let \( \mathcal{F}_{ik} = \{E \in \mathcal{F}_i : E \subseteq \text{supp} \ x_k\} \). For each \( k \), let

\[
I_k = \{i \notin I : \min E : E \in \mathcal{F}_{ik} \in S_{n_i-m}\}
\]

and \( I'_k = \{i \notin I : \min E : E \in \mathcal{F}_{ik} \notin S_{n_i-m}\} \).

Suppose that \( i \in I_k \). Choose \( n_1(i), \ldots, n_s(i) \) such that \( n_1(i) + \cdots + n_s(i) \geq n_i - m \),

\[
\frac{\theta_{m+n_i-m}}{\theta_{n_1(i)} \cdots \theta_{n_s(i)}} < \varepsilon
\]

and \( \sigma_{n_1(i)} = \cdots = \sigma_{n_s(i)} = M \) if \( \sigma_{n_i} = M \). This is possible since \( i \notin I \) implies that \( n_i - m \geq n_0 \) and hence \( \eta_{m,n_i-m} < \varepsilon \).

If \( \sigma_{n_i} = U \), then the sets in \( \mathcal{F}_i \) and hence \( \mathcal{F}_{ik} \) are successive. Since \( \{\min E : E \in \mathcal{F}_{ik}\} \in S_{n_i-m} \), \( \mathcal{F}_{ik} \) is \( S_{n_i-m} \)-admissible and hence \( S_{n_1(i) + \cdots + n_s(i)} \)-admissible. Then

\[
(6) \quad \sum_{E \in \mathcal{F}_{ik}} \theta_{n_1(i)} \cdots \theta_{n_s(i)} \|Ex_k\| = \sum_{E \in \mathcal{F}_{ik}} \theta_{n_1(i)} \cdots \theta_{n_s(i)} \|ED_i x_k\| \leq \|D_i x_k\|.
\]

If \( \sigma_{n_i} = M \), then \( \mathcal{F}_{ik} \) is \( S_{n_i-m} \)-allowable and hence \( S_{n_1(i) + \cdots + n_s(i)} \)-allowable. Since \( \sigma_{n_1(i)} = \cdots = \sigma_{n_s(i)} = M \), we obtain the same inequality as (6).
From inequality (6),
\[
\sum_{i \in I_k} \sum_{E \in F_{ik}} t(E) \|Ex_k\| = \sum_{i \in I_k} t(D_i) \theta_{n_i} \sum_{E \in F_{ik}} \|Ex_k\|
\leq \varepsilon \sum_{i \in I_k} t(D_i) \sum_{E \in F_{ik}} \|Ex_k\|
\leq \varepsilon \sum_{i \in I_k} t(D_i) \|D_ix_k\| \leq \varepsilon.
\]

Therefore,
\[
\sum_{\{(i,k): i \in I_k\}} \sum_{E \in F_{ik}} t(E) \|Ex\| \leq \varepsilon \|(a_k)\|_{\ell^1}.
\]

For each \(i \notin I\), set \(J_i = \{k: i \in I'_k\}\). Then \(\{\min E : E \in F_{ik}\} \notin S_{n_0-m}\) for each \(k \in J_i\) but \(\cup_k \{\min E : E \in F_{ik}\} = \{\min E : E \in F_i\} \in S_{n_0}\). By [17, Lemma 2], \((\min \cup_{E \in F_{ik}} E)_{k \in J_i} \in S_m\). Now \(ord(D_i) \leq k(\theta_N)\) for all \(i\) and \(D\) consist of pairwise disjoint sets. Thus by Lemma 6, \(D\) is \(S_{k(\theta_N)}\)-allowable. Therefore, \(\{q_k : k \in \cup_{i \notin I} J_i\} \in S_{k(\theta_N)+m}\). It follows that
\[
\sum_{\{(i,k): i \in I_k\}} \sum_{E \in F_{ik}} t(E) \|Ex\| \leq \sum_{k \in \cup_{i \notin I} J_i} |a_k| 
\leq \left\| \sum_{k \in \cup_{i \notin I} J_i} a_k e_{q_k} \right\|_{S_{k(\theta_N)+m}}.
\]

Combining (5), (7) and (8) yields
\[
\sum_{E \in E \setminus F_1} t(E) \|Ex\| = \varepsilon \|(a_k)\|_{\ell^1} + 2 \left\| \sum_{k \in \cup_{i \notin I} J_i} a_k e_{q_k} \right\|_{S_{k(\theta_N)+n_0+m}}.
\]

From Lemmas 18, 19, and 20 we have

**Proposition 21.** Suppose that \(\inf_n \limsup (\eta_{m,n} = 0).\) Then given any \(\varepsilon > 0\) and \(N\), there exist \(m\) and \(n_0\) such that
\[
\|x\| \leq \left( \varepsilon + \frac{\theta_N}{\theta_{p_0}} \right) \|(a_k)\|_{\ell^1} + 3 \left\| \sum_{k \in \cup_{i \notin I} J_i} a_k e_{q_k} \right\|_{S_{k(\theta_N)+n_0+m}}.
\]

**Theorem 22.** If \(\inf_n \limsup (\eta_{m,n} = 0),\) then \(I(Y) = \omega^\omega\) for all \(Y < X\).

**Proof.** Since \(Y\) contains \(\ell^1\)-\(S_n\)-spreading models with constant \(\theta_n^{-1}\) for all \(n\) and all \(Y < X\), it is clear that \(I(Y) \geq \omega^\omega\). Suppose that \(I(Y) > \omega^\omega\) for some \(Y < X\). Then \(I(X) > \omega^\omega\). There exist \(K > 1\) and an \(\ell^1\)-\(K\)-block tree \(T\) such that \(o(T) > \omega^\omega\). Let \(\mathcal{H}(T) = \{\text{max supp } x_j : (x_j)_{j=1}^r \in T\}\) and \(\mathcal{G} = \{G : G\ is \ a \ spreading \ of \ a \ subset \ of \ some \ H \in \mathcal{H}\}\). Then \(i(\mathcal{G}) \geq o(T) > \omega^\omega\). Choose \(\varepsilon\) and \(N\) such that \(\varepsilon + \frac{\theta_N}{\theta_p} < \frac{1}{2K}\) and let \(r = k(\theta_N) + n_0 + m\) where \(n_0, m\) are such that \(\eta_{m,n} < \varepsilon\) if \(n \geq n_0\). Applying [14, Theorem 1.1],
there exists $M \in \mathbb{N}$ such that $\mathcal{S}_n \cap [M]^{< \infty} \subseteq \mathcal{G}$. Hence, it follows from [21, Proposition 3.6] that there exist $G = (t_i) \in \mathcal{G}$ and $(a_i) \in c_{00}$ such that $\sum a_i = 1$ and $\|\sum a_i e_i\|_{\mathcal{S}_r} < \frac{1}{\sqrt{K}}$.

By definition, there exists a normalized $\ell^1$-$K$-block sequence $(x_i)_1^k$ in $X$ such that $(t_i)$ is a spreading of $(q_i) = (\max \text{ supp } x_i)$. By Proposition 21,

$$
\frac{1}{K} \leq \frac{1}{2K} \frac{1}{2K} \|a_i\|_{\ell^1} + 3 \left\| \sum a_i e_i \right\|_{\mathcal{S}_r} < \frac{1}{K},
$$
a contradiction. □

**Theorem 23.** If $\inf \limsup_{m} \inf_{n, n_1+\cdots+n_s \geq n} \frac{\theta_{m+n}}{\theta_{n_1} \cdots \theta_{n_s}} > 0$, then $X$ contains $\ell^1$-$\mathcal{S}_m$-spreading models with uniform constant. In particular, $I(X) = \omega^{\omega^2}$.

If $X$ satisfies the hypothesis of Theorem 23 and is subsequentially minimal, as happens for instance if $(\theta_n)$ is regular and $\sup \theta_{1/n} = 1$, then $I(Y) = \omega^{\omega^2}$ for all infinite dimensional closed subspaces $Y$ of $X$. This follows from the fact that the following construction can be carried out on any subsequence of the unit vector basis.

**Lemma 24.** For any $n \in \mathbb{N}$, $\varepsilon > 0$ and $L \in [\mathbb{N}]$, there exists $x \in c_{00}$ such that

$$
\|x\|_{\ell^1} = \frac{1}{\theta_n}, \quad \text{supp } x \in \mathcal{S}_{N+1} \cap [L]^{< \infty} \quad \text{and} \quad \|x\|_{X} \leq 1 + \frac{1}{\varepsilon},
$$

where $N = \max \{n_1 + \cdots + n_s : \varepsilon \theta_{n_1} \cdots \theta_{n_s} > \theta_n\}$. (We take $\max 0 = 0$.)

**Proof.** According to [21, Proposition 3.6], there exists $x \in c_{00}$ such that $\|x\|_{\ell^1} = \frac{1}{\theta_n}$, $\text{supp } x \in \mathcal{S}_{N+1} \cap [L]^{< \infty}$ and $\|x\|_{\mathcal{S}_N} \leq 1$. If $T$ is an adapted tree, then

$$
Tx = \sum_{E \in \mathcal{L}(T) \atop \varepsilon t(E) \leq \theta_n} t(E) \|Ex\|_{c_0} + \sum_{E \in \mathcal{L}(T) \atop \varepsilon t(E) > \theta_n} t(E) \|Ex\|_{c_0}
$$

$$
\leq \frac{\theta_n}{\varepsilon} \|x\|_{\ell^1} + \sum_{E \in \mathcal{L}(T) \atop \varepsilon t(E) > \theta_n} \|Ex\|_{c_0}.
$$

But $\varepsilon t(E) > \theta_N$ implies that $\text{ord } (E) \leq N$. It follows from Lemma 6 that $\{E \in \mathcal{L}(T) : \varepsilon t(E) > \theta_n\}$ is $\mathcal{S}_N$-allowable. Then $Tx \leq \frac{1}{\varepsilon} + \|x\|_{\mathcal{S}_N} \leq \frac{1}{\varepsilon} + 1$. □

**Proof of Theorem 23.** Let $\varepsilon > 0$ be such that

$$
\inf_{m} \limsup_{n} \inf_{n_1+\cdots+n_s \geq n-m} \frac{\theta_n}{\theta_{n_1} \cdots \theta_{n_s}} > \varepsilon.
$$

Given any $m$, choose $n > m$ such that

$$
\inf_{n_1+\cdots+n_s \geq n-m} \frac{\theta_n}{\theta_{n_1} \cdots \theta_{n_s}} > \varepsilon.
$$
Then \( N = \max \{ n_1 + \cdots + n_s : \varepsilon \theta_{n_1} \cdots \theta_{n_s} > \theta_n \} < n - m \). Choose a block sequence \( (x_k) \) such that \( \|x_k\|_{\ell^1} = \frac{1}{n_k} \), \( \text{supp} x_k \in S_{N+1} \) and \( \|x_k\|_X \leq 1 + \frac{1}{\varepsilon} \) for all \( k \). Let \( p_k = \min \text{supp} x_k \) for all \( k \). If \( F \in S_m \), then \( (p_k)_{k \in F} \in S_m \) and hence \( \bigcup_{k \in F} \text{supp} x_k \in S_{m+N+1} \subseteq S_n \). Thus for any \( (a_k) \in c_0 \),

\[
\left\| \sum_{k \in F} a_k x_k \right\|_{\ell^1} \geq n \left\| \sum_{k \in F} a_k x_k \right\|_{\ell^1} = \sum_{k \in F} |a_k|.
\]

This shows that \( (x_k/\|x_k\|) \) is an \( \ell^1 \)-\( S_m \)-spreading model with constant \( 1 + 1/\varepsilon \).

Let \( K \) be a fixed constant so that for each \( m \), there is a normalized block sequence \( (x_i^m)_{n=1}^\infty \) that is an \( \ell^1 \)-\( S_m \)-spreading model with constant \( K \). If \( F \) is a regular family, consider the tree \( T(F) \) in \( X \) consisting of all sequences of the form \( (x_i^m)_{i \in I_1} \cup \cdots \cup (x_i^m)_{i \in I_r} \) with \( I_k \in S_m \), \( 1 \leq k \leq r \), \( i_{k+1} > \max_{i \in I_k} \text{supp} x_i^m \) for all \( i_{k+1} \in I_{k+1}, 1 \leq k < r \), and \( (\min I_1, \ldots, \min I_r) \in F \). If \( (x_i^{m_1})_{i \in I_1} \cup \cdots \cup (x_i^{m_r})_{i \in I_r} \in T(F) \), take \( i_0 = \max_{i \in I_r} \text{supp} x_i^m \). There exists \( j_0 \) so that \( (\min I_1, \ldots, \min I_r, j_0) \in F \). Then \( (x_i^{m_1})_{i \in I_1} \cup \cdots \cup (x_i^{m_r})_{i \in I_r} \cup (x_i^m)_{i \in I} \in T(F) \) if \( I > \max(i_0, j_0) \). It follows easily that \( T(F^{(1)}) \subseteq T(F)^{\omega} \). Carrying on inductively, one deduces that \( o(T(S_n)) \geq \omega^m \cdot n \) for all \( n \). Finally, note that if \( (x_i^{m_1})_{i \in I_1} \cup \cdots \cup (x_i^{m_r})_{i \in I_r} \in T(S_n) \), then for all scalars \( (a_i^m) \),

\[
\left\| \sum_{k=1}^r \sum_{i \in I_k} a_i^m x_i^m \right\| \geq \theta_n \left\| \sum_{k=1}^r \sum_{i \in I_k} a_i^m x_i^m \right\| \geq \frac{\theta_n}{K} \sum_{k=1}^r \sum_{i \in I_k} |a_i^m|.
\]

Hence \( T(S_n) \) is an \( \ell^1-K\theta_n^{-1} \)-tree in \( X \) of order at least \( \omega^n \). Thus \( I(X) \geq \omega^{n+2} \). The reverse inequality holds by Theorem 17. \( \square \)

**Problem.** If there are \( n_1, n_2 \) so that \( \sigma_{n_1} = M \) and \( \sigma_{n_2} = U \), then Theorems 22 and 23 leave the value of \( I(X) \) undetermined when

\[
\inf \limsup_{m,n} \eta_{m,n} > 0 \quad \text{and} \quad \inf \limsup \limits_{m,n} \inf_{n_1 + \cdots + n_s \geq n} \frac{\theta_{m+n}}{\theta_{n_1} \cdots \theta_{n_s}} = 0.
\]

The gap is bridged if \( X \) is either "boundedly modified" or "(completely) modified".

**Corollary 25.** Suppose that there exists \( N \) such that \( \sigma_n = U \) for all \( n > N \), or that \( \sigma_n = M \) for all \( n \). Then

1. \( I(X) = \omega^m \) if \( \inf \limsup_{m,n} \inf_{n_1 + \cdots + n_s \geq n} \frac{\theta_{m+n}}{\theta_{n_1} \cdots \theta_{n_s}} = 0 \),
2. \( I(X) = \omega^{m+2} \) if \( \inf \limsup \limits_{m,n} \inf_{n_1 + \cdots + n_s \geq n} \frac{\theta_{m+n}}{\theta_{n_1} \cdots \theta_{n_s}} > 0 \). In this case \( X \) has \( \ell^1 \)-\( S_m \)-spreading models with uniform constant.
In the final section, it is shown that a large class of (unmodified) mixed Tsirelson spaces fails to be subsequentially minimal in a strong sense. We consider a mixed Tsirelson space \( X = T[(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}] = T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^{\infty}] \), where \( \sigma_n = U \) for all \( n \). In this case, we may assume without loss of generality that \( (\theta_n) \) is a regular sequence, i.e., \( (\theta_n) \) is a non-increasing null sequence in \( (0, 1) \) such that \( \theta_{m+n} \geq \theta_n \theta_m \) for all \( m, n \in \mathbb{N} \). By [21, Lemma 4.13], \( \theta = \lim_{n} \theta_n^{1/n} \) exists and is equal to \( \sup \theta_n^{1/n} \). Also, we let \( \varphi_n = \theta_n / \theta^n \).

**Definition.** We say that a Banach space \( X \) with a normalized basis \( (e_k) \) is strongly non-subsequentially minimal if for every normalized block basis \( (x_k) \) of \( (e_k) \), there exists \( (y_k) \prec (x_k) \) such that for all \( (z_k) \prec (y_k) \), \( (z_k) \) is not equivalent to any subsequence of \( (e_k) \).

The main result of this section is Theorem 35 where it is shown that \( X \) is strongly non-subsequentially minimal if \( \theta < 1 \) and \( 0 < \inf \varphi_n \leq \sup \varphi_n < 1 \).

**Proposition 26.** [19, Proposition 21] If \( \theta < 1 \) and \( \inf \varphi_n > 0 \), then \( (\theta_n) \) satisfies

\[
- (\dag) \quad \lim_{n} \lim_{n} \sup_{n} \frac{\theta_{m+n}}{\theta_n} = 0 \quad \text{and} \\
(\ddagger) \quad \text{There exists} \quad F : \mathbb{N} \to \mathbb{R} \quad \text{with} \quad \lim_{n \to \infty} F(n) = 0 \quad \text{such that for all} \quad R, t \in \mathbb{N} \quad \text{and any arithmetic progression} \ (s_i)_{i=1}^{R} \quad \text{in} \ \mathbb{N},
\]

\[
\max_{1 \leq i \leq R} \frac{\theta_{s_i+t}}{\theta_{s_i}} \leq F(R) \sum_{i=1}^{R} \frac{\theta_{s_i+t}}{\theta_{s_i}}.
\]

The main tool in our investigation is a construction of certain “layered repeated averages” that can be carried out under the assumptions \((-\dag)\) and \((\ddagger)\). The basic units of the construction are the repeated averages due to Argyros, Mercourakis and Tsarpalias [6] which we recall here. An \( S_0 \)-repeated average is a vector \( e_k \) for some \( k \in \mathbb{N} \). For any \( p \in \mathbb{N} \), an \( S_p \)-repeated average is a vector of the form \( \frac{1}{k} \sum_{i=1}^{k} x_i \), where \( x_1 < \cdots < x_k \) are repeated \( S_{p-1} \)-repeated averages and \( k = \min \supp x_1 \). Observe that any \( S_p \)-repeated average \( x \) is a convex combination of \( \{e_k : k \in \supp x\} \) such that \( \|x\|_{\infty} \leq (\min \supp x)^{-1} \) and \( \supp x \in S_p \).

**Construction of Layered Repeated Averages**

Assume that \((-\dag)\) and \((\ddagger)\) hold. Given \( N \in \mathbb{N} \) and \( V \in [\mathbb{N}] \), choose sequences \( (p_k)_{k=1}^{N} \) and \( (L_k)_{k=1}^{N} \) in \( \mathbb{N} \), \( L_k \geq 2 \), that satisfy the following conditions:

\[
(A) \quad \frac{\theta_{p_{M}+1}+n}{\theta_{p}} \leq \frac{\theta_{1}}{2N} \prod_{i=1}^{M} \theta_{L_{i}p_{i}} \quad \text{if} \quad 0 \leq M \leq N - 2 \quad \text{and} \quad n \geq p_{N} \quad \text{(the vacuous product} \ \prod_{i=1}^{0} \theta_{L_{i}p_{i}} \quad \text{is taken to be} \ 1),}
\]
(B) \( p_{M+1} > \sum_{i=1}^{M} L_i p_i \) if \( 0 < M \leq N - 2 \),
(C) \( F(L_{M+1}) \leq \frac{\theta_1}{M^{N^2}} \prod_{i=1}^{M} \theta_i p_i \) if \( 0 < M \leq N - 2 \).

If \( k \in \mathbb{N} \) and \( 1 \leq M \leq N \), define \( r_M(k) \) to be the integer in \( \{1, 2, \ldots, L_M\} \) such that \( L_M | (k - r_M(k)) \). We can construct sequences of vectors \( x^0, \ldots, x^N \) with the following properties. Let \( (e_k) \) be the unit vector basis of \( X = T[\{(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}\}] \).

(\( \alpha \)) \( x^0 \) is a subsequence of \( (e_k)_{k \in \mathbb{V}} \).
(\( \beta \)) Say \( x^M = (x_j^M) \) and \( m_j = \min \text{supp} x_j^M \). Then there is a sequence \( (I_k^{M+1}) \) of integer intervals such that \( I_k^{M+1} < I_{k+1}^{M+1} \), \( \bigcup_{k=1}^{\infty} I_k^{M+1} = \mathbb{N} \)
and each vector \( x_k^{M+1} \in \mathbb{R}^{M+1} \) is of the form
\[
x_k^{M+1} = \sum_{j \in I_k^{M+1}} a_j x_j^M,
\]
where \( \theta_{r_{M+1}(k)p_{M+1}} \sum_{j \in I_k^{M+1}} a_j e_j \) is an \( S_{r_{M+1}(k)p_{M+1}} \)-repeated average. Moreover, the sequence \( (a_j)_{j=1}^{\infty} \) is decreasing.

Each \( x_k^{M+1} \) is made up of components of diverse complexities. We analyze it by decomposing it into components of "pure forms" in the following manner. We adhere to the notation in (\( \beta \)).

"Pure Forms" Given \( 1 \leq r_i \leq L_i, 1 \leq M \leq N - 1 \), write
\[
x_k^{M+1}(r_M) = \sum_{j \in I_k^{M+1}, r_M(j)=r_M} a_j x_j^M.
\]

For \( 1 \leq s < M \), define
\[
x_k^{M+1}(r_s, \ldots, r_M) = \sum_{j \in I_k^{M+1}, r_M(j)=r_M} a_j x_j^M(r_s, \ldots, r_M-1).
\]
If \( 1 \leq s \leq M \), it is clear that \( x_k^{M+1} = \sum x_k^{M+1}(r_s, \ldots, r_M) \), where the sum is taken over all possible values of \( r_s, \ldots, r_M \).

Given \( r_1, \ldots, r_N \), write \( p(r_1, \ldots, r_j) = \sum_{i=1}^{j} r_i p_i \), \( 1 \leq j \leq N \). Set
\[
\Phi_k^N = \frac{\theta_1}{2} \sum_{r_1, \ldots, r_{N-1}} \theta_{p(r_1, \ldots, r_{N}(k))} \theta_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta_{r_i p_i} L_i^{-1}.
\]

If \( p \geq N \), define
\[
\Theta_p = \Theta_p(N) = \max \{ \prod_{i=1}^{N} \theta_{\ell_i} : \ell_i \in \mathbb{N}, \sum_{i=1}^{N} \ell_i = p \}.
\]
The following estimates are crucial for subsequent computations. From here on, we fix a \( k \) satisfying

\[
k \geq 42 N^2 \prod_{i=1}^{N} L_i \theta^{-1}_{L_i p_i}.
\]

**Proposition 27.** [19, Theorem 20; see also the remark following the proof of the theorem]

\[
\|x_k^N\| \leq \left( \frac{2}{N} + 4 \theta^{-1}_{1} \sup_{r_1, \ldots, r_{N-1}} \Theta_{p(r_1, \ldots, r_N)} \right) \Phi_N.
\]

**Proposition 28.** [19, Corollary 9]

\[
\|x_k^N (r_1, \ldots, r_{N-1})\|_{\ell_1} \geq \frac{1}{2} \theta^{-1}_{1} r_N \prod_{i=1}^{N-1} \theta^{-1}_{r_i p_i} L_i^{-1}.
\]

For all \( m \in \mathbb{N} \), \( z \in c_{00} \), define

\[
\|z\|_m = \theta_m \sup \left\{ \sum \|E \ell z\| : (E \ell) \text{ is } \mathcal{S}_{m} \text{-admissible} \right\}.
\]

**Proposition 29.** Suppose that \( x = x_k^N = \sum_{i=1}^{\ell} b_i e_{m_i} \), \((z_i)\) is a normalized block basis of \((e_k)\) with min\( \text{supp} \) \( z_i = m_i \), \( q = \sum_{j=1}^{N} L_i p_j \), and there exists \( K < \infty \) such that \( \|z_i\|_s \geq \frac{1}{K} \) for all \( 1 \leq s \leq q \), \( 1 \leq i \leq \ell \). Let \( z = \sum_{i=1}^{\ell} b_i z_i \).

Then

\[
\|x\| \leq \left( \frac{2}{N} + 4 \theta^{-1}_{1} \sup_{r_1, \ldots, r_{N-1}} \Theta_{p(r_1, \ldots, r_N)} \right) K \theta_1 \|z\|.
\]

**Proof.** According to Proposition 27, it suffices to show that \( \|z\| \geq (\theta_1 K)^{-1} \Phi_N \). For each \( 1 \leq i \leq \ell \), let \((r_1, \ldots, r_{N-1})\) be the unique \((N-1)\)-tuple such that \( m_i \in \text{supp} \ x_k^N (r_1, \ldots, r_{N-1}) \). Since \( \|z_i\|_t \geq \frac{1}{K} \) for \( t = q - p (r_1, \ldots, r_{N-1}) \), there exists an \( \mathcal{S}_t \) admissible family \( \mathcal{G}_t \) such that \( G \subseteq \text{supp} \ z_i \) for all \( G \in \mathcal{G}_t \) and

\[
\|z_i\|_t = \theta_t \sum_{G \in \mathcal{G}_t} \|Gz_i\| \geq \frac{1}{K}.
\]

We estimate the norm of \( z \) by means of a particular tree \( T \). If \( 0 \leq n \leq N \) and \( \text{supp} \ x_k^{N-n} \subseteq \text{supp} \ x_k^{N} \), let

\[
E_j^n = \bigcup \left\{ \text{supp} \ z_i : m_i \in \text{supp} \ x_j^{N-n} \right\}
\]

and

\[
E^n = \left\{ E_j^n : \text{supp} x_j^{N-n} \subseteq \text{supp} x_k^{N} \right\}.
\]
By (\(\beta\)) in the construction of \(x^{N-n}\), \(E^m_s\) is an \(S_{r_{N-n}(s)p_{N-n}}\)-admissible union of the sets \(\left\{ E^m_{j+1} : \text{supp} \, x_j^{N-n-1} \subseteq \text{supp} \, x_k^N \right\}\). Hence \(\bigcup_{n=1}^{N} E^n_s\) is an admissible tree so that

\[
\text{ord} \left( E^m_{n+1} \right) = \text{ord} \left( E^n_s \right) + r_{N-n}(s) \, p_{N-n} \text{ if } E^m_{n+1} \subseteq E^n_s.
\]

Note that \(\text{supp} \, x_j^0\) is a singleton \(\{m_i\}\) for some \(i\) and hence \(E^m_j = \text{supp} \, z_i\).

It follows from (11) that \(\text{ord} \left( E^N_j \right) = p(r_1, ..., r_{N-1}) + r_N(k) \, p_N\), where \((r_1, ..., r_{N-1})\) is the unique \((N-1)\)-tuple determined by \(m_i\). Set \(E^{N+1}_j = \bigcup_{i=1}^{r} G_i\). Since \(G_i\) is an \(S_{q-p(r_1, ..., r_{N-1})}\)-admissible family with \(\bigcup_{G \in G_i} G \subseteq \text{supp} \, z_i = E^N_j \in \mathcal{E}^N\), \(T = \bigcup_{n=1}^{N+1} E^n\) is an admissible tree such that \(\text{ord}(G) = q + r_N(k) \, p_N\) for each of the leaves \(G\) of \(T\). By Lemma 6, \(\bigcup_{i=1}^{r} G_i\) is \(S_{r_N(k) \, p_N + q}\)-admissible. Therefore,

\[
||z|| \geq \theta_{q+r_N(k) \, p_N} \sum_{i=1}^{r} b_i \sum_{G \in G_i} ||G z_i||
\]

\[
\geq \theta_{q+r_N(k) \, p_N} \sum_{i=1}^{r} b_i \left( K \theta_{q-p(r_1, ..., r_{N-1})} \right)^{-1} \text{ by (10)}
\]

\[
= \theta_{q+r_N(k) \, p_N} \sum_{r_1, ..., r_{N-1} = 1}^{r_1, ..., r_{N-1}} \left( K \theta_{q-p(r_1, ..., r_{N-1})} \right)^{-1} ||x(r_1, ..., r_{N-1})||_{G_i}.
\]

By the regularity of \(\theta_n\), \(\theta_{p(r_1, ..., r_{N-1})} \theta_{q-p(r_1, ..., r_{N-1})} \leq \theta_{q+r_N(k) \, p_N}\). Applying Proposition 28 to the above gives

\[
||z|| \geq \frac{\theta_{q+r_N(k) \, p_N}}{K} \sum_{r_1, ..., r_{N-1}}^{r_1, ..., r_{N-1}} \theta_{p(r_1, ..., r_N)} \left( \frac{1}{2} \theta_{r_N(k) \, p_N} \prod_{i=1}^{N-1} \theta_{r_i \, p_i} L_i^{-1} \right)
\]

\[
= \frac{1}{K} \left( \frac{1}{2} \sum_{r_1, ..., r_{N-1}}^{r_1, ..., r_{N-1}} \theta_{p(r_1, ..., r_N)} \theta_{r_N(k) \, p_N} \prod_{i=1}^{N-1} \theta_{r_i \, p_i} L_i^{-1} \right)
\]

\[
= (\theta_1 K)^{-1} \Phi_N^1.
\]

We need a few preparatory results in order to exploit the estimate established in Proposition 29.

**Lemma 30.** If \((x_k) < (e_k), \varepsilon > 0\) and \(p \in \mathbb{N}\), then there exists \(y \in \text{span} \, (x_k)\), \(||y|| = 1\) such that \(||y||_{S_p} < \varepsilon\).

**Proof.** Assume to the contrary. There exist \(\varepsilon > 0, p \in \mathbb{N}\) such that for all \(y \in \text{span} \, \{(x_k)\}, ||y||_{S_p} \geq \varepsilon \, ||y||\). On the other hand, \(||y|| \geq \theta_p \, ||y||_{S_p}\). Hence \(||-||\) and \(||-||_{S_p}\) are equivalent on \(\text{span} \, \{(x_k)\}\) . However, the Schreier space \(S_p\) is \(c_0\)-saturated. It follows that \(\{x_k\}\) and thus \(X\) contains a copy of \(c_0\), contradicting the reflexivity of \(X\). \(\square\)
Lemma 31. If \((z_k) \prec (y_k) \prec (e_k)\), and \(||y_k||_{S_k-1} \leq \frac{1}{2^{k+2}}\) for all \(k\), then

\[ ||z_k||_{S_k-1} \leq \frac{1}{2^{k+1}} \text{ for all } k. \]

Proof. Write \(z_k = \sum_{j \in J_k} a_j y_j\). Note that \(|a_j| = ||z_k|| = 1\) for all \(j \in J_k\). Therefore,

\[ ||z_k||_{S_k-1} \leq \sum_{j \in J_k} ||y_j||_{S_j-1} \]

\[ \leq \sum_{j \in J_k} ||y_j||_{S_j-1} \text{ since } k \leq \min J_k \leq j \]

\[ \leq \sum_{j \in J_k} \frac{1}{2^{j+2}} \leq \frac{1}{2^{k+1}}. \]

\( \square \)

Lemma 32. Assume that \(\theta < 1\) and \(\inf_n \varphi_n > 0\). If \((z_k) \prec (e_k)\) and

\[ ||z_k||_{S_k-1} \leq \frac{1}{2^{k+1}} \text{ for all } k, \]

then there is a constant \(K < \infty\) such that for all \(z \in \text{span } (z_k)_{k=n}^\infty\), we have

\[ ||z||_m \geq \frac{1}{2K} ||z|| \text{ for all } 1 \leq m \leq n. \]

Proof. First observe that

\[ \frac{\theta_{m+n}}{\theta_m \theta_n} = \frac{\varphi_{m+n}}{\varphi_m \varphi_n} \leq \frac{1}{(\inf \varphi_n)^2} \text{ for all } m, n. \]

Let \(K = \frac{1}{(\inf \varphi_n)^2}\). Suppose that \(z \in \text{span } (z_k)_{k=n}^\infty\), \(||z|| = 1\) and \(1 \leq m \leq n\).

Choose an admissible tree \(T\) of \(z\) so that

\[ 1 = ||z|| = Tz = \sum_{E \in \mathcal{L}(T)} t(E) ||Ez||_{c_0} \]

\[ = \sum_{E \in \mathcal{L}(T) \atop \text{ord}(E) \leq m} t(E) ||Ez||_{c_0} + \sum_{E \in \mathcal{L}(T) \atop \text{ord}(E) > m} t(E) ||Ez||_{c_0}. \]
Write $z = \sum_{k=n}^{\infty} a_k z_k$. Then $|a_k| \leq 1$ as $||z|| = 1$. Note that according to Lemma 6, the collection $\{ E \in \mathcal{L}(T) : \text{ord}(E) \leq m \}$ of leaves is $\mathcal{S}_m$-admissible. Therefore,

$$\sum_{E \in \mathcal{L}(T) : \text{ord}(E) \leq m} t(E) ||Ez||_{c_0} \leq \sum_{E \in \mathcal{L}(T) : \text{ord}(E) \leq m} ||Ez||_{c_0} \leq ||z||_{\mathcal{S}_m} \leq \sum_{k=n}^{\infty} ||z_k||_{\mathcal{S}_m} \leq \sum_{k=n}^{\infty} \frac{1}{2^{k+1}} \leq \frac{1}{2}.$$ 

Thus

$$\sum_{E \in \mathcal{L}(T) : \text{ord}(E) > m} t(E) ||Ez||_{c_0} \geq \frac{1}{2}.$$ 

Let $\mathcal{E}$ be the collection of all nodes $E$ in $T$ that are minimal subject to the condition $\text{ord}(E) > m$. Also, let $D$ be the set of all immediate predecessors of nodes in $\mathcal{E}$. If $D \in \mathcal{D}$, let $\mathcal{E}(D)$ be the collection of its immediate successors. For each $E \in \mathcal{E}(D)$, $\text{ord}(D) \leq m < \text{ord}(E)$. Therefore there exists an $\mathcal{S}_{\text{ord}(E)-m}$-admissible collection $\mathcal{G}_D$ of subsets of $D$ such that $\mathcal{E}(D) = \cup \{ E \in \mathcal{E}(D) : E \subseteq G \text{ for some } G \in \mathcal{G}_D \}$ and $\{ E \in \mathcal{E}(D) : E \subseteq G \}$ is $\mathcal{S}_{\text{ord}(E)-m}$-admissible for each $G \in \mathcal{G}_D$. Now $\mathcal{G} = \cup_{D \in \mathcal{D}} \mathcal{G}_D$ is $\mathcal{S}_m$-admissible and $\theta_{\text{ord}(E)} \geq t(E)$ by the regularity of $(\theta_n)$. Hence

$$||z||_m \geq \theta_m \sum_{G \in \mathcal{G}} ||Gz|| \geq \theta_m \sum_{G \in \mathcal{G}} \theta_{\text{ord}(E)-m} \sum_{E \in \mathcal{E} : E \subseteq G} ||Ez|| \geq \sum_{G \in \mathcal{G}} \theta_{\text{ord}(E)} \sum_{E \in \mathcal{E} : E \subseteq G} ||Ez|| \text{ by (12) and the definition of } K$$

$$\geq \frac{1}{K} \sum_{E \in \mathcal{E}} t(E) ||Ez|| \geq \frac{1}{K} \sum_{E \in \mathcal{L}(T) : \text{ord}(E) > m} t(E) ||Ez|| \geq \frac{1}{2K}.$$ 

We shall show that, for appropriate $(\theta_n)$, if $(z_k) \prec (e_k)$ satisfies the conclusion of Lemma 31, then it is not equivalent to a subsequence of $(e_k)$.

**Lemma 33.** If $0 < \inf_n \varphi_n \leq \sup_n \varphi_n < 1$, then $\lim_N \sup_{p \geq N} \frac{\Theta_p(N)}{\eta_p} = 0$. 

□
Proof. Let $\varepsilon > 0$. Choose $N$ such that $\frac{d^N}{c} < \varepsilon$, where $0 < c = \inf_n \varphi_n \leq \sup_n \varphi_n = d < 1$. Let $p \in \mathbb{N}$. If $(\ell_i)_{i=1}^N$ is a sequence of positive integers such that $\sum_{i=1}^N \ell_i = p$, then
\[
\prod_{i=1}^N \theta_{\ell_i} = \theta^p \prod_{i=1}^N \varphi_{\ell_i} \leq \theta^p d^N
\]
and
\[
\theta_p = \varphi_p \theta^p \geq c \theta^p.
\]
Thus
\[
\sup_{p \geq N} \frac{\Theta_p(N)}{\theta_p} \leq \frac{d^N}{c} < \varepsilon.
\]

**Proposition 34.** If $(z_k)$ is a normalized block basis that is equivalent to a subsequence of $(e_k)$, then there is a subsequence $(z_{k_j})$ of $(z_k)$ such that $(z_{k_j})$ is equivalent to $(e_{n_j})$, where $n_j = \min \supp z_{k_j}$.

**Proof.** By [17, Proposition 9], two subsequences $(e_{n_i})$ and $(e_{\ell_i})$ of $(e_k)$ are equivalent whenever $\max \{n_i, \ell_i\} < \min \{n_{i+1}, \ell_{i+1}\}$ for all $i$. If $(z_k)$ is equivalent to a subsequence of $(e_k)$, then there is a subsequence $(z_{k_j})$ of $(z_k)$ that is equivalent to a subsequence $(e_{n_j})$ of $(e_k)$ with
\[
\max \{\min \supp z_{k_j}, n_j\} < \min \{\min \supp z_{k_{j+1}}, n_{j+1}\} \text{ for all } j.
\]
Thus $\max \{n_j, m_j\} < \min \{n_{j+1}, m_{j+1}\}$, and hence $(e_{n_j})$ is equivalent to $(e_{m_j})$. Consequently, $(z_{k_j})$ is equivalent to $(e_{m_j})$. $\square$

We are now ready to prove the main result of the section.

**Theorem 35.** If $0 < \inf_n \varphi_n \leq \sup_n \varphi_n < 1$, then $X$ is strongly non-subsequentially minimal.

**Proof.** Let $(x_k)$ be a normalized block basis of $(e_k)$. By Lemma 30, there exists $(y_k) \prec (x_k)$ such that $\|y_k\|_{S_{k-1}} \leq \frac{1}{2K^{2}}$ for all $k$. Suppose that there exists $(z_k) \prec (y_k)$ that is equivalent to a subsequence of $(e_k)$. Applying Proposition 34, we may assume that $(z_k)$ is equivalent to $(e_{m_k})$, where $m_k = \min \supp z_k$. Pick $\varepsilon > 0$ so that
\[
\varepsilon \left\| \sum_k b_k z_k \right\| \leq \left\| \sum_k b_k e_{m_k} \right\| \text{ for all } (b_k) \in c_{00}.
\]

By a combination of Lemmas 31 and 32 there is a constant $K < \infty$ such that $\|z\|_s \geq \frac{1}{2K} \|z\|$, for all $z \in \text{span} \{z_k\}_{k=n}, 1 \leq s \leq n$. Use Lemma 33 to choose $N$ such that $\frac{2}{N} + 4 \theta^{-1} \sup_p \frac{\Theta_p(N)}{\theta_p} < \frac{\varepsilon}{2K \theta}$ if $p \geq N$. With the chosen $N$ and $V = (m_i)_{i=q}^\infty$ construct the layered repeated average vector $x = x_k^N = \sum_{i=q}^\ell b_i e_{m_i}$ with $k$ satisfying the inequality (9). Let $z = \sum_{i=q}^\ell b_i z_i$. 
(Recall that \( q = \sum_{j=1}^{N} L_j p_j \), where \((p_j)_{j=1}^{N}\) and \((L_j)_{j=1}^{N}\) are chosen to satisfy conditions (A), (B), and (C) once \( N \) is determined.) According to Proposition 29,

\[
\|x\| \leq \left( \frac{2}{N} + 4\theta_1^{-1} \sup_{r_1,\ldots,r_{N-1}} \frac{\Theta(p_{(r_1,\ldots,r_{N-1})})}{\theta(p_{(r_1,\ldots,r_{N-1})})} \right) 2K\theta_1 \|z\|
\]

\[
< \varepsilon \|z\|,
\]

contrary to the choice of \( \varepsilon \).

The following example shows that the condition \( \sup_n \varphi_n < 1 \) is not necessary for the conclusion of the theorem to hold.

**Example 36.** If \( \theta < 1 \), there exists a regular sequence \((\theta_n)\) with \( \sup_n \theta_n^{1/n} = \theta \) and \( \lim_n \varphi_n = 1 \) such that \( X \) is strongly non-subsequentially minimal.

**Proof.** Suppose that \( 0 < \theta < 1 \). In [19, Example 23], a regular sequence \((\theta_n)\) is constructed so that \( \sup_n \theta_n^{1/n} = \theta \), \( \lim_n \varphi_n = 1 \) and for all \( N \in \mathbb{N} \), there are sequences \((p_k)_{k=1}^{N}\) and \((L_k)_{k=1}^{N}\) satisfying conditions (A), (B), and (C) and

\[
\lim_{N \to \infty} \sup_{r_1,\ldots,r_{N-1}} \frac{\Theta(p_{(r_1,\ldots,r_{N-1})})}{\theta(p_{(r_1,\ldots,r_{N-1})})} = 0.
\]

Following the arguments in Theorem 35 with Lemma 33 replaced by (13) shows that \( X \) is strongly non-subsequentially minimal.

In view of Proposition 14, any subsequentially minimal partly modified mixed Tsirelson space is quasi-minimal. However, the existence of strongly non-subsequentially minimal mixed Tsirelson spaces prompts the following question.

**Question.** Does every (partly modified) mixed Tsirelson space \( T[(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}] \) (or \( T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^{\infty}] \)) contain a quasi-minimal subspace?

## References


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