

# THE CLASSIFICATION PROBLEM FOR NONATOMIC WEAK $L^p$ SPACES

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ABSTRACT. The aim of the paper is to study the isomorphic structure of the weak  $L^p$  space  $L^{p,\infty}(\Omega, \Sigma, \mu)$  when  $(\Omega, \Sigma, \mu)$  is a purely nonatomic measure space. Using Maharam's classification of measure algebras, it is shown that every such  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isomorphic to a weak  $L^p$  space defined on a weighted direct sum of product measure spaces of the type  $2^\kappa$ . Several isomorphic invariants are then obtained. In particular, it is found that there is a notable difference between the case  $1 < p < 2$  and the case where  $2 \leq p < \infty$ . Applying the methods developed, we obtain an isomorphic classification of the purely nonatomic weak  $L^p$  spaces in a special case.

## 1. INTRODUCTION

Let  $1 < p < \infty$  and let  $(\Omega, \Sigma, \mu)$  be a measure space. The *Weak  $L^p$*  space  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is the space of all measurable functions  $f$  on  $(\Omega, \Sigma, \mu)$  so that

$$\|f\| = \sup_{c>0} c(\mu\{\omega : |f(\omega)| > c\})^{1/p} < \infty.$$

A good source of information regarding the Weak  $L^p$  spaces, and more generally, the Lorentz spaces  $L^{p,q}$ , is [1]. While  $\|\cdot\|$  is only a quasinorm on  $L^{p,\infty}(\Omega, \Sigma, \mu)$ , it is equivalent to a norm; in fact, if we set

$$|||f||| = \sup_{\sigma} (\mu(\sigma))^{1-\frac{1}{p}} \int_{\sigma} |f|,$$

where the sup is taken over all sets  $\sigma \in \Sigma$  with  $0 < \mu(\sigma) < \infty$ , then  $|||\cdot|||$  is a norm on  $L^{p,\infty}(\Omega, \Sigma, \mu)$  so that

$$(1) \quad \|f\| \leq |||f||| \leq \frac{1}{1-\frac{1}{p}} \|f\|.$$

[To see the inequality, use the fact that  $\|f\| \leq 1$  if and only if the decreasing rearrangement  $f^*$  of  $|f|$  satisfies  $f^*(t) \leq t^{-\frac{1}{p}}$  for  $t \in (0, \infty)$ . See [1, Chapter 4, §4], in particular, Lemma 4.5.] As we will be concerned exclusively with the isomorphic structure of weak  $L^p$  spaces, we will primarily utilize

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the quasinorm  $\|\cdot\|$ . The weak  $L^p$  spaces arise naturally in interpolation theory, and find applications in harmonic analysis, probability theory and functional analysis. As a class, they share many of the properties of the classical Lebesgue  $L^p$  spaces and yet are different in many respects. Thus it is a natural and interesting problem to try to understand the isomorphic structure of the class of weak  $L^p$  spaces. In [7], the first author gave a complete isomorphic classification of the atomic weak  $L^p$  spaces. In [8], however, it was shown that, in general,  $L^{p,\infty}(\Omega, \Sigma, \mu)$  behaves differently for atomic and nonatomic measure spaces. In the present paper, we will attempt to classify isomorphically all purely nonatomic weak  $L^p$  spaces. While the attempt is only wholly successful for a special subclass, many interesting results have been thrown up along the way. In particular, the bifurcation in behavior between the cases where  $1 < p < 2$  and where  $2 \leq p < \infty$  is quite unexpected and does not occur for atomic weak  $L^p$  spaces.

The classification of the Lebesgue spaces  $L^p(\Omega, \Sigma, \mu)$  is classical (an exposition may be found in [5]) and is based on Maharam's classification of measure algebras [9]. In §2, we make use of Maharam's result to show that if  $(\Omega, \Sigma, \mu)$  is a purely nonatomic measure space, then  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isomorphic to  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ , where  $\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha}$  denotes a weighted direct sum of the product measure spaces  $2^{\kappa_\alpha}$ . The representation is further refined in Theorem 1.

In §3, several isomorphic invariants are obtained. By an isomorphic invariant, we mean a parameter, defined only in terms of sequences  $(a_\alpha)_{\alpha < \tau}$  and  $(\kappa_\alpha)_{\alpha < \tau}$  mentioned in the representation above, that depend solely on the isomorphism class of the space  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ . Essentially, the invariants obtained measure either the complexity ( $\sup \kappa_\alpha$ ) or the "width" ( $|\tau|$ ) of the measure space  $\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha}$ , or a combination of both. However, the surprising fact emerges that the width is only an invariant when  $2 \leq p < \infty$ . In §4, making use of some results of Carothers and Dilworth [2, 3, 4] and arguments of a probabilistic flavor, it is shown that the weak  $L^p$  spaces defined on  $\bigoplus_{\alpha < \kappa} 2^{\aleph_\alpha} \oplus 2^\kappa$  and  $2^\kappa$  respectively are isomorphic if  $1 < p < 2$ . In the final section, we make use of the methods developed in the preceding sections to give an isomorphic classification of nonatomic weak  $L^p$  spaces in a special case. The paper ends with a list of several open problems.

## 2. REDUCTION TO STANDARD FORM

The main objective of this section is to show that every weak  $L^p$  space is isomorphic to a weak  $L^p$  space defined on a measure space of a special form. This runs in parallel to the situation in the Lebesgue spaces  $L^p$ . The argument relies on Maharam's classification of measure algebras. Let us establish some notation regarding measure spaces that will be used throughout the rest of the paper. By  $2$  we denote the two point measure space  $\{-1, 1\}$ , where each of the one-point sets  $\{-1\}$  and  $\{1\}$  is assigned a measure of  $1/2$ . If  $\kappa$  is

a cardinal, let  $2^\kappa$  be the product measure space of  $\kappa$  copies of 2. If  $(\Omega, \Sigma, \mu)$  is a measure space and  $a$  is a positive real number, denote by  $a \cdot (\Omega, \Sigma, \mu)$  the measure space  $(\Omega, \Sigma, a\mu)$ . Given a family of measure spaces  $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ , let  $\oplus_\alpha (\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$  be the measure space  $(\Omega, \Sigma, \mu)$ , where  $\Omega = \cup_\alpha \Omega_\alpha$  (we assume here that the sets  $\Omega_\alpha$  are pairwise disjoint; otherwise, replace them with pairwise disjoint copies) and  $\Sigma$  is the smallest  $\sigma$ -algebra generated by  $\cup_\alpha \Sigma_\alpha$ . Note that  $\sigma \subseteq \Omega$  belongs to  $\Sigma$  if and only if  $\sigma \cap \Omega_\alpha \in \Sigma_\alpha$  for all  $\alpha$  and either  $\sigma \cap \Omega_\alpha = \emptyset$  for all but countably many  $\alpha$  or  $\sigma \cap \Omega_\alpha = \Omega_\alpha$  for all but countably many  $\alpha$ . For  $\sigma \in \Sigma$ ,  $\mu(\sigma)$  is defined to be  $\sum_\alpha \mu_\alpha(\sigma)$ . We can now state the main result of this section.

**Theorem 1.** *Let  $(\Omega, \Sigma, \mu)$  be a purely nonatomic measure space and let  $1 < p < \infty$ . Then  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isomorphic to a direct sum of at most three spaces  $E \oplus F \oplus G$ , where, if nontrivial,*

$$E = L^{p,\infty}(\oplus_{\alpha < \omega_1} 2^{\kappa_\alpha}); \quad F = L^{p,\infty}(\oplus_{n=1}^{\infty} 2^{\kappa'_n}); \quad G = L^{p,\infty}(\oplus_{n=1}^{\infty} 2^{-n} \cdot 2^{\kappa''_n}).$$

Here  $\kappa_\alpha$ ,  $\kappa'_n$  and  $\kappa''_n$  are infinite cardinals so that, when present,

$$\kappa_{\alpha_1} \leq \kappa_{\alpha_2} < \kappa'_{n_1} \leq \kappa'_{n_2} < \kappa''_{n_1} \leq \kappa''_{n_2}$$

if  $\alpha_1 < \alpha_2$  and  $n_1 < n_2$ .

Let  $(\Omega, \Sigma, \mu)$  be a measure space. Define an equivalence relation on  $\Sigma$  by  $\sigma_1 \sim \sigma_2$  if  $\mu(\sigma_1 \Delta \sigma_2) = 0$ , where  $\Delta$  denotes the symmetric difference. Write the equivalence class containing  $\sigma$  as  $\hat{\sigma}$  and let the set of equivalence classes be denoted by  $\hat{\Sigma}$ . Clearly,  $\hat{\Sigma}$  is a Boolean algebra under the operations

$$\hat{\sigma}_1 \vee \hat{\sigma}_2 = (\sigma_1 \cup \sigma_2), \quad \hat{\sigma}_1 \wedge \hat{\sigma}_2 = (\sigma_1 \cap \sigma_2), \quad \neg \hat{\sigma} = (\Omega \setminus \sigma).$$

We may also transfer the measure  $\mu$  over to  $\hat{\Sigma}$  by defining  $\hat{\mu}(\hat{\sigma}) = \mu(\sigma)$  for all  $\sigma \in \Sigma$ . The subset of  $\Sigma$  consisting of all  $\sigma$  with  $\mu(\sigma) < \infty$  is denoted by  $\Sigma_0$ . Let  $\hat{\Sigma}_0 = \{\hat{\sigma} : \sigma \in \Sigma_0\}$ . Now suppose that  $(\Omega', \Sigma', \mu')$  is another measure space, with the corresponding objects  $\hat{\Sigma}'$ ,  $\hat{\mu}'$  and  $\hat{\Sigma}'_0$ . A *finite measure isomorphism* is a bijection  $\Phi : \hat{\Sigma}_0 \rightarrow \hat{\Sigma}'_0$  such that

$$\Phi(\hat{\sigma}_1 \vee \hat{\sigma}_2) = \Phi(\hat{\sigma}_1) \vee \Phi(\hat{\sigma}_2), \quad \Phi(\hat{\sigma}_1 \wedge \hat{\sigma}_2) = \Phi(\hat{\sigma}_1) \wedge \Phi(\hat{\sigma}_2), \quad \hat{\mu}'(\Phi(\hat{\sigma})) = \hat{\mu}(\hat{\sigma}).$$

If such a finite measure isomorphism exists, we say that the measure spaces  $(\Omega, \Sigma, \mu)$  and  $(\Omega', \Sigma', \mu')$  are finitely measure isomorphic. The words “finite” and “finitely” are suppressed if the measure spaces under consideration are finite measure spaces. The first proposition is well known.

**Proposition 2.** *Suppose that  $1 < p < \infty$  and that the measure spaces  $(\Omega, \Sigma, \mu)$  and  $(\Omega', \Sigma', \mu')$  are finitely measure isomorphic. Then the spaces  $L^{p,\infty}(\Omega, \Sigma, \mu)$  and  $L^{p,\infty}(\Omega', \Sigma', \mu')$  are isometrically lattice isomorphic.*

*Sketch of Proof.* Suppose that  $f$  is a nonnegative function in  $L^{p,\infty}(\Omega, \Sigma, \mu)$ . For any  $C > 1$ , let  $g = \sum_{k=-\infty}^{\infty} C^k 1_{\{C^k \leq f < C^{k+1}\}}$  (pointwise sum). Then  $g \leq f \leq Cg$ . Therefore, the space  $X$  of all functions in  $L^{p,\infty}(\Omega, \Sigma, \mu)$  of the form  $\sum a_k 1_{\sigma_k}$ , where  $a_k \in \mathbb{R}$  and  $(\sigma_k)$  is a pairwise disjoint sequence

in  $\Sigma_0$ , is a dense sublattice of  $L^{p,\infty}(\Omega, \Sigma, \mu)$ . Let  $\Phi : \hat{\Sigma}_0 \rightarrow \hat{\Sigma}'_0$  be a finite measure isomorphism. The map  $T : \sum a_k 1_{\sigma_k} \mapsto \sum a_k 1_{\tau_k}$ , where  $\hat{\tau}_k = \Phi(\hat{\sigma}_k)$ , is an isometric lattice isomorphism from  $X$  onto a dense sublattice of  $L^{p,\infty}(\Omega', \Sigma', \mu')$ .  $\square$

**Theorem 3.** (Maharam) [9] *Let  $(\Omega, \Sigma, \mu)$  be a purely nonatomic finite measure space. Then there are a sequence of positive real numbers  $(a_n)$  and a sequence of infinite cardinals  $(\kappa_n)$  such that  $(\Omega, \Sigma, \mu)$  is measure isomorphic to  $\bigoplus a_n \cdot 2^{\kappa_n}$ .*

**Proposition 4.** *Let  $(\Omega, \Sigma, \mu)$  be a purely nonatomic measure space. There exist positive real numbers  $a_\alpha$  and infinite cardinals  $\kappa_\alpha$  so that  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isometrically lattice isomorphic to  $L^{p,\infty}(\bigoplus a_\alpha \cdot 2^{\kappa_\alpha})$ .*

*Proof.* By Zorn's Lemma, there exists a family of sets  $(\Omega_\alpha)$  in  $\Sigma_0$  so that (i)  $\mu(\Omega_\alpha \cap \Omega_\beta) = 0$  if  $\alpha \neq \beta$  and (ii) if  $\sigma \in \Sigma_0$  and  $\mu(\sigma \cap \Omega_\alpha) = 0$  for all  $\alpha$ , then  $\mu(\sigma) = 0$ . Let  $\Sigma_\alpha = \{\sigma \cap \Omega_\alpha : \sigma \in \Sigma\}$  and  $\mu_\alpha = \mu|_{\Sigma_\alpha}$ . Denote by  $(\Omega', \Sigma', \mu')$  the measure space  $\bigoplus_\alpha (\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ . It is straightforward to check that the map  $\Phi : \hat{\Sigma}_0 \rightarrow \hat{\Sigma}'_0$ ,  $\hat{\sigma} \mapsto (\bigoplus_{\alpha \in A} (\sigma \cap \Omega_\alpha))$ ,  $A = \{\alpha : \mu(\sigma \cap \Omega_\alpha) > 0\}$ , is a finite measure isomorphism. By Maharam's Theorem, each  $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$  is measure isomorphic to some  $\bigoplus_n a_{\alpha,n} \cdot 2^{\kappa_{\alpha,n}}$ . It follows easily that  $(\Omega, \Sigma, \mu)$  is finitely measure isomorphic to  $\bigoplus_\alpha \bigoplus_n a_{\alpha,n} \cdot 2^{\kappa_{\alpha,n}}$ . The desired conclusion follows from Proposition 2.  $\square$

Suppose  $((\Omega_\alpha, \Sigma_\alpha, \mu_\alpha))_\alpha$  and  $((\Omega'_\alpha, \Sigma'_\alpha, \mu'_\alpha))_\alpha$  are families of finite measure spaces. Assume that for each  $\alpha$ , there is a measure space  $(\Omega''_\alpha, \Sigma''_\alpha, \mu''_\alpha)$  measure isomorphic to  $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$  so that

- (1)  $\Omega''_\alpha$  is a subset of  $\Omega'_\alpha$  and  $\Omega''_\alpha \in \Sigma'_\alpha$ ;
- (2)  $\Sigma''_\alpha$  is a sub- $\sigma$ -algebra of  $\Sigma'_\alpha \cap \Omega''_\alpha = \{\sigma' \cap \Omega''_\alpha : \sigma' \in \Sigma'_\alpha\}$ ;
- (3)  $\mu''_\alpha = \mu'_\alpha|_{\Sigma''_\alpha}$ .

Then we write  $\bigoplus_\alpha (\Omega_\alpha, \Sigma_\alpha, \mu_\alpha) \hookrightarrow \bigoplus_\alpha (\Omega'_\alpha, \Sigma'_\alpha, \mu'_\alpha)$ . For each  $\alpha$ , the operator  $\mathbb{E}_\alpha$  that maps each  $f \in L^1(\Omega''_\alpha, \Sigma'_\alpha \cap \Omega''_\alpha, \mu'_\alpha|_{\Sigma'_\alpha \cap \Omega''_\alpha})$  to its Radon-Nikodym derivative with respect to the measure  $\mu''_\alpha$  (defined on the sub- $\sigma$ -algebra  $\Sigma''_\alpha$  of  $\Sigma'_\alpha \cap \Omega''_\alpha$ ) is a norm 1 projection from  $L^1(\Omega''_\alpha, \Sigma'_\alpha \cap \Omega''_\alpha, \mu'_\alpha|_{\Sigma'_\alpha \cap \Omega''_\alpha})$  onto  $L^1(\Omega''_\alpha, \Sigma''_\alpha, \mu''_\alpha)$ . Keeping in mind the form of the norm  $\|\cdot\|$  defined in §1, we see that  $\mathbb{E}_\alpha$  is also a norm 1 projection from  $L^{p,\infty}(\Omega''_\alpha, \Sigma'_\alpha \cap \Omega''_\alpha, \mu'_\alpha|_{\Sigma'_\alpha \cap \Omega''_\alpha})$  onto  $L^{p,\infty}(\Omega''_\alpha, \Sigma''_\alpha, \mu''_\alpha)$ , provided both spaces are equipped with the norm  $\|\cdot\|$ . When both of these spaces are equipped with the quasinorm  $\|\cdot\|$ , it follows from inequality (1) in §1 that  $\|\mathbb{E}_\alpha\| \leq (1 - \frac{1}{p})^{-1}$ . Denote by  $P_\alpha$  the operator from  $L^{p,\infty}(\Omega'_\alpha, \Sigma'_\alpha, \mu'_\alpha)$  onto  $L^{p,\infty}(\Omega''_\alpha, \Sigma''_\alpha, \mu''_\alpha)$  given by  $P_\alpha f = \mathbb{E}_\alpha(f|_{\Omega''_\alpha})$ . Then  $P_\alpha$  is a projection of norm  $\leq (1 - \frac{1}{p})^{-1}$ . It is easy to verify that the map  $\bigoplus f_\alpha \mapsto \bigoplus P_\alpha f_\alpha$  is a bounded projection from  $L^{p,\infty}(\bigoplus_\alpha (\Omega'_\alpha, \Sigma'_\alpha, \mu'_\alpha))$  onto  $L^{p,\infty}(\bigoplus_\alpha (\Omega''_\alpha, \Sigma''_\alpha, \mu''_\alpha))$ . By Proposition 2,  $L^{p,\infty}(\bigoplus_\alpha (\Omega_\alpha, \Sigma_\alpha, \mu_\alpha))$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_\alpha (\Omega'_\alpha, \Sigma'_\alpha, \mu'_\alpha))$ .

**Proposition 5.** *Consider the measure spaces  $\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}}$  and  $\bigoplus_{\beta} b_{\beta} \cdot 2^{\kappa'_{\beta}}$ , where  $(a_{\alpha}), (b_{\beta})$  are positive real numbers and  $(\kappa_{\alpha}), (\kappa'_{\beta})$  are infinite cardinals. Suppose that for each  $\alpha$ , there is a set  $J(\alpha)$  of indices  $\beta$  so that the sets  $(J(\alpha))_{\alpha}$  are pairwise disjoint,  $\sum_{\beta \in J(\alpha)} b_{\beta} \geq a_{\alpha}$  for all  $\alpha$  and  $\kappa'_{\beta} \geq \kappa_{\alpha}$  for all  $\beta \in J(\alpha)$ . Then  $L^{p,\infty}(\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}})$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_{\beta} b_{\beta} \cdot 2^{\kappa'_{\beta}})$ .*

*Proof.* For each  $\alpha$ , choose nonnegative real numbers  $(c_{\beta})_{\beta \in J(\alpha)}$  so that  $a_{\alpha} = \sum_{\beta \in J(\alpha)} c_{\beta}$  and that  $c_{\beta} \leq b_{\beta}$  for each  $\beta$ . Now  $a_{\alpha} \cdot 2^{\kappa_{\alpha}}$  is measure isomorphic to  $\bigoplus_{\beta \in J(\alpha)} c_{\beta} \cdot 2^{\kappa_{\alpha}}$  and  $\bigoplus_{\beta \in J(\alpha)} c_{\beta} \cdot 2^{\kappa_{\alpha}} \hookrightarrow \bigoplus_{\beta \in J(\alpha)} b_{\beta} \cdot 2^{\kappa_{\alpha}}$ . Hence  $\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}}$  is finitely measure isomorphic to  $\bigoplus_{\alpha} (\bigoplus_{\beta \in J(\alpha)} c_{\beta} \cdot 2^{\kappa_{\alpha}})$  and

$$\bigoplus_{\alpha} (\bigoplus_{\beta \in J(\alpha)} c_{\beta} \cdot 2^{\kappa_{\alpha}}) \hookrightarrow \bigoplus_{\alpha} (\bigoplus_{\beta \in J(\alpha)} b_{\beta} \cdot 2^{\kappa_{\alpha}}) \hookrightarrow \bigoplus_{\alpha} \bigoplus_{\beta \in J(\alpha)} b_{\beta} \cdot 2^{\kappa'_{\beta}},$$

where the final “ $\hookrightarrow$ ” follows from the fact that  $\kappa_{\alpha} \leq \kappa'_{\beta}$  for all  $\beta \in J(\alpha)$ . By Proposition 2 and the discussion above,  $L^{p,\infty}(\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}})$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_{\alpha} \bigoplus_{\beta \in J(\alpha)} b_{\beta} \cdot 2^{\kappa'_{\beta}})$ , which in turn is clearly isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_{\beta} b_{\beta} \cdot 2^{\kappa'_{\beta}})$ .  $\square$

By Proposition 4, given a purely nonatomic measure space, there exist an ordinal  $\tau$ , positive real numbers  $(a_{\alpha})_{\alpha < \tau}$  and infinite cardinals  $(\kappa_{\alpha})_{\alpha < \tau}$  so that  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isometrically lattice isomorphic to  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_{\alpha} \cdot 2^{\kappa_{\alpha}})$ . We may also assume that  $\kappa_{\alpha} \leq \kappa_{\beta}$  if  $\alpha \leq \beta < \tau$ . We now further refine this representation. Let  $\tau_1$  be the smallest ordinal such that  $\sum_{\tau_1 \leq \alpha < \tau} a_{\alpha} < \infty$  if such an ordinal exists. Otherwise, let  $\tau_1 = \tau$ . Write  $\tau_1 = \omega_1 \cdot \tau_2 + \gamma$ , where  $\gamma$  is a countable ordinal. Now  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isomorphic to the direct sum

$$(2) \quad L^{p,\infty}(\bigoplus_{\alpha < \omega_1 \cdot \tau_2} a_{\alpha} \cdot 2^{\kappa_{\alpha}}) \oplus L^{p,\infty}(\bigoplus_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} a_{\alpha} \cdot 2^{\kappa_{\alpha}}) \\ \oplus L^{p,\infty}(\bigoplus_{\tau_1 \leq \alpha < \tau} a_{\alpha} \cdot 2^{\kappa_{\alpha}}).$$

Any one (but not all) of the three terms may be trivial.

Let  $(\kappa_{\alpha})$  be a family of infinite cardinals. For each  $\alpha$ , the measure space  $2^{\kappa_{\alpha}}$  is measure isomorphic to  $2^{\kappa_{\alpha}} \oplus 2^{\kappa_{\alpha}}$ . Hence  $\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}}$  is finitely measure isomorphic to  $(\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}}) \oplus (\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}})$ . By Proposition 2,  $L^{p,\infty}(\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}})$  is isomorphic to  $L^{p,\infty}(\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}}) \oplus L^{p,\infty}(\bigoplus_{\alpha} a_{\alpha} \cdot 2^{\kappa_{\alpha}})$ . Combined with the representation (2), we see that  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isomorphic to its square  $L^{p,\infty}(\Omega, \Sigma, \mu) \oplus L^{p,\infty}(\Omega, \Sigma, \mu)$  for any purely nonatomic measure space  $(\Omega, \Sigma, \mu)$ .

**Lemma 6.** *Suppose that  $\eta$  is a nonzero ordinal and  $(\gamma_{\alpha})_{\alpha < \omega_1 \cdot \eta}$  is an increasing sequence of ordinals so that (i)  $\gamma_0 = 0$  and for all  $\alpha < \omega_1 \cdot \eta$ ,  $\gamma_{\alpha+1} = \gamma_{\alpha} + \lambda_{\alpha}$  for some countable ordinal  $\lambda_{\alpha}$ , and (ii)  $\gamma_{\alpha} = \sup_{\xi < \alpha} \gamma_{\xi}$  for all limit ordinals  $\alpha < \omega_1 \cdot \eta$ . Then  $\gamma_{\alpha} < \omega_1 \cdot \eta$  for all  $\alpha < \omega_1 \cdot \eta$ .*

*Proof.* Since the supremum of countably many countable ordinals is countable,  $\gamma_{\alpha} < \omega_1$  for all  $\alpha < \omega_1$ . It follows that the lemma holds for  $\eta = 1$ . Suppose that the lemma holds for all  $\eta < \eta_0$ . If  $\alpha < \omega_1 \cdot \eta_0$ , we can

write  $\alpha = \omega_1 \cdot \eta + \beta + m$ , where  $\eta < \eta_0$ ,  $\beta$  is either a countable limit ordinal or 0, and  $m < \omega$ . By the inductive hypothesis,  $\gamma_{\omega_1 \cdot \eta} \leq \omega_1 \cdot \eta$ . If  $\omega_1 \cdot \eta \leq \alpha' < \omega_1 \cdot \eta + \beta$ , let  $\gamma_{\alpha'} = \gamma_{\omega_1 \cdot \eta} + \xi_{\alpha'}$ . Then  $\xi_{\alpha'}$  is countable and hence  $\sup_{\omega_1 \cdot \eta \leq \alpha' < \omega_1 \cdot \eta + \beta} \xi_{\alpha'}$  is countable. Therefore,  $\gamma_{\omega_1 \cdot \eta + \beta} < \omega_1 \cdot (\eta + 1)$ . It follows that  $\gamma_\alpha < \omega_1 \cdot (\eta + 1) \leq \omega_1 \cdot \eta_0$ .  $\square$

Recall that a well known variant of Pełczyński's Decomposition Method states that if  $E$  and  $F$  are Banach spaces so that  $E$  is isomorphic to  $E \oplus E$ ,  $F$  is isomorphic to  $F \oplus F$ ,  $E$  is isomorphic to a complemented subspace of  $F$  and  $F$  is isomorphic to a complemented subspace of  $E$ , then  $E$  and  $F$  are isomorphic.

**Proposition 7.** *If  $\tau_2 > 0$ , then  $L^{p,\infty}(\bigoplus_{\alpha < \omega_1 \cdot \tau_2} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to  $L^{p,\infty}(\bigoplus_{\alpha < \omega_1 \cdot \tau_2} 2^{\kappa_\alpha})$ .*

*Proof.* Let  $(a_\alpha)_{\alpha < \omega_1 \cdot \tau_2}$  and  $(b_\alpha)_{\alpha < \omega_1 \cdot \tau_2}$  be any two transfinite sequences of positive real numbers. Set  $\gamma_0 = 0$ . If  $\gamma_\alpha < \omega_1 \cdot \tau_2$  has been chosen for some  $\alpha < \omega_1 \cdot \tau_2$ , there exists  $\gamma_{\alpha+1} = \gamma_\alpha + \lambda_\alpha$  for some countable  $\lambda_\alpha$  such that  $\sum_{\xi \in [\gamma_\alpha, \gamma_{\alpha+1})} b_\xi \geq a_\alpha$ . In particular,  $\gamma_{\alpha+1} < \omega_1 \cdot \tau_2$ . If  $\alpha < \omega_1 \cdot \tau_2$  is a limit ordinal and  $\gamma_\xi < \omega_1 \cdot \tau_2$  has been defined for all  $\xi < \alpha$ , let  $\gamma_\alpha = \sup_{\xi < \alpha} \gamma_\xi$ . By Lemma 6,  $\gamma_\alpha < \omega_1 \cdot \tau_2$ . Thus  $\gamma_\alpha$  is defined for all  $\alpha < \omega_1 \cdot \tau_2$ . Take  $J(\alpha)$  to be the set  $[\gamma_\alpha, \gamma_{\alpha+1})$  for each  $\alpha$ . Since  $\gamma_\alpha \geq \alpha$  for all  $\alpha$ ,  $\kappa_\beta \geq \kappa_{\gamma_0} \geq \kappa_\alpha$  for all  $\beta \in J(\alpha)$ . It follows from Proposition 5 that  $L^{p,\infty}(\bigoplus_{\alpha < \omega_1 \cdot \tau_2} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_{\alpha < \omega_1 \cdot \tau_2} b_\alpha \cdot 2^{\kappa_\alpha})$ . The conclusion of the proposition follows by symmetry and by Pełczyński's Decomposition Method.  $\square$

**Proposition 8.** *Suppose that  $\omega_1 \cdot \tau_2 < \tau_1$ .*

- (1) *If  $(\kappa_\alpha)_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1}$  has a maximum  $\kappa$ , then  $L^{p,\infty}(\bigoplus_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to the space  $L^{p,\infty}(\bigoplus_{n=1}^{\infty} 2^\kappa)$ ;*
- (2) *If  $(\kappa_\alpha)_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1}$  does not have a maximum, then for any sequence of cardinals  $\kappa'_1 < \kappa'_2 < \dots$  such that  $\sup_n \kappa'_n = \sup_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} \kappa_\alpha$ , the space  $L^{p,\infty}(\bigoplus_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to  $L^{p,\infty}(\bigoplus_{n=1}^{\infty} 2^{\kappa'_n})$ .*

*Proof.* (1) Since  $[\omega_1 \cdot \tau_2, \tau_1)$  is countable and each  $a_\alpha$  is finite, we can choose pairwise disjoint finite subsets of  $\mathbb{N}$ ,  $J(\alpha)$ ,  $\omega_1 \cdot \tau_2 \leq \alpha < \tau_1$ , so that  $|J(\alpha)| \geq a_\alpha$  for all  $\alpha$ . By Proposition 5,  $L^{p,\infty}(\bigoplus_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_{n=1}^{\infty} 2^\kappa)$ . On the other hand, let  $\tau' \in [\omega_1 \cdot \tau_2, \tau_1)$  be such that  $\kappa_{\tau'} = \kappa$ . By the definition of  $\tau_1$ ,  $\sum_{\tau' \leq \alpha < \tau_1} a_\alpha = \infty$ . Thus, there is a sequence  $(J(n))_{n \in \mathbb{N}}$  of pairwise disjoint subsets of  $[\tau', \tau_1)$  so that  $\sum_{\alpha \in J(n)} a_\alpha \geq 1$  for all  $n$ . By Proposition 5,  $L^{p,\infty}(\bigoplus_{n=1}^{\infty} 2^\kappa)$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} a_\alpha \cdot 2^{\kappa_\alpha})$ . The conclusion (1) follows by Pełczyński's Decomposition Method.

(2) Let  $j$  be an injection from  $[\omega_1 \cdot \tau_2, \tau_1)$  into  $\mathbb{N}$  and let  $(N_i)$  be a sequence of pairwise disjoint infinite subsets of  $\mathbb{N}$ . For each  $\alpha \in [\omega_1 \cdot \tau_2, \tau_1)$ , there exists a finite subset  $J(\alpha)$  of  $N_{j(\alpha)}$  such that  $\kappa'_n \geq \kappa_\alpha$  for all  $n \in J(\alpha)$  and

that  $|J(\alpha)| \geq a_\alpha$ . By Proposition 5,  $L^{p,\infty}(\oplus_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\oplus_{n=1}^\infty 2^{\kappa'_n})$ . Conversely, using the fact that  $\sum_{\tau' \leq \alpha < \tau_1} a_\alpha = \infty$  for all  $\tau' \in [\omega_1 \cdot \tau_2, \tau_1)$ , one can find pairwise disjoint intervals  $(J(n))_{n \in \mathbb{N}}$  in  $[\omega_1 \cdot \tau_2, \tau_1)$  such that  $\kappa_\alpha \geq \kappa'_n$  for all  $\alpha \in J(n)$  and that  $\sum_{\alpha \in J(n)} a_\alpha \geq 1$ . Again, by Proposition 5,  $L^{p,\infty}(\oplus_{n=1}^\infty 2^{\kappa'_n})$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\oplus_{\omega_1 \cdot \tau_2 \leq \alpha < \tau_1} a_\alpha \cdot 2^{\kappa_\alpha})$ . Apply Pełczyński's Decomposition Method to complete the proof.  $\square$

A similar idea applied to the last term in (2) results in a simplification of that term as well.

**Proposition 9.** *Suppose that  $\tau_1 < \tau$ .*

- (1) *If  $(\kappa_\alpha)_{\tau_1 \leq \alpha < \tau}$  has a maximum  $\kappa$ , then  $L^{p,\infty}(\oplus_{\tau_1 \leq \alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to the space  $L^{p,\infty}(2^\kappa)$ ;*
- (2) *If  $(\kappa_\alpha)_{\tau_1 \leq \alpha < \tau}$  does not have a maximum, then there exists a sequence of infinite cardinals  $\kappa'_1 \leq \kappa'_2 \leq \dots < \sup \kappa'_n = \sup \kappa_\alpha$  such that  $L^{p,\infty}(\oplus_{\tau_1 \leq \alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to  $L^{p,\infty}(\oplus_{n=1}^\infty 2^{-n} \cdot 2^{\kappa'_n})$ .*

*Proof.* (1) Suppose that  $\kappa = \kappa_{\alpha_0}$  and let  $b = \sum_{\tau_1 \leq \alpha < \tau} a_\alpha$ . Then  $b$  is finite by the choice of  $\tau_1$ . Clearly

$$a_{\alpha_0} \cdot 2^\kappa \hookrightarrow \oplus_{\tau_1 \leq \alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha} \hookrightarrow b \cdot 2^\kappa.$$

Since  $L^{p,\infty}(a \cdot 2^\kappa)$  is isomorphic to  $L^{p,\infty}(2^\kappa)$  for any positive real number  $a$ , the desired conclusion follows once again by Pełczyński's Decomposition Method.

(2) Let  $a = \sum_{\tau_1 \leq \alpha < \tau} a_\alpha$ . By definition of  $\tau_1$ ,  $a$  is a positive real number. If  $\lambda$  is an infinite cardinal, then  $(b+c) \cdot 2^\lambda$  is measure isomorphic to  $b \cdot 2^\lambda \oplus c \cdot 2^\lambda$  for any positive reals  $b$  and  $c$ . Splitting the appropriate terms  $a_\alpha \cdot 2^{\kappa_\alpha}$  in this manner if necessary, we may assume that there is an increasing sequence of ordinals  $(\beta_n)_{n=0}^\infty$  so that  $\beta_0 = \tau_1$  and  $\sum_{\beta_{n-1} \leq \alpha < \beta_n} a_\alpha = a/2^n$  for all  $n \in \mathbb{N}$ . The last condition ensures that  $\sup \beta_n = \tau$ . Define  $\kappa'_n = \kappa_{\beta_n}$  for all  $n \in \mathbb{N}$ . Then  $\sup \kappa'_n = \sup \kappa_\alpha > \kappa'_m$  for all  $m$ . We have the following chain of relationships:

$$\begin{aligned} \oplus_{\tau_1 \leq \alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha} &= \oplus_n \oplus_{\beta_{n-1} \leq \alpha < \beta_n} a_\alpha \cdot 2^{\kappa_\alpha} \hookrightarrow \oplus_n \oplus_{\beta_{n-1} \leq \alpha < \beta_n} a_\alpha \cdot 2^{\kappa_{\beta_n}} \\ &= \oplus_n \frac{a}{2^n} \cdot 2^{\kappa'_n} = \oplus_n \oplus_{\beta_n \leq \alpha < \beta_{n+1}} 2a_\alpha \cdot 2^{\kappa'_n} \\ &\hookrightarrow \oplus_n \oplus_{\beta_n \leq \alpha < \beta_{n+1}} 2a_\alpha \cdot 2^{\kappa_\alpha} \hookrightarrow \oplus_{\tau_1 \leq \alpha < \tau} 2a_\alpha \cdot 2^{\kappa_\alpha}. \end{aligned}$$

Since  $L^{p,\infty}(\oplus_{\tau_1 \leq \alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  and  $L^{p,\infty}(\oplus_{\tau_1 \leq \alpha < \tau} 2a_\alpha \cdot 2^{\kappa_\alpha})$  are isomorphic, Pełczyński's Decomposition Method yields that  $L^{p,\infty}(\oplus_{\tau_1 \leq \alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to  $L^{p,\infty}(\oplus_n \frac{a}{2^n} \cdot 2^{\kappa'_n})$ , which in turn is isomorphic to  $L^{p,\infty}(\oplus_n \frac{1}{2^n} \cdot 2^{\kappa'_n})$ .  $\square$

The representation given by (2) before Lemma 6, together with Propositions 7, 8 and 9, yield Theorem 1.

## 3. INVARIANTS

In this section, we only consider weak  $L^p$  spaces represented in the form  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ , where  $(a_\alpha)$  is a transfinite sequence of positive real numbers and  $(\kappa_\alpha)_{\alpha < \tau}$  is a nondecreasing sequence of infinite cardinals. Let  $\pi = \pi(\tau, (\kappa_\alpha), (a_\alpha))$  be a parameter that depends on the constants arising from the representation. We call  $\pi$  an *isomorphic invariant* if  $\pi(\tau, (\kappa_\alpha), (a_\alpha)) = \pi(\tau', (\kappa'_\alpha), (a'_\alpha))$  whenever  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  and  $L^{p,\infty}(\bigoplus_{\alpha < \tau'} a'_\alpha \cdot 2^{\kappa'_\alpha})$  are isomorphic as Banach spaces. In this section, we will show that the following parameters are isomorphic invariants. The symbol  $|\tau|$  denotes the cardinality of the ordinal  $\tau$ .

- (1)  $\max\{|\tau|, \sup \kappa_\alpha\}$ ;
- (2)  $\sup \kappa_\alpha$ ;
- (3) For  $2 \leq p < \infty$ ,  $\max\{|\tau|, \aleph_0\}$ .

Let us reiterate that the third parameter is only an isomorphic invariant for  $p$  in the range  $[2, \infty)$ . We will show in the next section that it fails to be an isomorphic invariant if  $1 < p < 2$ . Of course, if  $2 \leq p < \infty$ , the fact that the first parameter is an invariant is a consequence of the second and the third. However, the assertion is that the first parameter is an invariant for the entire range  $p \in (1, \infty)$ . It will also be shown that, subject to some constraints, whether the set of cardinals  $(\kappa_\alpha)_{\alpha < \tau}$  has a maximum element is also determined by the isomorphic class of the space  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ . In the rest of the section, we assume that  $1 < p < \infty$  unless expressly stated otherwise. The exception will only occur when we discuss the third parameter. By inequality (1) in §1, we have

$$\int_\sigma |f| d\mu \leq \|f\| \mu(\sigma)^{1-\frac{1}{p}} \leq (1 - \frac{1}{p})^{-1} \|f\| \mu(\sigma)^{1-\frac{1}{p}}$$

for all  $f \in L^{p,\infty}(\Omega, \Sigma, \mu)$  and all sets  $\sigma \in \Sigma$  of finite measure.

**Theorem 10.** *If  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to  $L^{p,\infty}(\bigoplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$ , then  $\max\{|\tau|, \sup \kappa_\alpha\} = \max\{|\tau'|, \sup \kappa'_\beta\}$ .*

*Proof.* Recall that  $\Sigma'_0$  denotes the set of all measurable subsets of  $\bigoplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta}$  with finite measure. Denote the measure on  $\bigoplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta}$  by  $\nu$ . For each  $\sigma \in \Sigma'_0$ ,  $x'_\sigma(f) = (\nu(\sigma))^{-1+1/p} \int_\sigma f d\nu$  defines a bounded linear functional on  $L^{p,\infty}(\bigoplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$ . Moreover,

$$(3) \quad \frac{1}{2} \|f\| \leq \sup_{\sigma \in \Sigma'_0} |x'_\sigma(f)| \leq (1 - \frac{1}{p})^{-1} \|f\|$$

for all  $f \in L^{p,\infty}(\bigoplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$ . There is a subset  $S'$  of  $\Sigma'_0$  of cardinality  $\max\{|\tau'|, \sup \kappa'_\beta\}$  so that for all  $\sigma \in \Sigma'_0$ ,  $\inf\{\nu(\sigma \Delta \theta) : \theta \in S'\} = 0$ . It follows that (3) holds with  $S'$  in place of  $\Sigma'_0$ . Transferring over to  $X = L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  via the assumed isomorphism and normalizing,

we obtain a normalized subset  $S$  of  $X'$ , the dual space of  $X$ , of cardinality  $\max\{|\tau'|, \sup \kappa'_\beta\}$  and  $c > 0$  so that  $\sup_{x' \in S} |x'(f)| > c\|f\|$  for all  $f \in X$ . Suppose that  $A$  is a subset of normalized elements of  $X$  of cardinality greater than  $\max\{|\tau'|, \sup \kappa'_\beta\}$ . For each  $f \in A$ , there exists  $x'_f \in S$  so that  $|x'_f(f)| > c$ . Now there is an  $x' \in S$  so that  $x'_f = x'$  for infinitely many  $f \in A$ . It follows that for any  $n \in \mathbb{N}$ , there is a subset  $F$  of  $A$  having exactly  $n$  elements and a choice of signs  $(\eta_f)_{f \in F}$  so that  $\|\sum_{f \in F} \eta_f f\| > nc$ .

Suppose that  $\tau > \max\{|\tau'|, \sup \kappa'_\beta\}$ . For each  $\alpha < \tau$ , let  $f_\alpha = a_\alpha^{-1/p} 1_\alpha$ , where  $1_\alpha$  denotes the characteristic function of the component  $a_\alpha \cdot 2^{\kappa_\alpha}$  in  $\bigoplus_{\gamma < \tau} a_\gamma \cdot 2^{\kappa_\gamma}$ . The elements  $(f_\alpha)_{\alpha < \tau}$  are pairwise disjoint and normalized. Hence  $\|\sum_{\alpha \in F} \eta_\alpha f_\alpha\| \leq n^{1/p}$  for any subset  $F$  of  $[0, \tau)$  with  $n$  elements and any choice of signs  $(\eta_\alpha)$ . This contradicts the previous paragraph.

Suppose that  $\sup_\alpha \kappa_\alpha > \max\{|\tau'|, \sup \kappa'_\beta\}$ . Choose  $\alpha_0 < \tau$  such that  $\kappa_{\alpha_0} > \max\{|\tau'|, \sup \kappa'_\beta\}$ . We identify  $\kappa_{\alpha_0}$  with the set of ordinals less than  $\kappa_{\alpha_0}$ . For each  $\gamma \in \kappa_{\alpha_0}$ , let  $\varepsilon_\gamma$  be the projection of  $2^{\kappa_{\alpha_0}}$  onto the  $\gamma$ -th component. We may regard  $\varepsilon_\gamma$  as a function on  $\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha}$  by defining it to be 0 on all components  $a_\alpha \cdot 2^{\kappa_\alpha}$ ,  $\alpha \neq \alpha_0$ . The set  $(a_{\alpha_0}^{-1/p} \varepsilon_\gamma)_{\gamma \in \kappa_{\alpha_0}}$  is a normalized set of functions in  $X$ . By Khintchine's inequality, it is equivalent to the unit vector basis of  $\ell^2(\kappa_{\alpha_0})$ . Once again, this contradicts the conclusion of the paragraph before last.

We have shown that  $\max\{|\tau|, \sup \kappa_\alpha\} \leq \max\{|\tau'|, \sup \kappa'_\beta\}$ . The desired conclusion follows by symmetry.  $\square$

For each  $\alpha_0 < \tau$ , let  $P_{\alpha_0} f$  denote the restriction of  $f \in L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  to the component  $a_{\alpha_0} \cdot 2^{\kappa_{\alpha_0}}$ .

**Lemma 11.** *Let  $\kappa$  be an uncountable cardinal and let  $(\kappa_\alpha)_{\alpha < \tau}$  be a set of infinite cardinals so that  $\sup_{\alpha \in A} \kappa_\alpha < \kappa$  for all countable subsets  $A$  of  $[0, \tau)$ . Suppose that  $(g_\gamma)_{\gamma \in \kappa}$  is a transfinite sequence in  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  that is dominated by the unit vector basis of  $\ell^2(\kappa)$ . For any countable subset  $A$  of  $[0, \tau)$ , there exists  $\gamma \in \kappa$  such that  $P_\alpha g_\gamma = 0$  for  $\alpha \in A$ . Consequently, there exists  $(\gamma_\eta)_{\eta < \omega_1}$  such that  $(g_{\gamma_\eta})_{\eta < \omega_1}$  is a pairwise disjoint sequence of functions.*

*Proof.* Assume that the first conclusion of the lemma fails for a countable subset  $A$  of  $[0, \tau)$ . For each  $\gamma$ , let  $h_\gamma$  be the restriction of  $g_\gamma$  to  $\bigoplus_{\alpha \in A} a_\alpha \cdot 2^{\kappa_\alpha}$ . Then  $(h_\gamma)_{\gamma \in \kappa}$  is a set of nonzero functions in  $L^{p,\infty}(\bigoplus_{\alpha \in A} a_\alpha \cdot 2^{\kappa_\alpha})$  that is dominated by the  $\ell^2(\kappa)$  basis. Note that  $\kappa > \kappa_0 \equiv \sup_{\alpha \in A} \kappa_\alpha \cdot |A|$ . There exists a subset  $S$  of cardinality  $\kappa_0$  in the dual  $X'$  of  $X = L^{p,\infty}(\bigoplus_{\alpha \in A} a_\alpha \cdot 2^{\kappa_\alpha})$  such that  $x'(h) = 0$  for all  $x' \in S$  implies  $h = 0$ . In particular, for each  $\gamma \in \kappa$ , there exists  $(x', n) \in S \times \mathbb{N}$  so that  $|x'(h_\gamma)| > 1/n$ . Since  $\kappa > |S \times \mathbb{N}|$ , there exists an infinite subset  $\Gamma$  of  $\kappa$  and an element  $(x'_0, n_0) \in S \times \mathbb{N}$  so that  $|x'_0(h_\gamma)| > 1/n_0$  for all  $\gamma \in \Gamma$ . For every finite subset  $F$  of  $\Gamma$ ,

$$\frac{|F|}{n_0} < x'_0 \left( \sum_{\gamma \in F} \operatorname{sgn} x'_0(h_\gamma) h_\gamma \right) \leq \|x'_0\| \left\| \sum_{\gamma \in F} \operatorname{sgn} x'_0(h_\gamma) h_\gamma \right\| \leq C|F|^{1/2}$$

for some fixed constant  $C$  since  $(h_\gamma)_{\gamma \in \kappa}$  is dominated by the unit vector basis of  $\ell^2(\kappa)$ . This is clearly impossible.

The second statement of the lemma follows from the first by induction. Indeed, choose  $\gamma_0$  arbitrarily. Assume that  $\gamma_\eta$  has been chosen for all  $\eta < \rho$  for some  $\rho < \omega_1$ . For each  $\gamma$ ,  $g_\gamma$  has  $\sigma$ -finite support and hence  $\{\alpha : P_\alpha g_\gamma \neq 0\}$  is countable. Thus  $A = \{\alpha : P_\alpha g_{\gamma_\eta} \neq 0 \text{ for some } \eta < \rho\}$  is countable. By the first part of the lemma, there exists  $\gamma_\rho$  such that  $P_\alpha g_{\gamma_\rho} = 0$  for all  $\alpha \in A$ . This completes the inductive choice of the sequence  $(\gamma_\eta)_{\eta < \omega_1}$ . It is clear from the inductive definition that the sequence consists of pairwise disjoint functions.  $\square$

**Lemma 12.** *Let  $(\Omega, \Sigma, \mu)$  be a measure space. Suppose that  $p \neq 2$ . Then no transfinite pairwise disjoint sequence  $(g_\gamma)_{\gamma < \omega_1}$  of length  $\omega_1$  can be equivalent in  $L^{p,\infty}(\Omega, \Sigma, \mu)$  to the unit vector basis of  $\ell^2(\omega_1)$ .*

*Proof.* Suppose that  $(g_\gamma)_{\gamma < \omega_1}$  is a pairwise disjoint sequence in  $L^{p,\infty}(\Omega, \Sigma, \mu)$  that is equivalent to the unit vector basis of  $\ell^2(\omega_1)$ . Every normalized pairwise disjoint sequence in  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is dominated by the unit vector basis of  $\ell^p$  of the appropriate dimension. On the other hand,  $(g_\gamma)_{\gamma < \omega_1}$  is equivalent to the unit vector basis of  $\ell^2(\omega_1)$ . Thus we must have  $p \leq 2$ . On the other hand,  $(\|g_\gamma\|)_{\gamma < \omega_1}$  is bounded away from 0. There exists  $\delta > 0$  so that for each  $\gamma$ , there exists a rational number  $c_\gamma$  such that  $c_\gamma(\mu\{|g_\gamma| > c_\gamma\})^{1/p} > \delta$ . Since  $\omega_1$  is uncountable, there exist  $c > 0$  and an infinite subset  $\Gamma$  of  $[0, \omega_1)$  so that  $c(\mu\{|g_\gamma| > c\})^{1/p} > \delta$  for all  $\gamma \in \Gamma$ . Recall that the sets  $\{|g_\gamma| > c\}$ ,  $\gamma < \omega_1$ , are pairwise disjoint. For any finite subset  $F$  of  $\Gamma$ ,

$$c(\mu\{|\sum_{\gamma \in F} g_\gamma| > c\})^{1/p} = c(\sum_{\gamma \in F} \mu\{|g_\gamma| > c\})^{1/p} > \delta|F|^{1/p}.$$

Hence  $\|\sum_{\gamma \in F} g_\gamma\| \geq \delta|F|^{1/p}$ . But  $(g_\gamma)_{\gamma \in \kappa}$  is equivalent to the unit vector basis of  $\ell^2(\kappa)$ . Thus,  $p \geq 2$ . Since we are assuming that  $p \neq 2$ , we have a contradiction.  $\square$

Denote by  $M^{p,\infty}(2^\kappa)$  the closure of  $L^\infty(2^\kappa)$  in the space  $L^{p,\infty}(2^\kappa)$ .

**Lemma 13.** *Let  $\kappa$  be an uncountable cardinal and let  $(\kappa_\alpha)_{\alpha < \tau}$  be a set of infinite cardinals so that  $\sup_{\alpha \in A} \kappa_\alpha < \kappa$  for all countable subsets  $A$  of  $[0, \tau)$ . Then  $M^{p,\infty}(2^\kappa)$  does not embed isomorphically into  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ .*

*Proof.* We first consider the case where  $p \neq 2$ . The sequence of Rademacher functions  $(\varepsilon_\gamma)_{\gamma \in \kappa}$  in  $M^{p,\infty}(2^\kappa)$  is equivalent to the  $\ell^2(\kappa)$  basis. If  $M^{p,\infty}(2^\kappa)$  embeds isomorphically in  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ , then the latter space contains a sequence  $(g_\gamma)_{\gamma \in \kappa}$  equivalent to the  $\ell^2(\kappa)$  basis. By Lemma 11, there is a sequence  $(\gamma_\eta)_{\eta < \omega_1}$  such that  $(g_{\gamma_\eta})_{\eta < \omega_1}$  is a pairwise disjoint sequence of functions. However, this contradicts Lemma 12.

Now suppose that  $T : M^{2,\infty}(2^\kappa) \rightarrow L^{2,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  is an isomorphic embedding. Let  $(\varepsilon_\gamma)_{\gamma \in \kappa}$  be the sequence of Rademacher functions in

$M^{2,\infty}(2^\kappa)$ . Let  $h_0$  be the constant function 1 on  $2^\kappa$ . There is a countable subset  $A_0$  of  $[0, \tau)$  such that  $P_\alpha Th_0 = 0$  if  $\alpha \notin A_0$ . Since  $(T\varepsilon_\gamma)_{\gamma \in \kappa}$  is equivalent to the unit vector basis of  $\ell^2(\kappa)$ , by Lemma 11, there exists  $\gamma_1$  so that  $P_\alpha T\varepsilon_{\gamma_1} = 0$  for all  $\alpha \in A_0$ . Set  $h_1 = \varepsilon_{\gamma_1}$ . Next, there exists a countable subset  $A_1$  of  $[0, \tau)$  such that  $P_\alpha Th_i = 0$  if  $i = 0, 1$  and  $\alpha \notin A_1$ . Then one can find  $\gamma_2 \neq \gamma_1$  so that  $P_\alpha T(1_{\{h_1=-1\}} \cdot \varepsilon_{\gamma_2}) = 0$  for all  $\alpha \in A_1$ . Set  $h_2 = 1_{\{h_1=-1\}} \cdot \varepsilon_{\gamma_2}$ . Choose a countable subset  $A_2$  of  $[0, \tau)$  such that  $P_\alpha Th_i = 0$  if  $i = 0, 1, 2$  and  $\alpha \notin A_2$ . Then one can find  $\gamma_3 \notin \{\gamma_1, \gamma_2\}$  so that  $P_\alpha T(1_{\{h_1=1\}} \cdot \varepsilon_{\gamma_3}) = 0$  for all  $\alpha \in A_2$ . Set  $h_3 = 1_{\{h_1=1\}} \cdot \varepsilon_{\gamma_3}$ . Continuing in this manner, we obtain a sequence  $(h_i)$  in  $M^{2,\infty}(2^\kappa)$  equivalent to the sequence of Haar functions in  $M^{2,\infty}[0, 1]$ . At the same time,  $(h_i)$  is equivalent to the pairwise disjoint sequence  $(Th_i)$  in  $L^{2,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ . This is impossible according to [8, Proposition 8].  $\square$

**Theorem 14.** *If  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to  $L^{p,\infty}(\oplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$ , then  $\sup \kappa_\alpha = \sup \kappa'_\beta$ .*

*Proof.* By symmetry, it suffices to show that  $\sup \kappa'_\beta \leq \sup \kappa_\alpha$ . Assume on the contrary that there exists  $\beta_0$  with  $\kappa'_{\beta_0} > \sup \kappa_\alpha$ . Note that, in particular,  $\kappa'_{\beta_0} > \aleph_0$ . By Lemma 13,  $L^{p,\infty}(2^{\kappa'_{\beta_0}})$  does not embed isomorphically into  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ . Consequently, the spaces  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  and  $L^{p,\infty}(\oplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$  cannot be isomorphic.  $\square$

We say that an infinite cardinal  $\kappa$  has *uncountable cofinality* if it is not equal to the supremum of a countable set of smaller cardinals.

**Theorem 15.** *Suppose that  $(\kappa_\alpha)_{\alpha < \tau}$  and  $(\kappa'_\beta)_{\beta < \tau'}$  are sequences of cardinals with uncountable cofinality. If the spaces  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  and  $L^{p,\infty}(\oplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$  are isomorphic and the sequence  $(\kappa'_\beta)_{\beta < \tau'}$  has a maximum, then so does the sequence  $(\kappa_\alpha)_{\alpha < \tau}$ .*

*Proof.* Let  $\kappa = \max_{\beta < \tau'} \kappa'_\beta$ . Assume that the spaces  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  and  $L^{p,\infty}(\oplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$  are isomorphic. Then  $\kappa = \sup_{\alpha < \tau} \kappa_\alpha$  by Theorem 14. Suppose that the sequence  $(\kappa_\alpha)_{\alpha < \tau}$  does not have a maximum. Then  $\kappa > \kappa_\alpha$  for all  $\alpha < \tau$ . Since  $\kappa$  is assumed to have uncountable cofinality,  $\kappa > \sup_{\alpha \in A} \kappa_\alpha$  for all countable subsets  $A$  of  $[0, \tau)$ . By Lemma 13,  $L^{p,\infty}(2^\kappa)$  does not embed isomorphically into  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ . This contradicts the isomorphism of the spaces  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  and  $L^{p,\infty}(\oplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$ .  $\square$

Let  $\kappa$  be an infinite cardinal. We have already encountered the projections  $\varepsilon_\gamma : 2^\kappa \rightarrow \{-1, 1\}$ ,  $\gamma \in \kappa$  (Rademacher functions). For any finite subset  $F$  of  $\kappa$ , we define the *Walsh function*  $W_F$  to be  $\prod_{\gamma \in F} \varepsilon_\gamma$ , where the empty product is taken to be the constant function 1. Denote the set of all Walsh functions  $W_F$ ,  $|F| \leq n$ , by  $\mathcal{W}_n$ .

**Lemma 16.** *Let  $1 \leq q < \infty$ . From any infinite set of Walsh functions in  $\mathcal{W}_n$ , one can extract an infinite sequence that is equivalent in the norm of  $L^q(2^\kappa)$  to the  $\ell^2$  basis.*

*Proof.* We prove the lemma by induction on  $n$ . The case  $n = 1$  is Khintchine's inequality. Assume that the lemma holds for some  $n$ . Let  $(F_k)$  be an infinite sequence of pairwise distinct finite subsets of  $\kappa$ , with  $|F_k| \leq n + 1$  for all  $k$ . By considering a suitable subsequence, we may assume that either  $(F_k)$  is pairwise disjoint, or that there exists  $\gamma \in \cap F_k$ . In the former case,  $(F_k)$  has the same joint distribution as the Rademacher functions and the result follows from Khintchine's inequality. In the latter case, for any finitely supported sequence  $(a_k)$ ,

$$\left\| \sum a_k W_{F_k} \right\| = \left\| \varepsilon_\gamma \sum a_k W_{F_k \setminus \{\gamma\}} \right\| = \left\| \sum a_k W_{F_k \setminus \{\gamma\}} \right\|$$

and the desired result follows by the inductive assumption.  $\square$

**Lemma 17.** *If  $f \in L^{p,\infty}(2^\kappa)$ , then  $\int f W_F = 0$  for all but countably many Walsh functions  $W_F$ .*

*Proof.* Suppose that the lemma fails. There exist  $f \in L^{p,\infty}(2^\kappa)$ , a rational  $\delta > 0$ ,  $n \in \mathbb{N}$  and an infinite set  $(W_{F_k})$  in  $\mathcal{W}_n$  so that  $|\int f W_{F_k}| > \delta$  for all  $k$ . Choose  $1 < p' < p$  and let  $q' = \frac{p'}{p'-1}$ . By Lemma 16, considered as a sequence in  $L^{q'}(2^\kappa)$ ,  $(W_{F_k})$  has a subsequence equivalent to the  $\ell^2$  basis. We may assume that  $(W_{F_k}) \subseteq L^{q'}(2^\kappa)$  is equivalent to the  $\ell^2$  basis. Let  $\eta_k = \text{sgn} \int f W_{F_k}$ . There is a constant  $C < \infty$  so that for any  $m \in \mathbb{N}$ ,

$$m\delta < \left| \int f \sum_{k=1}^m \eta_k W_{F_k} \right| \leq \|f\|_{p'} \left\| \sum_{k=1}^m \eta_k W_{F_k} \right\|_{q'} \leq C\sqrt{m} \|f\|_{p'}.$$

This is impossible since  $L^{p,\infty}(2^\kappa) \subseteq L^{p'}(2^\kappa)$ .  $\square$

Corresponding to each Walsh function  $W_F$ , we define a bounded linear functional  $x'_F$  on  $L^{p,\infty}(2^\kappa)$  via  $x'_F(f) = \int f W_F$ . Note that  $\|x'_F\| \leq 1 - \frac{1}{p}$ . A well known fact, easily verified, is that for every finite subset  $F$  of  $\kappa$ , each function on  $2^\kappa$  that is measurable with respect to the set of coordinates  $F$  lies in the span of  $\{W_G : G \subseteq F\}$ . As a result, the only function  $f \in L^{p,\infty}(2^\kappa)$  that satisfies  $x'_F(f) = 0$  for all finite subsets  $F$  of  $\kappa$  is the 0 function. For any set  $\Xi$ ,  $\ell^{p,\infty}(\Xi)$  is the weak  $L^p$  space defined on the measure space consisting of the set  $\Xi$  endowed with the counting measure.

**Lemma 18.** *Suppose that  $2 \leq p < \infty$  and that  $(f_\xi)_{\xi \in \Xi}$  is a set of functions in  $L^{p,\infty}(2^\kappa)$  dominated by the set of coordinate unit vectors in  $\ell^{p,\infty}(\Xi)$ . The set*

$$\Xi' = \left\{ \xi : \int f_\xi W_F \neq 0 \text{ for some finite } F \subseteq \kappa \right\}$$

*is countable.*

*Proof.* Suppose that the set  $\Xi'$  is uncountable. We will select inductively a transfinite sequence  $(\xi_\gamma)_{\gamma < \omega_1}$  from  $\Xi'$  so that the sets

$$\mathcal{F}_\gamma = \{F : F \text{ is a finite subset of } \kappa \text{ with } \int f_{\xi_\gamma} W_F \neq 0\}$$

are pairwise disjoint. Choose  $\xi_0 \in \Xi'$  arbitrarily. Assume that  $\zeta_0 < \omega_1$  and that  $\xi_\zeta$  has been chosen for all  $\zeta < \zeta_0$ . Since  $\zeta_0$  is countable, by Lemma 17, the set  $\cup_{\zeta < \zeta_0} \mathcal{F}_{\gamma_\zeta}$  is countable. Suppose that for each  $\xi \in \Xi' \setminus \{\xi_\zeta : \zeta < \zeta_0\}$ , there exists  $F \in \cup_{\zeta < \zeta_0} \mathcal{F}_{\gamma_\zeta}$  such that  $\int f_\xi W_F \neq 0$ . Then there are a particular  $F \in \cup_{\zeta < \zeta_0} \mathcal{F}_{\gamma_\zeta}$  and a rational  $\delta > 0$  so that  $|\int f_\xi W_F| > \delta$  for all  $\xi$  in an infinite subset  $\Xi''$  of  $\Xi' \setminus \{\xi_\zeta : \zeta < \zeta_0\}$ . If  $A$  is a finite subset of  $\Xi''$  and  $\eta_\xi = \text{sgn} \int f_\xi W_F$ , then

$$\delta|A| < \sum_{\xi \in A} \eta_\xi \int f_\xi W_F = x'_F \left( \sum_{\xi \in A} \eta_\xi f_\xi \right) \leq \left(1 - \frac{1}{p}\right) \left\| \sum_{\xi \in A} \eta_\xi f_\xi \right\|.$$

This contradicts the fact that  $(f_\xi)_{\xi \in \Xi}$  is dominated by the set of coordinate unit vectors in  $\ell^{p,\infty}(\Xi)$ . Thus we can choose  $\xi_{\zeta_0} \in \Xi' \setminus \{\xi_\zeta : \zeta < \zeta_0\}$  so that  $\int f_{\xi_{\zeta_0}} W_F \neq 0$  for all  $F \in \cup_{\zeta < \zeta_0} \mathcal{F}_{\gamma_\zeta}$ . The inductive selection is complete.

For each  $\gamma < \omega_1$ , choose  $F_\gamma \in \mathcal{F}_\gamma$  so that  $\int f_{\xi_\gamma} W_{F_\gamma} \neq 0$ . There exists a rational  $r > 0$  and an infinite subset  $\Gamma$  of  $\omega_1$  so that  $|\int f_{\xi_\gamma} W_{F_\gamma}| > r$  for all  $\gamma \in \Gamma$ . Since the sets  $\mathcal{F}_\gamma$  are pairwise disjoint, the sets  $F_\gamma$  are pairwise distinct. Let  $1 < p' < p$  and  $q' = \frac{p'}{p'-1}$ . By Lemma 16, there is a sequence  $(\gamma_k)$  in  $\Gamma$  so that, in the norm of  $L^{q'}(2^\kappa)$ ,  $(W_{F_{\gamma_k}})$  is equivalent to the  $\ell^2$  basis. Since the formal inclusion  $L^{p'}(2^\kappa) \subseteq L^{p,\infty}(2^\kappa)$  is bounded, it follows that the sequence  $(x'_{F_{\gamma_k}})$  is dominated by  $(W_{F_{\gamma_k}}) \subseteq L^{q'}(2^\kappa)$ , and hence dominated by the  $\ell^2$  basis. Let  $(b_k)$  and  $(c_k)$  be finitely supported real sequences. For each  $k$ , let  $s_k = \text{sgn}(b_k c_k \int f_{\xi_{\gamma_k}} W_{F_{\gamma_k}})$ . Note that if  $k \neq j$ , then  $F_k \notin \mathcal{F}_{\gamma_j}$  and hence  $\int f_{\xi_{\gamma_j}} W_{F_k} = 0$ . Now

$$\begin{aligned} r \left| \sum b_k c_k \right| &< \sum s_k b_k c_k \int f_{\xi_{\gamma_k}} W_{F_{\gamma_k}} \\ &= \sum s_k b_k x'_{F_{\gamma_k}} \left( \sum_j c_j f_{\xi_{\gamma_j}} \right) \\ &\leq \left\| \sum s_k b_k x'_{F_{\gamma_k}} \right\| \left\| \sum_j c_j f_{\xi_{\gamma_j}} \right\| \\ &\leq C \|(b_k)\|_2 \left\| \sum_j c_j f_{\xi_{\gamma_j}} \right\| \end{aligned}$$

for some fixed constant  $C$ . This implies that  $\left\| \sum_j c_j f_{\xi_{\gamma_j}} \right\| \geq \frac{r}{C} \|(c_j)\|_2$ . We have a contradiction since  $(f_{\xi_{\gamma_j}})$  is dominated by the unit vectors in  $\ell^{p,\infty}$  and  $2 \leq p < \infty$ .  $\square$

**Theorem 19.** *Suppose that  $2 \leq p < \infty$ . If  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  is isomorphic to  $L^{p,\infty}(\oplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$ , then  $\max\{|\tau|, \aleph_0\} = \max\{|\tau'|, \aleph_0\}$ .*

*Proof.* By symmetry, it suffices to show that  $\max\{|\tau'|, \aleph_0\} \leq \max\{|\tau|, \aleph_0\}$ . Assume on the contrary that  $|\tau'| > \max\{|\tau|, \aleph_0\}$ . There exists  $n \in \mathbb{N}$  such that  $|\Xi| > \max\{|\tau|, \aleph_0\}$ , where  $\Xi = \{\beta < \tau' : 1/n \leq a'_\beta \leq n\}$ . Note that  $\ell^{p,\infty}(\Xi)$  is isomorphic to a subspace of  $L^{p,\infty}(\oplus_{\beta < \tau'} a'_\beta \cdot 2^{\kappa'_\beta})$  and hence to a subspace of  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ . Let  $(f_\xi)_{\xi \in \Xi}$  be a set of functions in  $L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  equivalent to the set of coordinate unit vectors in  $\ell^{p,\infty}(\Xi)$ . For each  $\alpha_0 < \tau$ , let  $P_{\alpha_0} f$  be the restriction of  $f \in L^{p,\infty}(\oplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$  to the component  $a_{\alpha_0} \cdot 2^{\kappa_{\alpha_0}}$ . By Lemma 18, for each  $\alpha < \tau$ , there is a countable subset  $\Xi_\alpha$  of  $\Xi$  so that  $\int P_\alpha f_\xi W_F = 0$  for all  $\xi \in \Xi \setminus \Xi_\alpha$  and all Walsh functions  $W_F$ . Now  $|\cup_{\alpha < \tau} \Xi_\alpha| \leq |\tau| \cdot \aleph_0 < |\Xi|$ . Thus there exists  $\xi \in \Xi \setminus \cup_{\alpha < \tau} \Xi_\alpha$ . For this  $\xi$ ,  $P_\alpha f_\xi = 0$  for all  $\alpha < \tau$  and hence  $f_\xi = 0$ , which is absurd.  $\square$

**Remark.** It is known [6] that  $L^{p,\infty}(\oplus_{\alpha < \omega} 2^{\aleph_0})$  is isomorphic to  $L^{p,\infty}(2^{\aleph_0})$ . Thus it is not possible in Theorem 19 to conclude that  $|\tau| = |\tau'|$ .

#### 4. THE BIG SQUEEZE: $1 < p < 2$

In this section, we will show that the third invariant from the last section does not apply in the range  $1 < p < 2$ ; equivalently, Theorem 19 does not hold for  $1 < p < 2$ . For a Banach space  $E$ , denote by  $\ell^\infty(E)$  the space of all bounded sequences  $(x_n)$  in  $E$  with the norm  $\|(x_n)\| = \sup_n \|x_n\|$ . We will first show that for  $1 < p < 2$  and any infinite cardinal  $\kappa$ ,  $\ell^\infty(\ell^{p,\infty}(\kappa))$  is isomorphic to a complemented subspace of  $L^{p,\infty}(2^\kappa)$ . Then, viewing  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$  as a “limit” of the spaces  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^n)$ , it is shown that  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$  is isomorphic to a complemented subspace of  $\ell^\infty(\ell^{p,\infty}(\kappa))$ . It is then easy to deduce that the spaces  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0}) \oplus L^{p,\infty}(2^\kappa)$  and  $L^{p,\infty}(2^\kappa)$  are isomorphic. The main idea is to use probabilistic independence to replicate disjointness (in the lattice sense). The strategy has been used, for example, to embed  $\ell^q$  isometrically isomorphically into  $L^p$ ,  $1 \leq p < q < 2$ . In our case, simply using iid random variables will not do the job. (This assertion can be formulated in a precise way.) Instead, we use random variables that are independent in sections. The argument relies vitally on certain norm estimates of square functions in Lorentz spaces due to Carothers and Dilworth [2, 3, 4]. A set of random variables  $(f_i)$  on  $[0, 1]$  is said to be *symmetric* if for each finite subset  $F$ , the joint distribution of  $(f_i)_{i \in F}$  is the same as that of  $(\pm f_i)_{i \in F}$ .

**Theorem 20.** [2, Lemma 2.2] *Let  $X$  be a rearrangement invariant space on  $[0, 1]$ . There is a finite positive constant  $D$  so that for every symmetric sequence  $(f_i)$  in  $X$ , and all scalars  $(a_i)_{i=1}^n$ ,*

$$D^{-1} \left\| \left( \sum_{i=1}^n |a_i f_i|^2 \right)^{1/2} \right\| \leq \left\| \sum_{i=1}^n a_i f_i \right\| \leq D \left\| \left( \sum_{i=1}^n |a_i f_i|^2 \right)^{1/2} \right\|.$$

For the remainder of this section, let  $1 < p < 2$  and let  $\kappa$  be an infinite cardinal. Denote by  $\mu$  the usual product measure on  $2^\kappa$ . Given a finite set of functions  $f_1, \dots, f_n$  in  $L^{p,\infty}(2^\kappa)$ , the *disjoint sum*  $\sum_{i=1}^n \oplus f_i$  is any function  $f$  on  $L^{p,\infty}[0, \infty)$  so that  $\lambda\{|f| > t\} = \sum_{i=1}^n \mu\{|f_i| > t\}$  for all  $t > 0$ , where  $\lambda$  denotes Lebesgue measure.

**Theorem 21.** [3, Corollary 2.7] *For  $1 < p < 2$ , there is a finite positive constant  $C$  such that*

$$\left\| \left( \sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_{p,\infty} \leq C \left\| \sum_{i=1}^n \oplus f_i \right\|_{p,\infty}$$

for any  $f_1, \dots, f_n \in L^{p,\infty}[0, 1]$ .

For any probability space  $(\Omega, \Sigma, \nu)$ , Theorems 20 and 21 apply to functions in  $L^{p,\infty}(\Omega, \Sigma, \nu)$ , since any finite set of measurable functions on  $(\Omega, \Sigma, \nu)$  has a copy on  $[0, 1]$  with the same joint distribution. Let  $\eta_0 > 0$  be an absolute constant so that  $1 - e^{-x} \geq x/2$  for all  $x \in [0, \eta_0]$ .

**Proposition 22.** *Suppose that  $N \in \mathbb{N}$ ,  $0 < b < 1/2$ ,  $Nb \leq \eta_0$  and that  $(f_\alpha)_{\alpha < \kappa}$  is a set of iid random variables in  $L^{p,\infty}(2^\kappa)$  so that each  $f_\alpha$  has the same distribution as  $b^{-1/p}(1_{[0,b/2)} - 1_{[b/2,b)})$ . Then for all scalars  $(a_\alpha)$  and all subsets  $J$  of  $\kappa$  of cardinality at most  $N$ ,*

$$2^{-1/p} \|(a_\alpha)_{\alpha \in J}\|_{p,\infty} = 2^{-1/p} \left\| \sum_{\alpha \in J} \oplus a_\alpha f_\alpha \right\| \leq \left\| \left( \sum_{\alpha \in J} |a_\alpha f_\alpha|^2 \right)^{1/2} \right\|.$$

*Proof.* Let  $I$  be a finite subset of  $\kappa$  with cardinality at most  $N$ . For any  $0 < c < b^{-1/p}$ ,

$$\mu(\cap_{\alpha \in I} \{|f_\alpha| \leq c\}) = \prod_{\alpha \in I} \mu\{|f_\alpha| \leq c\} = (1 - b)^{|I|} \leq e^{-b|I|}.$$

Hence

$$\mu(\cup_{\alpha \in I} \{|f_\alpha| > c\}) \geq 1 - e^{-b|I|} \geq \frac{b|I|}{2}$$

since  $b|I| < \eta_0$ . Then

$$\begin{aligned} \left\| \left( \sum_{\alpha \in I} |f_\alpha|^2 \right)^{1/2} \right\| &\geq \sup_{0 < c < b^{-1/p}} c (\mu\{(\sum_{\alpha \in I} |f_\alpha|^2)^{1/2} > c\})^{1/p} \\ &\geq b^{-1/p} \left( \frac{b|I|}{2} \right)^{1/p} = 2^{-1/p} |I|^{1/p}. \end{aligned}$$

It is clear that  $\left\| \sum_{\alpha \in J} \oplus a_\alpha f_\alpha \right\| = \|(a_\alpha)_{\alpha \in J}\|_{p,\infty}$ . Thus, if  $\left\| \sum_{\alpha \in J} \oplus a_\alpha f_\alpha \right\| > 1$ , then there exists a subset  $I$  of  $J$  so that  $|a_\alpha| \geq |I|^{-1/p}$  for all  $\alpha \in I$ . Therefore,

$$\left\| \left( \sum_{\alpha \in J} |a_\alpha f_\alpha|^2 \right)^{1/2} \right\| \geq |I|^{-1/p} \left\| \left( \sum_{\alpha \in I} |f_\alpha|^2 \right)^{1/2} \right\| \geq 2^{-1/p}.$$

□

Choose positive sequences  $(w_n)$  and  $(b_n)$  so that

- (1)  $\sum w_n = 1$ ,
- (2)  $\sum_{n=m+1}^{\infty} w_n \leq w_m b_m$  for all  $m$ ,
- (3)  $n b_n \leq \eta_0$  for all  $n$ .

For each  $n \in \mathbb{N}$ , let  $(\tilde{F}_\alpha(n))_{\alpha \in \kappa}$  be iid random variables on  $2^\kappa$  (with respect to the product measure  $\mu$ ) so that each  $\tilde{F}_\alpha(n)$  has the same distribution as  $(w_n b_n)^{-1/p} (1_{[0, b_n/2]} - 1_{[b_n/2, b_n]})$ . Let  $F_\alpha(n)$  be formally the same function as  $\tilde{F}_\alpha(n)$ , but regarded as a function on the space  $w_n \cdot 2^\kappa$ . We view  $2^\kappa$  as the direct sum  $\oplus_n w_n \cdot 2^\kappa$ . With respect to the direct sum, a function  $f$  on  $2^\kappa$  will be written as  $\oplus f_n$ , where each  $f_n$  is a function on the component  $w_n \cdot 2^\kappa$ . In the proof of the next proposition, we will use the fact that since  $1 < p < 2$ , any  $(a_\alpha)$  in the ball of  $\ell^{p, \infty}(\kappa)$  satisfies  $(\sum |a_\alpha|^2)^{1/2} \leq (\sum k^{-2/p})^{1/2} \leq (\frac{2}{2-p})^{1/2}$ .

**Proposition 23.** *Assume that for each  $n \in \mathbb{N}$ , a real sequence  $(a_\alpha(n))_{\alpha \in \kappa}$  is given so that  $\sup_n \|(a_\alpha(n))_{\alpha \in \kappa}\|_{p, \infty} < \infty$ . Let  $I$  be a finite subset of  $\kappa$  and set  $f = \oplus_n \sum_{\alpha \in I} a_\alpha(n) F_\alpha(n)$ . Then*

$$2^{-1/p} D^{-1} \sup_{n \geq |I|} \|(a_\alpha(n))_{\alpha \in I}\|_{p, \infty} \leq \|f\| \leq C' D \sup_{n \in \mathbb{N}} \|(a_\alpha(n))_{\alpha \in I}\|_{p, \infty},$$

where  $C' = [C^p + (\frac{2}{2-p})^{p/2}]^{1/p}$  and  $D, C$  are the constants from Theorems 20 and 21 respectively.

*Proof.* By homogeneity, it suffices to prove the proposition assuming that  $\sup_n \|(a_\alpha(n))_{\alpha \in I}\|_{p, \infty} = 1$ . For each  $\alpha \in \kappa$ , let  $G_\alpha = \oplus_n a_\alpha(n) F_\alpha(n)$ . First we estimate  $\|(\sum_{\alpha \in I} |G_\alpha|^2)^{1/2}\|$ . Given  $c > 0$ , let  $n_0$  be the smallest natural number so that  $\mu\{(\sum_{\alpha \in I} |a_\alpha(n_0) F_\alpha(n_0)|^2)^{1/2} > c\} \neq 0$ . In particular,

$$c < (\sum |a_\alpha(n_0)|^2)^{1/2} (w_{n_0} b_{n_0})^{-1/p} \leq (\frac{2}{2-p})^{1/2} (w_{n_0} b_{n_0})^{-1/p}.$$

By Theorem 21,

$$\begin{aligned} \|(\sum_{\alpha \in I} |a_\alpha(n_0) F_\alpha(n_0)|^2)^{1/2}\| &\leq C \|\sum_{\alpha \in I} \oplus a_\alpha(n_0) F_\alpha(n_0)\| \\ &\leq C \|(a_\alpha(n_0))\|_{p, \infty} \leq C \end{aligned}$$

and hence

$$\mu\{(\sum_{\alpha \in I} |a_\alpha(n_0) F_\alpha(n_0)|^2)^{1/2} > c\} \leq (\frac{C}{c})^p.$$

Now

$$\begin{aligned} \mu\{(\sum_{\alpha \in I} |G_\alpha|^2)^{1/2} > c\} &\leq \mu\{(\sum_{\alpha \in I} |a_\alpha F_\alpha(n_0)|^2)^{1/2} > c\} + \sum_{n=n_0+1}^{\infty} w_n \\ &\leq (\frac{C}{c})^p + w_{n_0} b_{n_0} \leq (\frac{C}{c})^p + (\frac{2}{2-p})^{p/2} \frac{1}{c^p}. \end{aligned}$$

Thus

$$c(\mu\{(\sum_{\alpha \in I} |G_\alpha|^2)^{1/2} > c\})^{1/p} \leq [C^p + (\frac{2}{2-p})^{p/2}]^{1/p}.$$

Since  $c > 0$  is arbitrary, we have  $\|(\sum_{\alpha \in I} |G_\alpha|^2)^{1/2}\| \leq C'$ . In particular, this shows that  $G_\alpha \in L^{p,\infty}(2^\kappa)$  for all  $\alpha$  and hence  $f \in L^{p,\infty}(2^\kappa)$ . Since  $(G_\alpha)_{\alpha \in \kappa}$  is a set of symmetric random variables in  $L^{p,\infty}(2^\kappa)$ ,

$$D^{-1}\|(\sum_{\alpha \in I} |G_\alpha|^2)^{1/2}\| \leq \|\sum_{\alpha \in I} G_\alpha\| = \|f\| \leq D\|(\sum_{\alpha \in I} |G_\alpha|^2)^{1/2}\| \leq C'D$$

by Theorem 20. Suppose that  $n \geq |I|$ . With respect to the measure  $\mu$ ,  $w_n^{1/p}\|(\sum_{\alpha \in I} |a_\alpha(n)\tilde{F}_\alpha(n)|^2)^{1/2}\| \geq 2^{-1/p}\|(a_\alpha(n))_{\alpha \in I}\|$  by Proposition 22. Therefore,

$$\begin{aligned} \|f\| &= \|\sum_{\alpha \in I} G_\alpha\| \geq D^{-1}\|(\sum_{\alpha \in I} |G_\alpha|^2)^{1/2}\| \\ &\geq D^{-1}\|(\sum_{\alpha \in I} |a_\alpha(n)F_\alpha(n)|^2)^{1/2}\|_{L^{p,\infty}(w_n \cdot \mu)} \\ &= D^{-1}w_n^{1/p}\|(\sum_{\alpha \in I} |a_\alpha(n)\tilde{F}_\alpha(n)|^2)^{1/2}\|_{L^{p,\infty}(\mu)} \\ &\geq 2^{-1/p}D^{-1}\|(a_\alpha(n))_{\alpha \in I}\|. \end{aligned}$$

□

For  $1 < p < \infty$ ,  $q = p/(p-1)$  and an infinite cardinal  $\kappa$ , we refer to [1, Definition 4.1] for the definition of the Lorentz space  $L^{q,1}(2^\kappa)$ . By [1, Corollary 4.8],  $L^{p,\infty}(2^\kappa)$  is naturally isomorphic to the dual of  $L^{q,1}(2^\kappa)$ . The span of the characteristic functions of measurable subsets of  $2^\kappa$  is dense in  $L^{q,1}(2^\kappa)$ . If  $\mu(\sigma_n \Delta \sigma) \rightarrow 0$ , then  $1_{\sigma_n} \rightarrow 1_\sigma$  in  $L^{q,1}(2^\kappa)$ . Let  $(f_\alpha)_{\alpha \in \kappa}$  be a family of functions in  $L^{p,\infty}(2^\kappa)$ . By  $w^* \sum f_\alpha$  we mean the weak\* limit of the finite sums  $\sum_{\alpha \in I} f_\alpha$ ,  $I \subseteq \kappa$ ,  $|I| < \infty$ , along the Fréchet filter on  $\kappa$ .

**Proposition 24.** *Let  $((a_\alpha(n))_{\alpha \in \kappa})_{n=1}^\infty \in \ell^\infty(\ell^{p,\infty}(\kappa))$ . For all  $\alpha \in \kappa$ , let  $G_\alpha = \oplus_n a_\alpha(n)F_\alpha(n)$ . Then the sum  $w^* \sum G_\alpha$  exists and  $\|w^* \sum G_\alpha\| \leq C'D \sup_n \|(a_\alpha(n))_{\alpha \in \kappa}\|_{p,\infty}$ . Moreover,*

$$\|w^* \sum G_\alpha\| \geq 2^{-1/p}D^{-1}\|(a_\alpha(n))_{\alpha \in I}\|$$

for all  $n \in \mathbb{N}$  and all subsets  $I$  of  $\kappa$  with  $|I| \leq n$ .

*Proof.* For all finite subsets  $I$  of  $\kappa$ ,  $\|\sum_{\alpha \in I} G_\alpha\| \leq C'D \sup_n \|(a_\alpha(n))\|_{p,\infty}$  by Proposition 23. Suppose that  $I$  is a finite subset of  $\kappa$  and  $g \in L^{q,1}(2^\kappa)$  is measurable with respect to the functions  $(G_\alpha)_{\alpha \in I}$ . Since  $(G_\alpha)_{\alpha \in \kappa}$  is symmetric,  $\int G_\beta g = 0$  for all  $\beta \in \kappa \setminus I$ . Hence  $\int \sum_{\alpha \in J} G_\alpha g = \int \sum_{\alpha \in I} G_\alpha g$  for all finite subsets  $J$  of  $\kappa$  containing  $I$ . Thus, letting  $Y$  be the space of all functions  $g \in L^{q,1}(2^\kappa)$  measurable with respect to  $(G_\alpha)_{\alpha \in I}$  for some finite set  $I \subseteq \kappa$ , we see that  $\sum \int G_\alpha g$  exists for all  $g \in Y$ . Since  $\{\sum_{\alpha \in I} G_\alpha : I \subseteq \kappa, |I| < \infty\}$  is bounded,  $\sum \int G_\alpha g$  exists for all  $g \in \bar{Y}$ . Denote the  $\sigma$ -algebra generated by  $(G_\alpha)_{\alpha \in \kappa}$  as  $\tilde{\Sigma}$ . If  $\sigma \in \tilde{\Sigma}$ , then there exist a sequence  $(\sigma_n)$  such that  $1_{\sigma_n} \in Y$

for all  $n$  and  $\mu(\sigma\Delta\sigma_n) \rightarrow 0$ . Thus  $1_\sigma \in \bar{Y}$ . Hence all  $\tilde{\Sigma}$ -measurable functions in  $L^{q,1}(2^\kappa)$  belong to  $\bar{Y}$ . Let  $h \in L^{q,1}(2^\kappa)$  and let  $g$  be the conditional expectation of  $h$  with respect to  $\tilde{\Sigma}$ . Then  $g \in \bar{Y}$  and  $\int G_\alpha h = \int G_\alpha g$  for all  $\alpha \in \kappa$ . Hence  $\sum \int G_\alpha h$  exists. This proves that  $w^* \sum G_\alpha$  exists. Obviously,  $\|w^* \sum G_\alpha\| \leq C'D \sup_n \|(a_\alpha(n))\|_{p,\infty}$ .

For any finite subset  $I$  of  $\kappa$  and any  $\delta > 0$ , there exists a normalized  $g \in L^{q,1}(2^\kappa)$ , measurable with respect to  $(G_\alpha)_{\alpha \in I}$ , such that  $\int \sum_{\alpha \in I} G_\alpha g \geq \|\sum_{\alpha \in I} G_\alpha\| - \delta$ . Then

$$\int \sum_{\alpha \in J} G_\alpha g = \int \sum_{\alpha \in I} G_\alpha g \geq \|\sum_{\alpha \in I} G_\alpha\| - \delta$$

for all  $J \supseteq I$ ,  $|J| < \infty$ . Hence

$$\|w^* \sum G_\alpha\| \geq \|\sum_{\alpha \in I} G_\alpha\| \geq 2^{-1/p} D^{-1} \sup_{n \geq |I|} \|(a_\alpha(n))_{\alpha \in I}\|_{p,\infty}$$

by Proposition 23. □

Partition  $\mathbb{N}$  into a sequence of infinite subsets  $(M_m)_{m=1}^\infty$ . Suppose that  $a = ((a_\alpha(n))_{\alpha \in \kappa})_{n=1}^\infty \in \ell^\infty(\ell^{p,\infty}(\kappa))$ . Define  $H_\alpha(a) = \bigoplus_m \bigoplus_{n \in M_m} a_\alpha(n) F_\alpha(n)$  for all  $\alpha \in \kappa$ .

**Theorem 25.** *Suppose that  $1 < p < 2$  and that  $\kappa$  is an infinite cardinal. Then  $L^{p,\infty}(2^\kappa)$  contains a complemented subspace isomorphic to  $\ell^\infty(\ell^{p,\infty}(\kappa))$ .*

*Proof.* By Proposition 24, for all  $a \in \ell^\infty(\ell^{p,\infty}(\kappa))$ ,  $w^* \sum H_\alpha(a)$  exists and  $\|w^* \sum H_\alpha(a)\| \leq C'D \sup_n \|(a_\alpha(n))_{\alpha \in \kappa}\|_{p,\infty}$ . The map  $T : \ell^\infty(\ell^{p,\infty}(\kappa)) \rightarrow L^{p,\infty}(2^\kappa)$ ,  $Ta = w^* \sum H_\alpha(a)$  is a bounded linear operator. Let  $I$  be a finite subset of  $\kappa$ . For any  $m \in \mathbb{N}$ , there exists  $n \in M_m$  with  $n \geq |I|$ . By Proposition 24,  $\|w^* \sum H_\alpha(a)\| \geq 2^{-1/p} D^{-1} \|(a_\alpha(n))_{\alpha \in I}\|$ . It follows that  $\|Ta\| \geq 2^{-1/p} D^{-1} \sup_m \|(a_\alpha(m))_{\alpha \in I}\|_{p,\infty}$ . Thus  $T$  is an (into) isomorphism.

To complete the proof, we require a bounded linear map  $Q : L^{p,\infty}(2^\kappa) \rightarrow \ell^\infty(\ell^{p,\infty}(\kappa))$  such that  $QT$  is the identity on  $\ell^\infty(\ell^{p,\infty}(\kappa))$ . For each  $m$ , fix a free ultrafilter  $\mathcal{U}_m$  on  $M_m$ . Assume that  $\bigoplus f_n \in L^{p,\infty}(\bigoplus w_n \cdot 2^\kappa)$  has norm at most 1. For each  $n$  and each  $\alpha \in \kappa$ , observe that  $\mu\{F_\alpha(n) \neq 0\} = w_n b_n$ . Hence

$$\begin{aligned} \left| \int f_n F_\alpha(n) \right| &\leq (w_n b_n)^{-1/p} \left(1 - \frac{1}{p}\right)^{-1} \|f_n\| \mu\{F_\alpha(n) \neq 0\} \\ &\leq q (w_n b_n)^{\frac{1}{q} - \frac{1}{p}}. \end{aligned}$$

Thus  $x'_{\alpha,n}(\bigoplus f_m) = (w_n b_n)^{\frac{1}{p} - \frac{1}{q}} \int f_n F_\alpha(n)$  defines a bounded linear functional on  $L^{p,\infty}(\bigoplus w_m \cdot 2^\kappa)$  of norm at most  $q$ . For each  $(\alpha, m) \in \kappa \times \mathbb{N}$ , let  $y'_{\alpha,m}$  be the weak\* limit  $\lim_{n \rightarrow \mathcal{U}_m} x'_{\alpha,n}$ . We claim that the map  $Qf = ((y'_{\alpha,m}(f))_{\alpha \in \kappa})_{m=1}^\infty$  is the map we desire.

Let  $f = \bigoplus f_n$  be a normalized element in  $L^{p,\infty}(\bigoplus w_n \cdot 2^\kappa)$ . Suppose that  $m \in \mathbb{N}$ ,  $c > 0$  and  $I = \{\alpha_1, \dots, \alpha_k\}$  is a finite subset of  $\kappa$  so that

$|y'_{\alpha,m}(f)| > c$  for all  $\alpha \in I$ . There is a sufficiently large  $n \geq 2$  in  $M_m$  so that  $q(2b_n k)^{1/q} \leq c/2$  and that  $|x'_{\alpha_j,n}(f)| > c$  for all  $j \leq k$ . For each  $j \leq k$  and each  $n \in \mathbb{N}$ , let  $\sigma_{j,n}$ , respectively,  $\tilde{\sigma}_{j,n}$ , be the support of  $F_{\alpha_j}(n)$ , respectively,  $\tilde{F}_{\alpha_j}(n)$ . (The two sets are the same; but they are associated with different measures.) If  $1 \leq j \leq k$ , then using the independence of  $(\tilde{F}_{\alpha}(n))_{\alpha \in \kappa}$ ,

$$\begin{aligned} \mu(\tilde{\sigma}_{j,n} \cap (\cup_{i=1}^{j-1} \tilde{\sigma}_{i,n})) &= \mu(\tilde{\sigma}_{j,n}) \left(1 - \prod_{i=1}^{j-1} (1 - \mu(\tilde{\sigma}_{i,n}))\right) \\ &= b_n(1 - (1 - b_n)^{j-1}) \\ &\leq b_n(1 - e^{-2b_n(j-1)}) \text{ since } 2b_n \leq \eta_0 \\ &\leq 2b_n^2(j-1) \leq 2b_n^2 k. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\sigma_{j,n} \cap (\cup_{i=1}^{j-1} \sigma_{i,n})} |f| &\leq q(\mu(\sigma_{j,n} \cap (\cup_{i=1}^{j-1} \sigma_{i,n})))^{1/q} \\ &\leq q(w_n 2b_n^2 k)^{1/q} \leq \frac{c}{2}(w_n b_n)^{1/q}. \end{aligned}$$

Now

$$c < |x'_{\alpha_j,n}(f)| \leq (w_n b_n)^{-1/q} \int_{\sigma_{j,n}} |f_n|.$$

Hence

$$\int_{\sigma_{j,n} \setminus \cup_{i=1}^{j-1} \sigma_{i,n}} |f| > \frac{c}{2}(w_n b_n)^{1/q}.$$

Therefore,

$$\begin{aligned} (k w_n b_n)^{1/q} &= \left(\sum_{j=1}^k \mu(\sigma_{j,n})\right)^{1/q} \geq \left(\mu(\cup_{j=1}^k \sigma_{j,n})\right)^{1/q} \\ &\geq \frac{1}{q} \int_{\cup_{j=1}^k \sigma_{j,n}} |f| = \frac{1}{q} \sum_{j=1}^k \int_{\sigma_{j,n} \setminus \cup_{i=1}^{j-1} \sigma_{i,n}} |f| \\ &> \frac{c}{2q} k (w_n b_n)^{1/q}. \end{aligned}$$

So we have  $c|I|^{1/p} = ck^{1/p} < 2q$ . This proves that  $\|Qf\| \leq 2q$ .

Observe that  $x'_{\alpha,n}$  can be identified with the function  $(w_n b_n)^{\frac{1}{p}-\frac{1}{q}} F_{\alpha}(n)$  in the predual of  $L^{p,\infty}(2^\kappa)$ . Hence, if  $n \in M_m$ , then

$$\begin{aligned} x'_{\alpha,n}(Ta) &= \sum_{\beta \in \kappa} x'_{\alpha,n}(H_\beta(a)) \\ &= \sum_{\beta \in \kappa} (w_n b_n)^{\frac{1}{p}-\frac{1}{q}} a_\beta(m) \int F_\beta(n) F_\alpha(n) = a_\alpha(m). \end{aligned}$$

Thus  $y'_{\alpha,m}(Ta) = a_\alpha(m)$  and  $QT$  is the identity map.  $\square$

Identify  $\aleph_0$  with  $\mathbb{N}$ . For each  $n \in \mathbb{N}$ , denote by  $\varepsilon_n$  the projection from  $2^{\aleph_0}$  onto the  $n$ -th component. Suppose that  $\kappa$  is an infinite cardinal. A function  $f \in L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$  is written as  $\oplus_{\alpha < \kappa} f_\alpha$ , where each  $f_\alpha$  is a function on the  $\alpha$ -th copy of  $2^{\aleph_0}$ . For each  $\beta < \kappa$  and each  $n \in \mathbb{N}$ , let  $\varepsilon_{\beta,n}$  be the function  $\oplus_{\alpha < \kappa} f_\alpha$ , where  $f_\beta = \varepsilon_n$  and  $f_\alpha = 0$  if  $\alpha \neq \beta$ . If  $n \in \mathbb{N}$ ,  $\varphi = (\varphi_k)_{k=1}^n \in 2^n$  and  $\alpha < \kappa$ , denote the set  $\cap_{k=1}^n \{\varepsilon_{\alpha,k} = \varphi_k\}$  by  $\sigma_{\alpha,\varphi}$ . If  $\varphi = (\varphi_k)_{k=1}^n \in 2^n$  and  $\psi = (\psi_k)_{k=1}^m \in 2^m$ , let  $(\varphi, \psi)$  be the element  $(\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m)$  in  $2^{n+m}$ .

**Theorem 26.** *Suppose that  $1 < p < 2$  and that  $\kappa$  is an infinite cardinal. Then  $\ell^\infty(\ell^{p,\infty}(\kappa))$  contains a complemented subspace isomorphic to  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$  and  $L^{p,\infty}(2^\kappa)$  contains a complemented subspace isomorphic to  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ .*

*Proof.* The second statement follows from the first because of Theorem 25. Identify  $\ell^\infty(\ell^{p,\infty}(\kappa))$  with  $(\oplus_n \ell^{p,\infty}(\kappa \times 2^n))_{\ell^\infty}$ . Suppose that  $f = \oplus_{\alpha < \kappa} f_\alpha \in L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ . For each  $n \in \mathbb{N}$  and each  $(\alpha, \varphi) \in \kappa \times 2^n$ , let  $a_{\alpha,\varphi} = a_{\alpha,\varphi}(f) = 2^{n/q} \int_{\sigma_{\alpha,\varphi}} f_\alpha$ . If  $I$  is a finite subset of  $\kappa \times 2^n$  for some  $n \in \mathbb{N}$  so that  $|a_{\alpha,\varphi}| > c$  for all  $(\alpha, \varphi) \in I$ , then

$$\begin{aligned} c|I| &< \sum_{(\alpha,\varphi) \in I} 2^{n/q} \int_{\sigma_{\alpha,\varphi}} f_\alpha = 2^{n/q} \int_{\cup_{(\alpha,\varphi) \in I} \sigma_{\alpha,\varphi}} |f| \\ &\leq 2^{n/q} q \mu(\cup_{(\alpha,\varphi) \in I} \sigma_{\alpha,\varphi})^{1/q} \|f\| = q|I|^{1/q} \|f\|. \end{aligned}$$

Thus  $\|(a_{\alpha,\varphi})_{(\alpha,\varphi) \in \kappa \times 2^n}\|_{p,\infty} \leq q\|f\|$ . Hence the map  $T : L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0}) \rightarrow (\oplus_n \ell^{p,\infty}(\kappa \times 2^n))_{\ell^\infty}$ ,  $Tf = ((a_{\alpha,\varphi}(f))_{(\alpha,\varphi) \in \kappa \times 2^n})_{n=1}^\infty$  is bounded.

Now suppose that  $b = ((b_{\alpha,\varphi})_{(\alpha,\varphi) \in \kappa \times 2^n})_{n=1}^\infty \in (\oplus_n \ell^{p,\infty}(\kappa \times 2^n))_{\ell^\infty}$ . Let  $n \in \mathbb{N}$  and consider the function

$$g_n = \oplus_{\alpha \in \kappa} 2^{n/p} \sum_{\varphi \in 2^n} b_{\alpha,\varphi} 1_{\sigma_{\alpha,\varphi}}.$$

Then  $\|g_n\|_{L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})} = \|(b_{\alpha,\varphi})_{(\alpha,\varphi) \in \kappa \times 2^n}\|_{p,\infty}$ . Let  $\mathcal{U}$  be a free ultrafilter on  $\mathbb{N}$  and define  $g = w^* \lim_{n \rightarrow \mathcal{U}} g_n$ . Here  $w^*$  refers to the weak\* topology on  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ , identified as the dual to  $L^{q,1}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ . The map  $Q : (\oplus_n \ell^{p,\infty}(\kappa \times 2^n))_{\ell^\infty} \rightarrow L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ ,  $Qb = g$ , is a bounded linear operator. Suppose that  $b = Tf$  for some  $f \in L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ . Assume that  $(\beta, \psi) \in \kappa \times 2^m$  for some  $m$ . If  $n \geq m$ , we have by direct computation that  $\int_{\sigma_{\beta,\psi}} g_n = \int_{\sigma_{\beta,\psi}} f_\beta = \int_{\sigma_{\beta,\psi}} f$ . Since  $(g_n)$  is bounded and the functions  $1_{\sigma_{\beta,\psi}}$ ,  $(\beta, \psi) \in \cup_m (\kappa \times 2^m)$ , span a dense subspace of  $L^{q,1}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ , we deduce that  $w^* \lim g_n = f$ . Therefore,  $QT$  is the identity map on  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0})$ .  $\square$

By Theorem 26 and Pełczyński's Decomposition Method,  $L^{p,\infty}(2^\kappa)$  and  $L^{p,\infty}(\oplus_{\alpha < \kappa} 2^{\aleph_0}) \oplus L^{p,\infty}(2^\kappa)$  are isomorphic if  $1 < p < 2$ . In particular, Theorem 19 does not extend to the range  $1 < p < 2$ .

## 5. ISOMORPHIC CLASSIFICATION: A SPECIAL CASE

We have seen in Theorem 1 that if  $1 < p < \infty$  and  $(\Omega, \Sigma, \mu)$  is a purely nonatomic measure space, then  $L^{p,\infty}(\Omega, \Sigma, \mu)$  is isomorphic to the weak  $L^p$  space defined on a measure space of the form  $\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha}$ . In this section, we give, for a subclass of measure spaces in such “standard form” and  $2 \leq p < \infty$ , a complete isomorphic classification of the corresponding weak  $L^p$  spaces. Precisely, we prove the following theorem.

**Theorem 27.** *Suppose that  $(\kappa_\alpha)_{\alpha < \tau}$  and  $(\kappa'_\beta)_{\beta < \tau'}$  are sequences of cardinals. Assume that each  $\kappa_\alpha$  has uncountable cofinality and that if the sequence  $(\kappa'_\beta)_{\beta < \tau'}$  has a maximum, then the maximum is attained an infinite number of times. Consider the following statements:*

- (1) *There is an injection  $i : [0, \tau) \rightarrow [0, \tau')$  such that  $\kappa'_{i(\alpha)} \geq \kappa_\alpha$  for all  $\alpha < \tau$ .*
- (2) *The space  $L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha})$  is isomorphic to a complemented subspace of the space  $L^{p,\infty}(\bigoplus_{\beta < \tau'} 2^{\kappa'_\beta})$ .*
- (3) *The space  $L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha})$  is isomorphic to a subspace of the space  $L^{p,\infty}(\bigoplus_{\beta < \tau'} 2^{\kappa'_\beta})$ .*

*If  $1 < p < \infty$ , then (1)  $\implies$  (2)  $\implies$  (3). If  $2 \leq q < \infty$ , then all three statements are equivalent.*

**Corollary 28.** *Suppose that  $(\kappa_\alpha)_{\alpha < \tau}$  and  $(\kappa'_\beta)_{\beta < \tau'}$  are sequences of cardinals with uncountable cofinality. Assume that if either sequence has a maximum, then the maximum is attained an infinite number of times. If  $2 \leq p < \infty$ , then the spaces  $L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha})$  and  $L^{p,\infty}(\bigoplus_{\beta < \tau'} 2^{\kappa'_\beta})$  are isomorphic if and only if there are injections  $i : [0, \tau) \rightarrow [0, \tau')$  and  $j : [0, \tau') \rightarrow [0, \tau)$  such that  $\kappa'_{i(\alpha)} \geq \kappa_\alpha$  for all  $\alpha < \tau$  and  $\kappa_{j(\beta)} \geq \kappa'_\beta$  for all  $\beta < \tau'$ .*

Suppose that condition (1) holds. Then it follows from Proposition 5 that the space  $L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha})$  is isomorphic to a complemented subspace of  $L^{p,\infty}(\bigoplus_{\beta < \tau'} 2^{\kappa'_\beta})$ . The implication (2)  $\implies$  (3) is obvious. Thus, to complete the proof of Theorem 27, it suffices to show that (3)  $\implies$  (1) when  $2 \leq p < \infty$ . Also, Corollary 28 is an immediate consequence of Theorem 27.

Recall that  $M^{p,\infty}(2^\kappa)$  is the closure of  $L^\infty(2^\kappa)$  in  $L^{p,\infty}(2^\kappa)$ .

**Lemma 29.** *Suppose that  $(\kappa_\alpha)_{\alpha < \tau}$  and  $(\kappa'_\beta)_{\beta < \tau'}$  are sequences of cardinals, where each  $\kappa_\alpha$  has uncountable cofinality. If  $2 \leq p < \infty$  and  $L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha})$  isomorphically embeds into the space  $L^{p,\infty}(\bigoplus_{\beta < \tau'} b_\beta \cdot 2^{\kappa'_\beta})$ , then there exists a map  $k : [0, \tau) \rightarrow [0, \tau')$  such that  $\kappa'_{k(\alpha)} \geq \kappa_\alpha$  for all  $\alpha < \tau$  and  $|k^{-1}\{\beta\}| \leq \aleph_0$  for all  $\beta < \tau'$ .*

*Proof.* Let  $T : L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha}) \rightarrow L^{p,\infty}(\bigoplus_{\beta < \tau'} b_\beta \cdot 2^{\kappa'_\beta})$  be an isomorphic embedding. For each  $g \in L^{p,\infty}(\bigoplus_{\beta < \tau'} b_\beta \cdot 2^{\kappa'_\beta})$  and each  $\beta < \tau'$ , let  $P_\beta g$  denote the restriction of  $g$  to the component  $b_\beta \cdot 2^{\kappa'_\beta}$ . Let  $\alpha < \tau$  and consider

the set  $B_\alpha$  consisting of all  $\beta < \tau'$  such that  $\kappa'_\beta < \kappa_\alpha$ . By Lemma 13,  $M^{p,\infty}(2^{\kappa_\alpha})$  does not embed isomorphically into  $L^{p,\infty}(\bigoplus_{\beta \in B_\alpha} b_\beta \cdot 2^{\kappa'_\beta})$ . Thus, for each  $\alpha < \tau$ , there exist  $f_\alpha \in M^{p,\infty}(2^{\kappa_\alpha})$ , regarded as a subspace of  $L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha})$ , and  $\beta_\alpha \notin B_\alpha$ , so that  $P_{\beta_\alpha} T f_\alpha \neq 0$ . By a small perturbation, we may as well assume that  $f_\alpha \in L^\infty(2^{\kappa_\alpha})$ . Consider the correspondence  $k : \alpha \mapsto \beta_\alpha$ . By definition of  $B_\alpha$ ,  $\kappa'_{k(\alpha)} \geq \kappa_\alpha$  for all  $\alpha < \tau$ . Suppose there exists  $\beta_0 < \tau'$  such that  $|k^{-1}\{\beta_0\}| > \aleph_0$ . Then there exists  $M < \infty$  and an uncountable subset  $A$  of  $k^{-1}\{\beta_0\}$  such that  $\|f_\alpha\|_\infty \leq M$  for all  $\alpha \in A$ . Since the functions are also pairwise disjoint,  $(f_\alpha)_{\alpha \in A}$  (as a sequence of functions in  $L^{p,\infty}(\bigoplus_{\alpha < \tau} 2^{\kappa_\alpha})$ ) is dominated by the unit vectors in  $\ell^{p,\infty}(A)$ . Denote by  $(W_F)$  the set of Walsh functions on  $2^{\kappa'_{\beta_0}}$ . By Lemma 18, the set

$$A' = \{\alpha \in A : \int P_{\beta_0} T f_\alpha \cdot W_F \neq 0 \text{ for some finite } F \subseteq \kappa\}$$

is countable. Let  $\alpha_0 \in A \setminus A'$ . Then  $P_{\beta_0} T f_{\alpha_0} = 0$ , contrary to the fact that  $k(\alpha_0) = \beta_0$ .  $\square$

**Lemma 30.** *Let  $A$  be a set,  $\tau'$  be a limit ordinal and let  $k : A \rightarrow [0, \tau')$  be a function so that  $|k^{-1}\{\beta\}| \leq \aleph_0$  for all  $\beta < \tau'$ . Then there is an injection  $i : A \rightarrow [0, \tau')$  such that  $i(\alpha) \geq k(\alpha)$  for all  $\alpha \in A$ .*

*Proof.* Suppose that  $\gamma \in [0, \tau')$  is either 0 or a limit ordinal. The set  $A_\gamma = \bigcup_{n < \omega} k^{-1}\{\gamma + n\}$  is countable. Hence there is an injection  $i_\gamma : A_\gamma \rightarrow [\gamma, \gamma + \omega)$  such that  $i_\gamma(\alpha) \geq k(\alpha)$  for all  $\alpha \in A_\gamma$ . Consider the map  $i = \bigcup i_\gamma : A = \bigcup A_\gamma \rightarrow [0, \tau')$ , where the unions are taken over all  $\gamma$  that is either 0 or a limit ordinal in  $[0, \tau')$ . Clearly  $i(\alpha) \geq k(\alpha)$  for all  $\alpha \in A$ . Suppose that  $\alpha, \alpha' \in A$  and that  $i(\alpha) = i(\alpha')$ . Express  $\alpha$  uniquely as  $\gamma_\alpha + n_\alpha$ , where  $\gamma_\alpha$  is either 0 or a limit ordinal, and  $n_\alpha < \omega$ . Similarly, let  $\alpha' = \gamma_{\alpha'} + n_{\alpha'}$ . Then  $i(\alpha) = i(\alpha') \in [\gamma_\alpha, \gamma_\alpha + \omega) \cap [\gamma_{\alpha'}, \gamma_{\alpha'} + \omega)$  and hence  $\gamma_\alpha = \gamma_{\alpha'}$ . But then  $i(\alpha) = i_{\gamma_\alpha}(\alpha)$  and  $i(\alpha') = i_{\gamma_\alpha}(\alpha')$ . Since  $i_{\gamma_\alpha}$  is injective,  $\alpha = \alpha'$ .  $\square$

*Completion of proof of Theorem 27.* As discussed above, it suffices to prove the implication (3)  $\implies$  (1). Suppose that condition (3) holds. First of all, we may assume that the sequence of cardinals  $(\kappa'_\beta)_{\beta < \tau'}$  is arranged in nondecreasing order. Taking note of the condition on the maximum (if any) of the sequence  $(\kappa'_\beta)_{\beta < \tau'}$ , we may further assume that  $\tau'$  is a limit ordinal. By Lemma 29, there exists a map  $k : [0, \tau) \rightarrow [0, \tau')$  such that  $\kappa'_{k(\alpha)} \geq \kappa_\alpha$  for all  $\alpha < \tau$  and  $|k^{-1}\{\beta\}| \leq \aleph_0$  for all  $\beta < \tau'$ . By Lemma 30, there is an injection  $i : [0, \tau) \rightarrow [0, \tau')$  such that  $i(\alpha) \geq k(\alpha)$  for all  $\alpha \in [0, \tau)$ . In particular,  $\kappa'_{i(\alpha)} \geq \kappa'_{k(\alpha)} \geq \kappa_\alpha$  for all  $\alpha < \tau$ .  $\square$

In Theorem 1, it was shown that if  $(\Omega, \Sigma, \mu)$  is a purely nonatomic measure space, then  $L^{p,\infty}(\Omega, \Sigma, \mu)$  has a representation  $E \oplus H$ , where  $E$  has the form  $L^{p,\infty}(\bigoplus_{\alpha < \omega_1 \cdot \tau} 2^{\kappa_\alpha})$  for a nondecreasing sequence of cardinals  $(\kappa_\alpha)_{\alpha < \omega_1 \cdot \tau}$  and some ordinal  $\tau$  ( $\tau = 0$  is allowed here, in which case  $E = \{0\}$ ) and  $H$  is either  $\{0\}$  or has the form  $L^{p,\infty}(\bigoplus_{n=1}^\infty a_n \cdot 2^{\rho_n})$ , with  $\rho_n \geq \kappa_\alpha$  for all  $n$  and

all  $\alpha$ . Making use of the method of proof of Theorem 27, we show that the factor  $E$  in the representation is uniquely determined up to isomorphism if  $2 \leq p < \infty$  and the ordinals  $\kappa_\alpha$  have uncountable cofinality.

**Theorem 31.** *Suppose that the spaces  $E_1 \oplus H_1$  and  $E_2 \oplus H_2$  are isomorphic, where  $E_1 = L^{p,\infty}(\bigoplus_{\alpha < \omega_1 \cdot \tau} 2^{\kappa_\alpha})$ ,  $E_2 = L^{p,\infty}(\bigoplus_{\beta < \omega_1 \cdot \tau'} 2^{\kappa'_\beta})$ , with non-decreasing sequences of infinite cardinals  $(\kappa_\alpha)_{\alpha < \omega_1 \cdot \tau}$  and  $(\kappa'_\beta)_{\beta < \omega_1 \cdot \tau'}$ , and  $H_1$ , respectively  $H_2$ , are either  $\{0\}$  or  $L^{p,\infty}(\bigoplus_{n=1}^\infty a_n \cdot 2^{\rho_n})$  ( $\rho_n \geq \kappa_\alpha$ ) and  $L^{p,\infty}(\bigoplus_{n=1}^\infty b_n \cdot 2^{\rho'_n})$  ( $\rho'_n \geq \kappa'_\beta$ ) respectively. If  $2 \leq p < \infty$  and the cardinals  $\kappa_\alpha$  and  $\kappa'_\beta$  have uncountable cofinality, then  $E_1$  is isomorphic to  $E_2$ . Moreover, if  $H_1 = \{0\}$ , then so is  $H_2$ .*

*Proof.* First, suppose that  $E_1 = \{0\}$ , so that  $H_1$  is isomorphic to  $E_2 \oplus H_2$ . By Theorem 19,  $\aleph_0 \geq |\omega_1 \cdot \tau'|$ . Thus  $\tau' = 0$ , i.e.,  $E_2 = \{0\}$ . Now suppose that  $E_1 \neq \{0\}$  and thus  $\tau > 0$ . By the foregoing argument, we must have  $\tau' > 0$  as well. Since  $E_1$  isomorphically embeds into  $E_2 \oplus H_2$ , by Lemma 29, there exists a map  $k : [0, \omega_1 \cdot \tau) \rightarrow \{\kappa'_\beta : \beta < \omega_1 \cdot \tau'\} \cup \{\rho'_n : n < \omega\}$  such that  $k(\alpha) \geq \kappa_\alpha$  for all  $\alpha < \omega_1 \cdot \tau$  and that  $|k^{-1}\{\kappa'_\beta\}|, |k^{-1}\{\rho'_n\}| \leq \aleph_0$  for each  $\beta$  and each  $\rho'_n$ . Suppose that there exists  $\alpha_0 < \omega_1 \cdot \tau$  so that  $\kappa_\alpha > \kappa'_\beta$  for all  $\beta < \omega_1 \cdot \tau'$ . Then  $k(\alpha) \in \{\rho'_n : n < \omega\}$  for all  $\alpha \in [\alpha_0, \omega_1 \cdot \tau)$ . Hence  $[\alpha_0, \omega_1 \cdot \tau) \subseteq \bigcup_n k^{-1}\{\rho'_n\}$ . Since the latter set is countable, we have a contradiction. Therefore, for each  $\alpha < \omega_1 \cdot \tau$ , there exists  $\beta_\alpha < \omega_1 \cdot \tau'$  such that  $\kappa'_{\beta_\alpha} \geq \kappa_\alpha$ . Define  $j : [0, \omega_1 \cdot \tau) \rightarrow \{\kappa'_\beta : \beta < \omega_1 \cdot \tau'\}$  by

$$j(\alpha) = \begin{cases} k(\alpha) & \text{if } k(\alpha) \in \{\kappa'_\beta : \beta < \omega_1 \cdot \tau'\} \\ \kappa_{\beta_\alpha} & \text{if } \alpha \in \bigcup_n k^{-1}\{\rho'_n\} \end{cases}.$$

Since  $j$  differs from  $k$  at only countably many  $\alpha$ ,  $|j^{-1}\{\kappa'_\beta\}| \leq \aleph_0$  for all  $\beta$ . By Lemma 30, there exists an injection  $i : [0, \omega_1 \cdot \tau) \rightarrow \{\kappa'_\beta : \beta < \omega_1 \cdot \tau'\}$  such that  $i(\alpha) \geq j(\alpha) \geq \kappa_\alpha$  for all  $\alpha < \omega_1 \cdot \tau$ . It follows from Proposition 5 (or Theorem 27) that  $E_1$  is isomorphic to a complemented subspace of  $E_2$ . By symmetry and Pełczyński's Decomposition Method,  $E_1$  and  $E_2$  are isomorphic.

Now suppose that  $H_1 = \{0\} \neq H_2$ . It was shown in the previous paragraph that for each  $\alpha < \omega_1 \cdot \tau$ , there exists  $\beta_\alpha < \omega_1 \cdot \tau'$  such that  $\kappa'_{\beta_\alpha} \geq \kappa_\alpha$ . In particular,  $\sup \kappa_\alpha \leq \sup \kappa'_\beta < \sup(\{\kappa'_\beta\} \cup \{\rho'_n\})$ . However, this contradicts Theorem 14.  $\square$

We conclude the paper with several of the main open problems that need to be resolved on the way to a complete isomorphic classification of nonatomic weak  $L^p$  spaces.

### Open problems.

- (1) Let  $\kappa$  be an uncountable cardinal. Are the spaces  $L^{p,\infty}(\bigoplus_{n=1}^\infty 2^\kappa)$  and  $L^{p,\infty}(2^\kappa)$  isomorphic? Note that the answer is yes if  $\kappa = \aleph_0$  [6].

- (2) Let  $\kappa$  be an uncountable ordinal of countable cofinality. Can  $L^{p,\infty}(2^\kappa)$  be isomorphic to a space of the form  $L^{p,\infty}(\bigoplus_{\alpha < \tau} a_\alpha \cdot 2^{\kappa_\alpha})$ , where  $\kappa_\alpha < \kappa$  for all  $\alpha$ ?
- (3) If  $1 < p < 2$  and  $\kappa$  is an uncountable cardinal, are the space  $L^{p,\infty}(2^\kappa)$  and  $L^{p,\infty}(\bigoplus_{\alpha < \kappa} 2^\alpha)$  isomorphic? More generally, does  $L^{p,\infty}(2^\kappa)$  contain a complemented subspace of the form  $L^{p,\infty}(\bigoplus_{\alpha < \kappa} 2^\rho)$  for some uncountable cardinal  $\rho$ ?

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