EXTENSION OF FUNCTIONS WITH SMALL OSCILLATION

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Abstract. A classical theorem of Kuratowski says that every Baire one function on a G_δ subspace of a Polish (= separable completely metrizable) space X can be extended to a Baire one function on X. Kechris and Louveau introduced a finer gradation of Baire one functions into small Baire classes. A Baire one function f is assigned into a class in this hierarchy depending on its oscillation index $\beta(f)$. We prove a refinement of Kuratowski's theorem: if Y is a subspace of a metric space X and f is a real-valued function on Y such that $\beta_Y(f) < \omega^\alpha$, $\alpha < \omega_1$, then f has an extension F to X so that $\beta_X(F) \leq \omega^\alpha$. We also show that if f is a continuous real valued function on Y, then f has an extension F to X so that $\beta_X(F) \leq 3$. An example is constructed to show that this result is optimal.

Let X be a topological space. A real-valued function on X belongs to Baire class one if it is the pointwise limit of a sequence of continuous functions. If X is a Polish (= separable completely metrizable) space, then a classical theorem of Kuratowski [7] states that every Baire one function on a G_δ subspace of X can be extended to a Baire one function on X. In [5], Kechris and Louveau introduced a finer gradation of Baire one functions into small Baire classes using the oscillation index β, whose definition we now recall.

Let X be a topological space and let $\mathcal{C}$ denote the collection of all closed subsets of X. A *derivation* on $\mathcal{C}$ is a map $\mathcal{D} : \mathcal{C} \to \mathcal{C}$ such that $\mathcal{D}(P) \subseteq P$ for all $P \in \mathcal{C}$. The oscillation index β is associated with a family of derivations. Let $\varepsilon > 0$ and a function $f : X \to \mathbb{R}$ be given. For any $P \in \mathcal{C}$, let $\mathcal{D}^0(f, \varepsilon, P) = P$ and $\mathcal{D}^1(f, \varepsilon, P)$ be the set of all $x \in P$ such that for any neighborhood U of x, there exist $x_1, x_2 \in P \cap U$ such that $|f(x_1) - f(x_2)| \geq \varepsilon$. The derivation $\mathcal{D}^1(f, \varepsilon, \cdot)$ may be iterated in the usual manner. For all $\alpha < \omega_1$, let

$$\mathcal{D}^{\alpha+1}(f, \varepsilon, P) = \mathcal{D}^1(f, \varepsilon, \mathcal{D}^\alpha(f, \varepsilon, P)).$$

If α is a countable limit ordinal, set

$$\mathcal{D}^\alpha(f, \varepsilon, P) = \cap_{\gamma<\alpha}\mathcal{D}^\gamma(f, \varepsilon, P).$$

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If $\mathcal{D}^\alpha(f,\varepsilon,P)\neq\emptyset$ for all $\alpha<\omega_1$, let $\beta_X(f,\varepsilon)=\omega_1$. Otherwise, let $\beta_X(f,\varepsilon)$ be the smallest countable ordinal α such that $\mathcal{D}^\alpha(f,\varepsilon,P)=\emptyset$. The *oscillation index* of f is $\beta_X(f)=\sup_{\varepsilon>0}\beta_X(f,\varepsilon)$.

The main result of §1 is that if Y is a subspace of a metric space X and $f:Y\to\mathbb{R}$ satisfies $\beta_Y(f)<\omega^\alpha$ for some $\alpha<\omega_1$, then f can be extended to a function F on X with $\beta_X(F)\leq\omega^\alpha$. It follows readily from the Baire Characterization Theorem [2, 10.15] that a real-valued function on a Polish space is Baire one if and only if its oscillation index is countable. (See, e.g. [5].) Also, a theorem of Alexandroff says that a G_δ subspace of a Polish space is Polish [2, 10.18]. Hence our result refines Kuratowski's theorem in terms of the oscillation index. Let us mention that if X is a metric space, then every real-valued function with countable oscillation index on a closed subspace of X may be extended to X with preservation of the index [8, Theorem 3.6]. (Note that the proof of [8, Theorem 3.6] does not require the compactness of the ambient space.) More recent results on the extension of Baire one functions on general topological spaces are found in [6].

It is well known that if a function is continuous on a *closed* subspace of a metric space, then there exists a continuous extension to the whole space. §2 is devoted to the study of extensions of continuous functions from an *arbitrary* subspace of a metric space. It is shown that if f is a continuous function on a subspace Y of a metric space X, then f has an extension F to X with $\beta_X(F)\leq 3$. An example is given to show that the result is optimal. The criteria for continuous extension on dense subspaces had been studied by several authors. (See, e.g., [1], [4].)

1. Functions of Small Oscillation

Given a real-valued function defined on a set S, let $\|f\|_S=\sup_{s\in S}|f(s)|$. Since we do not assume that the function f is bounded, $\|f\|_S$ may take the value $+\infty$. For any topological space X, the support $\operatorname{supp} f$ of a function $f:X\to\mathbb{R}$ is the closed set $\overline{\{x\in X:f(x)\neq 0\}}$. A family $\{\varphi_\alpha:\alpha\in\mathcal{A}\}$ of nonnegative, continuous real-valued functions on X is called a *partition of unity on* X if

(1) The support of the φ_α's form a locally finite closed covering of X,
(2) $\sum_{\alpha\in\mathcal{A}}\varphi_\alpha(x)=1$ for all $x\in X$.

If $\{U_\beta:\beta\in\mathcal{B}\}$ is an open covering of X, we say that a partition of unity $\{\varphi_\beta:\beta\in\mathcal{B}\}$ on X is subordinated to $\{U_\beta:\beta\in\mathcal{B}\}$ if the support of each φ_β lies in the corresponding U_β. It is well known that if X is paracompact (in particular, if X is a metric space [3, Theorem IX 5.3]), then for each open covering $\{U_\beta:\beta\in\mathcal{B}\}$ of X there is a partition of unity on X subordinated to $\{U_\beta:\beta\in\mathcal{B}\}$. (See, for example, [3, Theorem VIII 4.2].)

Proposition 1. *Let X be a metric space and Y be a subspace of X. Suppose that $f:Y\to\mathbb{R}$ is a function such that $\beta_Y(f,\varepsilon)\leq\alpha$ for some $\varepsilon>0$,*

$\alpha < \omega_1$. *Then there exists a function* $\tilde{f} : X \to \mathbb{R}$ *such that* $\beta_X\left(\tilde{f}\right) \leq \alpha + 1$, $\left\|\tilde{f}\right\|_X \leq \|f\|_Y$ *and* $\left\|f - \tilde{f}\right\|_Y \leq \varepsilon$.

In the following, denote $\mathcal{D}^\beta\left(f, \varepsilon, Y\right)$ by Y^β for all $\beta < \omega_1$. Proposition 1 is proved by working on each of the pieces $Y^\beta \smallsetminus Y^{\beta+1}$, $\beta < \alpha$, and gluing together the results.

Lemma 2. *For all* $0 \leq \beta < \alpha$, *there exist an open set* Z_β *in* X *such that* $Y^\beta \smallsetminus Y^{\beta+1} \subseteq Z_\beta \subseteq \left(Y^{\beta+1}\right)^c$, *and a continuous function* $f_\beta : Z_\beta \to \mathbb{R}$ *such that* $\|f - f_\beta\|_{Y^\beta \smallsetminus Y^{\beta+1}} \leq \varepsilon$ *and* $\|f_\beta\|_{Z_\beta} \leq \|f\|_Y$.

Proof. If $0 \leq \beta < \alpha$ and $y \in Y^\beta \smallsetminus Y^{\beta+1}$, there exists a set U_y that is an open neighborhood of y in X so that U_y is disjoint from $Y^{\beta+1}$ and that $f\left(U_y \cap Y^\beta\right) \subseteq \left(f\left(y\right) - \varepsilon, f\left(y\right) + \varepsilon\right)$. Let

$$Z_\beta = \bigcup_{y \in Y^\beta \smallsetminus Y^{\beta+1}} U_y.$$

Each Z_β is open in X. Clearly, $Y^\beta \smallsetminus Y^{\beta+1} \subseteq Z_\beta \subseteq \left(Y^{\beta+1}\right)^c$. There exists a partition of unity $\left(\varphi_y\right)_{y \in Y^\beta \smallsetminus Y^{\beta+1}}$ on Z_β subordinated to the open covering $\mathcal{U} = \left\{U_y : y \in Y^\beta \smallsetminus Y^{\beta+1}\right\}$. Define $f_\beta : Z_\beta \to \mathbb{R}$ by

$$f_\beta\left(z\right) = \sum_{y \in Y^\beta \smallsetminus Y^{\beta+1}} f\left(y\right) \varphi_y\left(z\right).$$

Then f_β is well-defined, continuous and $\|f_\beta\|_{Z_\beta} \leq \|f\|_Y$. If $x \in Y^\beta \smallsetminus Y^{\beta+1}$, set $V_x = \left\{y \in Y^\beta \smallsetminus Y^{\beta+1} : \varphi_y\left(x\right) \neq 0\right\}$. Then $\sum_{y \in V_x} \varphi_y\left(x\right) = 1$. If $y \in V_x$, then $x \in U_y$; thus $\left|f\left(x\right) - f\left(y\right)\right| < \varepsilon$. Hence

$$\begin{aligned} \left|f\left(x\right) - f_\beta\left(x\right)\right| &= \left|\sum_{y \in V_x} \left(f\left(x\right) - f\left(y\right)\right) \varphi_y\left(x\right)\right| \\ &\leq \sum_{y \in V_x} \left|f\left(x\right) - f\left(y\right)\right| \varphi_y\left(x\right) \leq \varepsilon. \end{aligned}$$

Therefore, $\|f - f_\beta\|_{Y^\beta \smallsetminus Y^{\beta+1}} \leq \varepsilon$, as required. □

Proof of Proposition 1. Define a function $\tilde{f} : X \to \mathbb{R}$ by

$$\tilde{f}\left(x\right) = \begin{cases} f_\beta\left(x\right) & \text{if} \quad x \in Z_\beta \smallsetminus \cup_{\gamma < \beta} Z_\gamma,\ \beta < \alpha, \\ 0 & \text{if} \quad x \notin \cup_{\gamma < \alpha} Z_\gamma. \end{cases}$$

Clearly, $\left\|\tilde{f}\right\|_X = \sup_{\beta < \alpha} \|f_\beta\|_{Z_\beta} \leq \|f\|_Y$. If $x \in Y$, then $x \in Y^\beta \smallsetminus Y^{\beta+1}$ for some $\beta < \alpha$. In particular, $x \in Z_\beta \smallsetminus \cup_{\gamma < \beta} Z_\gamma$. Hence $\left|f\left(x\right) - \tilde{f}\left(x\right)\right| = \left|f\left(x\right) - f_\beta\left(x\right)\right| \leq \|f - f_\beta\|_{Y^\beta \smallsetminus Y^{\beta+1}} \leq \varepsilon$ according to Lemma 2. Since this is true for all $x \in Y$, $\left\|f - \tilde{f}\right\|_Y \leq \varepsilon$.

It remains to show that $\beta_X\left(\tilde{f}\right) \leq \alpha+1$. To this end, we claim that $\mathcal{D}^{\beta}\left(\tilde{f},\delta,X\right) \cap Z_\gamma = \emptyset$ for all $\delta > 0$, $\gamma < \beta \leq \alpha$. We prove the claim by induction. Let $\delta > 0$. Since f_0 is continuous on the open set Z_0, we have $\mathcal{D}^{1}\left(\tilde{f},\delta,X\right) \cap Z_0 = \emptyset$. Suppose that the claim holds for all ordinals less than β. By the inductive hypothesis, $\mathcal{D}^{\xi}\left(\tilde{f},\delta,X\right) \cap (\cup_{\gamma<\xi} Z_\gamma) = \emptyset$ for all $\xi < \beta$. Therefore,

$$\mathcal{D}^{\xi}\left(\tilde{f},\delta,X\right) \cap [Z_\xi \setminus (\cup_{\gamma<\xi} Z_\gamma)] = \mathcal{D}^{\xi}\left(\tilde{f},\delta,X\right) \cap Z_\xi.$$

Now $\tilde{f} = f_\xi$ is continuous on this set, which is open in $\mathcal{D}^{\xi}\left(\tilde{f},\delta,X\right)$. Therefore $\mathcal{D}^{\xi+1}\left(\tilde{f},\delta,X\right) \cap Z_\xi = \emptyset$. Also since $\mathcal{D}^{\beta}\left(\tilde{f},\delta,X\right) \subseteq \mathcal{D}^{\gamma+1}\left(\tilde{f},\delta,X\right)$ for all $\gamma < \beta$,

$$\mathcal{D}^{\beta}\left(\tilde{f},\delta,X\right) \cap Z_\gamma = \emptyset$$

for all $\gamma < \beta$. This proves the claim. It follows from the claim that

$$\mathcal{D}^{\alpha}\left(\tilde{f},\delta,X\right) \subseteq (\cup_{\gamma<\alpha} Z_\gamma)^c$$

for any $\delta > 0$. Since $\tilde{f} = 0$ on the latter set, $\mathcal{D}^{\alpha+1}\left(\tilde{f},\delta,X\right) = \emptyset$. □

In order to iterate Proposition 1 to obtain an extension of f, we need the following result.

Proposition 3. *Let Y be a subspace of a metric space X. If $\beta_Y(f) < \omega^\xi$ and $\beta_Y(g) < \omega^\xi$, then $\beta_Y(f+g) < \omega^\xi$.*

Proposition 3 is proved by the method used in [5, Lemma 5]. This requires a slight modification in the derivation $\mathcal{D}$ associated with the index β.

Given a real valued function $f : Y \to \mathbb{R}$, $\varepsilon > 0$, and a closed subset P of Y, define $G(f,\varepsilon,P)$ to be the set of all $y \in P$ such that for every neighborhood U of y, there exists $y' \in P \cap U$ such that $|f(y) - f(y')| \geq \varepsilon$. Let $\mathcal{G}^0(f,\varepsilon,P) = P$ and

$$\mathcal{G}^1(f,\varepsilon,P) = \overline{G(f,\varepsilon,P)},$$

where the closure is taken in Y. This defines a derivation $\mathcal{G}$ on the closed subsets of Y which may be iterated in the usual manner. If $\alpha < \omega_1$, let

$$\mathcal{G}^{\alpha+1}(f,\varepsilon,P) = \mathcal{G}^1(f,\varepsilon,\mathcal{G}^{\alpha}(f,\varepsilon,P)).$$

If $\alpha < \omega_1$ is a limit ordinal, let

$$\mathcal{G}^{\alpha}(f,\varepsilon,P) = \bigcap_{\alpha'<\alpha} \mathcal{G}^{\alpha'}(f,\varepsilon,P).$$

Clearly, the derivation $\mathcal{G}$ is closely related to $\mathcal{D}$. The precise relationship between $\mathcal{D}$ and $\mathcal{G}$ is given in part (c) of the next lemma.

Lemma 4. *If f and g are real-valued functions on Y, $\varepsilon > 0$, and P, Q are closed subsets of Y, then*

(a) $\mathcal{G}^1(f+g,\varepsilon,P) \subseteq \mathcal{G}^1(f,\varepsilon/2,P) \cup \mathcal{G}^1(g,\varepsilon/2,P)$,
(b) $\mathcal{G}^1(f,\varepsilon,P\cup Q) \subseteq \mathcal{G}^1(f,\varepsilon,P) \cup \mathcal{G}^1(f,\varepsilon,Q)$,
(c) $\mathcal{D}^1(f,2\varepsilon,P) \subseteq \mathcal{G}^1(f,\varepsilon,P) \subseteq \mathcal{D}^1(f,\varepsilon,P)$.

We leave the simple proofs to the reader. Note that it follows from part (c) that for all $\alpha < \omega_1$,

(d)
$$\mathcal{D}^\alpha(f,2\varepsilon,P) \subseteq \mathcal{G}^\alpha(f,\varepsilon,P) \subseteq \mathcal{D}^\alpha(f,\varepsilon,P).$$

Proof of Proposition 3. Parts (a) and (b) of Lemma 4 correspond to (*) and (**) in [5, Lemma 5] respectively. From the proof of that result, we obtain for all $n \in \mathbb{N}$ and $\zeta < \omega_1$,

(1)
$$\mathcal{G}^{\omega^\zeta\cdot 2n}(f+g,\varepsilon,Y) \subseteq \mathcal{G}^{\omega^\zeta\cdot n}(f,\varepsilon/2,Y) \cup \mathcal{G}^{\omega^\zeta\cdot n}(g,\varepsilon/2,Y).$$

Since $\beta_Y(f) < \omega^\xi$ and $\beta_Y(g) < \omega^\xi$, there exist $\zeta < \xi$ and $n \in \mathbb{N}$ such that $\beta_Y(f) < \omega^\zeta \cdot n$ and $\beta_Y(g) < \omega^\zeta \cdot n$. By (d), for any $\varepsilon > 0$,

$$\mathcal{G}^{\omega^\zeta\cdot n}(f,\varepsilon/2,Y) = \mathcal{G}^{\omega^\zeta\cdot n}(g,\varepsilon/2,Y) = \emptyset.$$

By (d) and (1) ,

$$\mathcal{D}^{\omega^\zeta\cdot 2n}(f+g,2\varepsilon,Y) = \emptyset.$$

Since this is true for all $\varepsilon > 0$, we have

$$\beta_Y(f+g) \le \omega^\zeta \cdot 2n < \omega^\xi.$$

□

Theorem 5. *Let X be a metric space and let Y be an arbitrary subspace of X. If $f : Y \to \mathbb{R}$ satisfies $\beta_Y(f) < \omega^\alpha$ for some $\alpha < \omega_1$, then there exists $F : X \to \mathbb{R}$ with $\beta_X(F) \le \omega^\alpha$ and $F_{|Y} = f$.*

Proof. Applying Proposition 1 to $f : Y \to \mathbb{R}$ with $\varepsilon = \frac{1}{2}$, we obtain $g_1 : X \to \mathbb{R}$, with $\|f - g_1\|_Y \le \frac{1}{2}$, and $\beta_X(g_1) < \omega^\alpha$. By Proposition 3, $\beta_Y(f - g_1) < \omega^\alpha$. Now apply Proposition 1 to $(f-g_1)_{|Y}$ with $\varepsilon = \frac{1}{2^2}$. We obtain $g_2 : X \to \mathbb{R}$, with $\|g_2\|_X \le \|f - g_1\|_Y \le \frac{1}{2}$, $\|f - g_1 - g_2\|_Y \le \frac{1}{2^2}$, and $\beta_X(g_2) < \omega^\alpha$. Continuing in this way, we obtain a sequence (g_n) of real-valued functions on X such that for all $n \in \mathbb{N}$,

(i) $\|g_{n+1}\|_X \le \|f - \sum_{i=1}^n g_i\|_Y \le \frac{1}{2^n}$,
(ii) $\beta_X(g_n) < \omega^\alpha$.

Let $F = \sum_{n=1}^\infty g_n$. Note that the series converges uniformly on X and $g_{|Y} = f$ by (i). Finally, suppose that $\varepsilon > 0$. Choose N such that $\sum_{n=N+1}^\infty \|g_n\|_X < \varepsilon/4$. Then

$$\mathcal{D}^{\omega^\alpha}(F,\varepsilon,X) \subseteq \mathcal{D}^{\omega^\alpha}\left(\sum_{n=1}^N g_n, \frac{\varepsilon}{2}, X\right) = \emptyset,$$

since $\beta_X\left(\sum_{n=1}^N g_n\right) < \omega^\alpha$ by Proposition 3. Thus $\beta_X(F) \le \omega^\alpha$. □

Corollary 6 (Kuratowski, [7, §31, VI]). *Let X be a Polish space and Y be a G_δ subset of X. Then every real-valued function of Baire class one on Y can be extended to a function of Baire class one on X.*

Remarks.
1. Kuratowski's theorem holds for functions with arbitrary Polish ranges. We do not know if our theorem is true in this more general context.
2. In general, the condition $\beta_Y(f) < \omega_1$ implies that f is of Baire class one on Y but not *vice versa*. Indeed, if Y is a subspace of a metric space X, then $\beta_Y(f) < \omega_1$ if and only if f has an extension f' to a G_δ-subset Y' of X such that $\beta_{Y'}(f') = \beta_Y(f)$. The two conditions coincides if Y is Polish.
3. Theorem 5 may be viewed as follows: For any $\beta < \omega_1$, there exists $\sigma(\beta) < \omega_1$ such that if f is a real-valued function defined on a subspace Y of a metric space X with $\beta_Y(f) = \beta$, then there exists $F : X \to \mathbb{R}$ with $\beta_X(F) \le \sigma(\beta)$ and $F_{|Y} = f$. (In fact, Theorem 5 shows that if $\beta = \omega^\alpha$, then $\sigma(\beta) = \omega^{\alpha+1}$ works.) A natural question is to ask for the optimal (i.e., minimal) value of $\sigma(\beta)$. Theorem 14 and Example 15 together show that $\sigma(1) = 3$ is optimal. We do not know the optimal value of $\sigma(\beta)$ for $1 < \beta < \omega_1$.

2. Extension of Continuous Functions

In this section, we study the extension of a continuous function on a subspace of a metric space to the whole space. To begin with, we consider the extension of a continuous function from a dense subspace.

Consider a metric space X with a dense subspace Y. Suppose that $f : Y \to \mathbb{R}$ is continuous on Y. An obvious way of extending f to X (if f is locally bounded) is to consider the limit superior (or limit inferior) of f, i.e.,

$$\tilde{f}(x) = \limsup_{y \to x,\ y \in Y} f(y) = \inf_{\delta > 0} \sup_{\substack{d(x,y) < \delta \\ y \in Y}} f(y).$$

The extended function, which is upper semi-continuous (lower semi-continuous in the case of limit inferior), is not optimal as far as the oscillation index is concerned. In fact, the lim sup extension $\tilde{f}$ of the continuous function f in Example 15 below has oscillation index $\beta_X\left(\tilde{f}\right) = \omega$. The following is an alternative algorithm that produces an extension with the smallest possible oscillation index. If $A \subseteq \operatorname{dom} f$, $\operatorname{osc}(f, A)$ is defined to be $\sup\{|f(x) - f(x')| : x, x' \in A\}$. If x belongs to the closure of $\operatorname{dom} f$, then define

$$\operatorname{osc}(f, x) = \lim_{\delta \to 0} \operatorname{osc}(f, B(x, \delta) \cap \operatorname{dom} f).$$

We first define layers of approximate extensions inductively. Precisely, for each $k \ge 0$, we will choose open sets S_k and X_k such that $Y \subseteq S_k \subseteq X_k$, nonnegative integer $(n_k(s))_{s \in S_k}$ and a function $F_k : X_k \to \mathbb{R}$. Let $S_0 = X$ and $n_0(s) = 0$ for all $s \in S_0$. Assume that S_k has been chosen and $n_k(s)$ is defined for all $s \in S_k$. Let $\mathcal{U}_k = \{B(s, 2^{-n_k(s)}) : s \in S_k\}$ and $X_k = \cup \mathcal{U}_k$.

Choose a partition of unity $(\varphi_s^k)_{s\in S_k}$ on X_k subordinated to $\mathcal{U}_k$. For each $s \in S_k$, choose $y_s^k \in Y \cap B(s, 2^{-n_k(s)})$. Define $F_k : X_k \to \mathbb{R}$ by $F_k(x) = \sum_{s\in S_k} \varphi_s^k(x) f(y_s^k)$. For each $x \in X_k$, let $S_k(x) = \{s \in S_k : x \in \operatorname{supp} \varphi_s^k\}$ and $l_k(x) = \max\{n_k(s) : s \in S_k(x)\} + 1$. Note that $S_k(x)$ is a finite set since $\left(\operatorname{supp} \varphi_s^k\right)_{s\in S_k}$ is locally finite. Let S_{k+1} be the set of all $x \in X_k$ such that $\operatorname{osc}(f, x) < 2^{-l_k(x)}$. If $x \in S_{k+1}$, choose $n_{k+1}(x) \ge l_k(x)$ so that

(1) $\operatorname{osc}(f, B(x, 2^{1-n_{k+1}(x)}) \cap Y) < 2^{-l_k(x)}$,
(2) $B(x, 2^{-n_{k+1}(x)}) \subseteq B(s, 2^{-n_k(s)})$ for all $s \in S_k(x)$,
(3) $B(x, 2^{1-n_{k+1}(x)}) \cap \operatorname{supp} \varphi_s^k = \emptyset$ if $s \in S_k \backslash S_k(x)$.

The extension F (defined after Lemma 8) is obtained by pasting the layers (F_k) one after another. Observe that $X_{k+1} \subseteq X_k$ because of condition (2).

Lemma 7. *Suppose that $s \in S_k$, $t \in S_m$ for some $m > k$, and that $\operatorname{supp} \varphi_s^k \cap \operatorname{supp} \varphi_t^m \neq \emptyset$. Then $B(t, 2^{-n_m(t)}) \subseteq B(s, 2^{-n_k(s)})$.*

Proof. Let $x \in \operatorname{supp} \varphi_s^k \cap \operatorname{supp} \varphi_t^m$. Then $x \in X_j$ for all $j \le m$. In particular, if $m > j > k$, then there exists $s_j \in S_j$ such that $x \in \operatorname{supp} \varphi_{s_j}^j$. Thus it suffices to prove the lemma for $m = k + 1$. Assume that $x \in \operatorname{supp} \varphi_s^k \cap \operatorname{supp} \varphi_t^{k+1}$. Note that $s \in S_k(t)$. For otherwise, $B(t, 2^{1-n_{k+1}(t)}) \cap \operatorname{supp} \varphi_s^k = \emptyset$ by (3). Since x belongs to this set, we have reached a contradiction. It now follows from (2) that $B(t, 2^{-n_{k+1}(t)}) \subseteq B(s, 2^{-n_k(s)})$. □

Lemma 8. *Suppose that $x \in X_m$ and $m > k \ge 1$. Then there exists $s \in S_k(x)$ such that $|F_k(x) - F_m(x)| < 2^{1-l_{k-1}(s)}$. Moreover, if $x \in Y$, then $|F_k(x) - f(x)| < 2^{-l_{k-1}(s)}$ for some $s \in S_k(x)$.*

Proof. Denote by S the set of all $t \in S_m$ such that $\varphi_t^m(x) > 0$ and choose a point $y \in \cap_{t\in S} B(t, 2^{-n_m(t)}) \cap Y$. Let s be an element where $l_{k-1}(s)$ attains its minimum over $S_k(x)$. By Lemma 7, $B(t, 2^{-n_m(t)}) \subseteq B(s, 2^{-n_k(s)})$ for all $t \in S$. Hence $|f(y) - f(y_t^m)| < 2^{-l_{k-1}(s)}$ for any $t \in S$. By Lemma 7 again, $y \in B(t, 2^{-n_m(t)}) \subseteq B(s', 2^{-n_k(s')})$ for all $t \in S$ and all $s' \in S_k(x)$. Hence

$$|f(y) - f(y_{s'}^k)| < 2^{-l_{k-1}(s')} \le 2^{-l_{k-1}(s)}$$

for all $s' \in S_k(x)$. Therefore

$$\begin{aligned} |F_k(x) - F_m(x)| &\le |F_k(x) - f(y)| + |f(y) - F_m(x)| \\ &< 2^{-l_{k-1}(s)} + 2^{-l_{k-1}(s)} = 2^{1-l_{k-1}(s)}. \end{aligned}$$

Moreover, if $x \in Y$, then the above applies for $y = x$. Hence $|F_k(x) - f(x)| < 2^{-l_{k-1}(s)}$. □

Observe that $l_k(s) \ge k+1$ for all $s \in S_k$, $k \ge 0$. It follows from Lemma 8 that (F_k) converges pointwise on $\cap X_k$ and that the limit is f on Y. Define $F : X \to \mathbb{R}$ by

$$F(x) = \begin{cases} \lim_k F_k(x) & \text{if } x \in \cap X_k, \\ F_k(x) & \text{if } x \in X_k \backslash X_{k+1},\ k \ge 0. \end{cases}$$

Then F is an extension of f to X.

Lemma 9. *Suppose that $x \in X_k$ for some $k \geq 1$. There exists an open neighborhood U of x and $s \in S_k(x)$ such that $|F(z) - F(x)| < 2^{3-l_{k-1}(s)}$ for all $z \in U$.*

Proof. Let s be an element where $l_{k-1}(s)$ attains its minimum over $S_k(x)$. Note that F_k is continuous on the open set X_k. Hence there is an open neighborhood U of x such that

(1) $\operatorname{osc}(F_k, U) < 2^{-l_{k-1}(s)}$,
(2) $U \subseteq X_k$,
(3) $U \cap \operatorname{supp} \varphi_s^k = \emptyset$ if $s \in S_k \backslash S_k(x)$.

We claim that $S_k(z) \subseteq S_k(x)$ for all $z \in U$. Indeed, if $z \in U$ and $s \in S_k(z) \backslash S_k(x)$, then $z \in U \cap \operatorname{supp} \varphi_s^k = \emptyset$, a contradiction. Now if $z \in U$, then either $z \in X_m$ for all m or $z \in X_m \backslash X_{m+1}$ for some $m \geq k$. In either case, $|F_k(z) - F(z)| \leq 2^{1-l_{k-1}(s)}$ by Lemma 8. Therefore,

$$\begin{aligned} |F(z) - F(x)| &\leq |F(z) - F_k(z)| + |F_k(z) - F_k(x)| + |F_k(x) - F(x)| \\ &< 2^{1-l_{k-1}(s)} + 2^{-l_{k-1}(s)} + 2^{1-l_{k-1}(s)} < 2^{3-l_{k-1}(s)}. \end{aligned}$$

□

The next proposition is an immediate consequence of Lemma 9.

Proposition 10. *Every $x \in \cap X_k$ is a point of continuity of F.*

Proposition 11. *If $x \in \mathcal{D}^1(F, 2^{-m}, X) \cap X_k$, $k \geq 1$, then there exists $s \in S_k(x)$ such that $l_{k-1}(s) \leq m + 3$.*

Proof. Since $x \in X_k$, by Lemma 9, there exist an open neighborhood U of x and $s \in S_k(x)$ such that for all $z \in U$, $|F(z) - F(x)| < 2^{3-l_{k-1}(s)}$. Hence $|F(z_1) - F(z_2)| < 2^{4-l_{k-1}(s)}$ for all $z_1, z_2 \in U$. As $x \in \mathcal{D}^1(F, 2^{-m}, X)$, $-m < 4 - l_{k-1}(s)$. Thus $l_{k-1}(s) \leq m + 3$. □

Proposition 12. *Suppose that $x \in X_k \cap \mathcal{D}^2(F, 2^{-m}, X)$, $k \geq 0$. Then $n_k(s) \leq m + 2$ for all $s \in S_k$ such that $\varphi_s^k(x) > 0$.*

Proof. Choose an open neighborhood U_1 of x such that $U_1 \subseteq \{\varphi_s^k > 0\}$ for all $s \in S_k$ such that $\varphi_s^k(x) > 0$. Note that, in particular, $U_1 \subseteq X_k$. Then choose an open neighborhood U_2 of x such that $\operatorname{osc}(F_k, U_2) < 2^{-m}$. Let $U = U_1 \cap U_2$. There exist $z_1, z_2 \in U \cap \mathcal{D}^1(F, 2^{-m}, X)$ such that $|F_k(z_1) - F_k(z_2)| \geq 2^{-m}$. If $z_1, z_2 \notin X_{k+1}$, then $F(z_i) = F_k(z_i)$, $i = 1, 2$. This leads to a contradiction with the fact that $\operatorname{osc}(F_k, U_2) < 2^{-m}$. Thus at least one of z_1, z_2 belongs to X_{k+1}. Denote it by z. By the previous proposition, there exists $t \in S_{k+1}(z)$ such that $l_k(t) \leq m + 3$. Let $s \in S_k$ be such that $\varphi_s^k(x) > 0$. We claim that $s \in S_k(t)$. For otherwise, $B(t, 2^{1-n_{k+1}(t)}) \cap \operatorname{supp} \varphi_s^k = \emptyset$. This is absurd since the intersection contains the point z. It follows from that claim that $l_k(t) \geq n_k(s) + 1$. Hence $n_k(s) \leq m + 2$, as required. □

Proposition 13. $\beta_X(F) \leq 3$.

Proof. Suppose that $x \in \mathcal{D}^3(F, 2^{-m}, X)$ for some m. Then there exists k such that $x \in X_k \backslash X_{k+1}$. Choose a neighborhood U of x such that $U \subseteq B(x, 2^{-m-2}) \cap X_k$ and $\operatorname{osc}(F_k, U) < 2^{-m}$. There exist $z_1, z_2 \in U \cap \mathcal{D}^2(F, 2^{-m}, X)$ such that $|F(z_1) - F(z_2)| \geq 2^{-m}$. If $z_1, z_2 \notin X_{k+1}$, then $F(z_i) = F_k(z_i)$, $i = 1, 2$. This contradicts the fact that $\operatorname{osc}(F_k, U) < 2^{-m}$. Hence there exists $z \in U \cap X_{k+1} \cap \mathcal{D}^2(F, 2^{-m}, X)$. By Proposition 12, $n_{k+1}(t) \leq m + 2$ for all $t \in S_{k+1}$ such that $\varphi_t^{k+1}(z) > 0$. Fix such a t. Note that

$$\begin{aligned} d(x,t) &\leq d(x,z) + d(z,t) \\ &< 2^{-m-2} + 2^{-n_{k+1}(t)} \leq 2^{1-n_{k+1}(t)}. \end{aligned}$$

Thus

$$\operatorname{osc}(f, x) \leq \operatorname{osc}(f, B(t, 2^{1-n_{k+1}(t)}) \cap Y) < 2^{-l_k(t)}.$$

We claim that $S_k(x) \subseteq S_k(t)$. For otherwise, there exists $s \in S_k(x) \backslash S_k(t)$. Then $B(t, 2^{1-n_{k+1}(t)}) \cap \operatorname{supp} \varphi_s^k = \emptyset$. This is absurd since the intersection contains the point x. It follows from the claim that $l_k(t) \geq l_k(x)$. Hence $\operatorname{osc}(f, x) < 2^{-l_k(x)}$. Then $x \in S_{k+1} \subseteq X_{k+1}$, a contradiction. □

Theorem 14. *Let X be a metric space and Y be a subspace of X. Every continuous function f on Y can be extended to a function F on X with $\beta_X(F) \leq 3$.*

Proof. Applying the preceding lemmas and propositions, we obtain an extension $\tilde{f}$ of f to $\overline{Y}$ such that $\beta_{\overline{Y}}\left(\tilde{f}\right) \leq 3$. By [8, Theorem 3.6], there is a further extension F of $\tilde{f}$ to X such that $\beta_X(F) = \beta_{\overline{Y}}\left(\tilde{f}\right) \leq 3$. (Note that the proof of [8, Theorem 3.6] does not require the compactness of X.) □

The following example shows that Theorem 14 is optimal.

Example 15. *There is a subspace $Y \subseteq \{0,1\}^\omega = X$ and a continuous real-valued function f on Y such that for any extension F of f to X, $\beta_X(F) \geq 3$.*

Proof. For any integer n, denote $n \pmod 2$ by $\hat{n}$. Let

$$Y = \{(\varepsilon_i) \in X : \varepsilon_i = 0 \text{ for infinitely many } i\text{'s}\}.$$

We denote elements in X of the form

$$\left(\underbrace{1,1,...,1}_{n_1}, 0, \underbrace{1,1,...,1}_{n_2}, 0, ..., \underbrace{1,1,...,1}_{n_k}, 0, ...\right)$$

by

$$(1^{n_1}, 0, 1^{n_2}, 0, ..., 1^{n_k}, 0, ...).$$

Also write $(\varepsilon_1, ..., \varepsilon_k, \varepsilon, \varepsilon, ...)$ as $(\varepsilon_1, ..., \varepsilon_k, \varepsilon^\omega)$, $\varepsilon_i, \varepsilon \in \{0,1\}$. Define $g : Y \to X$ by

$$g(1^{n_1}, 0, 1^{n_2}, 0, ..., 1^{n_k}, 0, ...) = (\hat{n}_1, \hat{n}_2, ...), \ n_1, n_2, ... \in \mathbb{N} \cup \{0\},$$

and let $h : X \to \mathbb{R}$ be the canonical embedding of X into $\mathbb{R}$, $h(\varepsilon_1, \varepsilon_2, ...) = \sum_{k=1}^{\infty} \frac{2\varepsilon_k}{3^k}$. Then the function $f = h \circ g : Y \to \mathbb{R}$ is continuous. Suppose that F is an extension of f to X such that $\beta_X(F) \leq 2$. First observe that for any $n_1, ..., n_k \in \mathbb{N} \cup \{0\}$ and all $n \in \mathbb{N}$,

$$\left|F\left(1^{n_1}, 0, ..., 1^{n_k}, 0, 1^{2n}, 0^{\omega}\right) - F\left(1^{n_1}, 0, ..., 1^{n_k}, 0, 1^{2n-1}, 0, 1, 0, 1, ...\right)\right| = \frac{1}{3^k}.$$

Hence $(1^{n_1}, 0, ..., 1^{n_k}, 0, 1^{\omega}) \in \mathcal{D}^1\left(F, \frac{1}{3^k}, X\right)$. Let $F(1^{\omega}) = a$. Either $|a| \geq \frac{1}{2}$ or $|1-a| \geq \frac{1}{2}$. We assume the former; the proof is similar for the latter case. Since $(1^{\omega}) \notin \mathcal{D}^2\left(F, \frac{1}{3}, X\right)$, there exists a neighborhood U of (1^{ω}) such that $|F(x) - a| < \frac{1}{3}$ if $x \in U \cap \mathcal{D}^1\left(F, \frac{1}{3}, X\right)$. In particular, there exists $n_1 \in \mathbb{N}$ such that

$$\left|F\left(1^{2n_1}, 0, 1^{\omega}\right) - a\right| = \frac{1}{3} - \delta \text{ for some } \delta > 0.$$

Similarly, using the fact that $\left(1^{2n_1}, 0, 1^{\omega}\right) \notin \mathcal{D}^2\left(F, \frac{1}{3^2}, X\right)$, we obtain $n_2 \in \mathbb{N}$ such that

$$\left|F\left(1^{2n_1}, 0, 1^{2n_2}, 0, 1^{\omega}\right) - F\left(1^{2n_1}, 0, 1^{\omega}\right)\right| < \frac{1}{3^2}.$$

Continuing, we choose $n_1, n_2, ... \in \mathbb{N}$ such that

$$\left|F\left(1^{2n_1}, 0, ..., 1^{2n_{k+1}}, 0, 1^{\omega}\right) - F\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right)\right| < \frac{1}{3^{k+1}}, \ k \in \mathbb{N}.$$

In particular,

$$\left|F\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right) - a\right| \leq \frac{1}{3} + \frac{1}{3^2} + ... - \delta = \frac{1}{2} - \delta, \ k \in \mathbb{N}.$$

Since $|a| \geq \frac{1}{2}$, we have $\left|F\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right)\right| \geq \delta$ for all $k \in \mathbb{N}$. But

$$F\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{2n}, 0^{\omega}\right) = f\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{2n}, 0^{\omega}\right) = 0$$

for all $n \in \mathbb{N}$. Hence $\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right) \in \mathcal{D}^1(F, \delta, X)$ for all $k \in \mathbb{N}$. However, note that the sequence $\left(\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right)\right)_{k \in \mathbb{N}}$ converges to the point $\left(1^{2n_1}, 0, ..., 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, ...\right)$ and

$$\begin{aligned} &\left|F\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right) - F\left(1^{2n_1}, 0, ..., 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, ...\right)\right| \\ &= \left|F\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right) - f\left(1^{2n_1}, 0, ..., 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, ...\right)\right| \\ &= \left|F\left(1^{2n_1}, 0, ..., 1^{2n_k}, 0, 1^{\omega}\right)\right| \geq \delta \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $\left(1^{2n_1}, 0, ..., 1^{2n_j}, 0, 1^{2n_{j+1}}, 0, ...\right) \in \mathcal{D}^2(F, \delta, X)$, contrary to the assumption that $\beta_X(F) \leq 2$. □

Remark.
With regard to the question raised in Remark 3 in §1, we have been able to show that if Y is a subspace of a countable ordinal X (not necessarily compact), and $f : Y \to \mathbb{R}$ satisfies $\beta_Y(f) \leq 3$, then there is an extension $F : X \to \mathbb{R}$ of f such that $\beta_X(F) \leq \beta_Y(f) + 1$.

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