SEMILATTICE STRUCTURES OF SPREADING MODELS

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Abstract. Given a Banach space $X$, denote by $SP_w(X)$ the set of equivalence classes of spreading models of $X$ generated by normalized weakly null sequences in $X$. It is known that $SP_w(X)$ is a semilattice, i.e., it is a partially ordered set in which every pair of elements has a least upper bound. We show that every countable semilattice that does not contain an infinite increasing sequence is order isomorphic to $SP_w(X)$ for some separable Banach space $X$.

Given a normalized basic sequence $(y_i)$ in a Banach space and $\varepsilon_n \downarrow 0$, using Ramsey’s Theorem, one can find a subsequence $(x_i)$ and a normalized basic sequence $(\tilde{x}_i)$ such that for all $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \subseteq [-1, 1],

$$|| \sum a_i x_{k_i} || - || \sum a_i \tilde{x}_i || < \varepsilon_n$$

for all $n \leq k_1 < \cdots < k_m$. The sequence $(\tilde{x}_i)$ is called a spreading model of $(x_i)$. It is well-known that if $(x_i)$ is in addition weakly null, then $(\tilde{x}_i)$ is 1-spreading and suppression 1-unconditional. See [3, 5] for more about spreading models. A spreading model $(\tilde{x}_i)$ is said to (C-) dominate another spreading model $(\tilde{y}_i)$ if there is a $C < \infty$ such that for all $(a_i) \subseteq \mathbb{R},$

$$|| \sum a_i \tilde{y}_i || \leq C \sum a_i \tilde{x}_i ||.$$

The spreading models $(\tilde{x}_i)$ and $(\tilde{y}_i)$ are said to be equivalent if they dominate each other. Let $[(\tilde{x}_i)]$ denote the class of all spreading models which are equivalent to $(\tilde{x}_i)$. Let $SP_w(X)$ denote the set of all $[(\tilde{x}_i)]$ generated by normalized weakly null sequences in $X$. If $[(\tilde{x}_i)], [(\tilde{y}_i)] \in SP_w(X)$, we write $[(\tilde{x}_i)] \leq [(\tilde{y}_i)]$ if $(\tilde{y}_i)$ dominates $(\tilde{x}_i)$. $(SP_w(X), \leq)$ is a partially ordered set. The paper [2] initiated the study of the order structures of $SP_w(X)$. It was established that every countable subset of $(SP_w(X), \leq)$ admits an upper bound ([2, Proposition 3.2]). Moreover, from the proof of this result, it follows that every pair of elements in $(SP_w(X), \leq)$ has a least upper bound. In other words, $(SP_w(X), \leq)$ is a semilattice. In [6], it was shown that if $SP_w(X)$ is countable, then it cannot admit a strictly increasing infinite sequence $(\tilde{x}_1) < (\tilde{x}_2) < \cdots$. In [4], two methods of construction, utilizing Lorentz sequence spaces and Orlicz sequence spaces respectively, were used to produce Banach spaces $X$ so that $SP_w(X)$ has certain prescribed order.

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structures. In the present paper, building on the techniques employed in [4, §2], we show that every countable semilattice that has no infinite increasing sequence is order isomorphic to $SP_w(X)$ for some Banach space $X$. This gives an affirmative answer to Problem 1.15 in [4]. (See, however, the remark at the end of the paper.)

1. A Representation Theorem for Semilattices

Any collection of subsets of a set $V$ that is closed under the taking of finite unions is a semilattice under the order of set inclusion. In this section, we show that any countable semilattice that does not admit an infinite increasing sequence may be represented in such a way using a countable set $V$. The result may be of independent interest.

**Theorem 1.** Let $L$ be a countable semilattice with no infinite increasing sequences. Then there exist a countable set $V$ and an injective map $T : L \rightarrow 2^V \setminus \{\emptyset\}$ that preserves the semilattice structure of $L$, i.e., $T(x \lor y) = T(x) \cup T(y)$ for all $x, y \in L$.

Suppose that $L$ is a semilattice that satisfies the hypothesis of Theorem 1. Note that every nonempty subset of $L$ has at least one maximal element; for otherwise, it will admit an infinite increasing sequence. Set $L_0 = L$. If $L_\alpha$ is defined for some countable ordinal $\alpha$ and $L_\alpha \neq \emptyset$, let $L_{\alpha+1} = L_\alpha \setminus \{\text{maximal elements in } L_\alpha\}$. If $\alpha$ is a countable limit ordinal such that $L_{\alpha'} \neq \emptyset$ for all $\alpha' < \alpha$, let $L_{\alpha} = \bigcap_{\alpha' < \alpha} L_{\alpha'}$. Since $(L_{\alpha})$ is a strictly decreasing transfinite sequence of subsets of the countable set $L$, $L_\alpha = \emptyset$ for some countable ordinal $\alpha$. Let $\alpha_0$ be the smallest ordinal such that $L_{\alpha_0} = \emptyset$.

Enumerate $L$ as a transfinite sequence $(e_\beta)_{\beta < \beta_0}$ so that if $e_\beta_1 \in L_{\alpha_1} \setminus L_{\alpha_1+1}$ and $e_\beta_2 \in L_{\alpha_2} \setminus L_{\alpha_2+1}$ for some $\alpha_1 < \alpha_2 < \alpha_0$, then $\beta_1 < \beta_2$. If $1 \leq \beta \leq \beta_0$, let $U_\beta = \{e_\beta' : \beta' < \beta\}$. Note that $L = U_{\beta_0}$.

**Lemma 2.**

(a) $e_\beta$ is a minimal element in $U_{\beta+1}$.

(b) If $e_\beta = e_{\beta_1} \lor e_{\beta_2}$ (least upper bound taken in $L$), then $\beta \leq \min \{\beta_1, \beta_2\}$.

(c) If $e_{\beta_1}, e_{\beta_2} \in U_\beta$, then $e_{\beta_1} \lor e_{\beta_2}$ belongs to $U_\beta$.

**Proof.** (a) Suppose on the contrary that $e_\beta$ is not a minimal element in $U_{\beta+1}$. Then there exists $e_{\beta'} \in U_{\beta+1}$ with $e_{\beta'} < e_\beta$. It follows from the definition of $U_{\beta+1}$ that $\beta' < \beta$. If $e_\beta \in L_\alpha \setminus L_{\alpha+1}$ and $e_{\beta'} \in L_{\alpha'} \setminus L_{\alpha'+1}$, then $\alpha' \leq \alpha$ and hence $L_\alpha \subseteq L_{\alpha'}$. Since $e_\beta, e_{\beta'} \in L_{\alpha'}$ and $e_{\beta'} < e_\beta$, $e_{\beta'}$ is not maximal in $L_{\alpha'}$. Thus $e_{\beta'} \in L_{\alpha'+1}$, a contradiction.

(b) Suppose that $\beta_1 < \beta$. Then $e_{\beta_1} \in U_{\beta+1}$ and $e_{\beta_1} < e_\beta$, contrary to the minimality of $e_\beta$ in $U_{\beta+1}$. Thus $\beta \leq \beta_1$. Similarly, $\beta \leq \beta_2$.

(c) Follows immediately from (b). \qed

If $1 \leq \beta < \omega_1$, write $\beta = \gamma + n$, where $\gamma$ is a limit ordinal, $n < \omega$, and let $V_\beta$ denote the ordinal interval $[0, \gamma + 2n)$. We define a family of maps $T_\beta : U_\beta \rightarrow 2^{V_\beta} \setminus \{\emptyset\}$, $1 \leq \beta \leq \beta_0$, inductively so that $T = T_{\beta_0}$ is the map
sought for in Theorem 1. Let \( T_1 : U_1 = \{ e_0 \} \to 2^{V_1} \setminus \{ \emptyset \} \) be defined by
\[ T_1(e_0) = \{ 0, 1 \} . \]
If \( T_\beta \) has been defined, \( 1 \leq \beta < \beta_0 \), let
\[ T_{\beta+1}(x) = \begin{cases} T_\beta(x) \cup \{ \gamma + 2n, \gamma + 2n + 1 \} & \text{if } x \in U_{\beta+1} \setminus \{ e_\beta \}, \\ \bigcap_{e_\beta < z \in U_\beta} T_\beta(z) \cup \{ \gamma + 2n + 1 \} & \text{if } x = e_\beta. \end{cases} \]

When \( \beta \leq \beta_0 \) is a limit ordinal and \( e_\beta' \in U_\beta \), let \( T_\beta(e_\beta') = \bigcup_{\beta' < \xi < \beta} T_\xi(e_\beta') \). The next result, which shows the compatibility of the definitions of \( T_\beta \) for different \( \beta \)’s, is the key to the subsequent arguments.

**Lemma 3.** If \( 1 \leq \beta_1 < \beta_2 \leq \beta_0 \) and \( \beta_i = \gamma_i + n_i, \ i = 1, 2 \), then
\[ T_{\beta_2}(e_{\beta_i}) = T_{\beta_1+1}(e_{\beta_i}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2). \]

**Proof.** If \( \beta_2 = \beta_1 + 1 \), the assertion holds clearly. Suppose that the assertion holds for some \( \beta_2 > \beta_1 \). By the definition of \( T_{\beta_2+1} \),
\[ T_{\beta_2+1}(e_{\beta_i}) = T_{\beta_2}(e_{\beta_i}) \cup \{ \gamma_2 + 2n_2, \gamma_2 + 2n_2 + 1 \} \]
\[ = T_{\beta_1+1}(e_{\beta_i}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2 + 1] \]
\[ = T_{\beta_1+1}(e_{\beta_i}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2 + 2). \]

Suppose that \( \beta_2 \leq \beta_0 \) is a limit ordinal and the assertion holds for all \( \beta_1 < \xi < \beta_2 \). For such \( \xi \), let \( \xi = \gamma_\xi + n_\xi \). By the inductive hypothesis,
\[ T_{\xi}(e_{\beta_i}) = T_{\beta_1+1}(e_{\beta_i}) \cup [\gamma_1 + 2n_1 + 2, \gamma_\xi + 2n_\xi \). \]
Since \( \beta_2 \) is a limit ordinal, we have
\[ T_{\beta_2}(e_{\beta_i}) = \bigcup_{\beta_1 < \xi < \beta_2} T_{\xi}(e_{\beta_i}) \]
\[ = T_{\beta_1+1}(e_{\beta_i}) \cup [\gamma_1 + 2n_1 + 2, \beta_2) \]
\[ = T_{\beta_1+1}(e_{\beta_i}) \cup [\gamma_1 + 2n_1 + 2, \gamma_2 + 2n_2), \]
as required. (Note that \( n_2 = 0 \) since \( \beta_2 \) is a limit ordinal). \( \square \)

**Lemma 4.** The map \( T_\beta : U_\beta \to 2^{V_\beta} \setminus \{ \emptyset \} \) is injective if \( 1 \leq \beta \leq \beta_0 \).

**Proof.** Suppose that \( e_\beta_1 \) and \( e_\beta_2 \) are distinct elements in \( U_\beta \), with \( \beta_1 < \beta_2 < \beta \). Write \( \beta_2 = \gamma_2 + n_2 \). It follows from Lemma 3 that \( \gamma_2 + 2n_2 \in T_\beta(e_{\beta_1}) \setminus T_\beta(e_{\beta_2}) \). \( \square \)

**Proposition 5.** If \( 1 \leq \beta \leq \beta_0 \), then \( T_\beta(x \lor y) = T_\beta(x) \cup T_\beta(y) \) for all \( x, y \in U_\beta \). In particular, \( T_\beta(x) \subseteq T_\beta(y) \) if \( x \leq y \).

**Proof.** The second statement follows easily from the first. We prove the first statement by induction on \( \beta \). The result is clear if \( \beta = 1 \). Suppose that the assertion is true for some \( \beta, 1 \leq \beta < \beta_0 \). Let \( x = e_\beta_1, y = e_\beta_2 \in U_{\beta+1} \). We may assume that \( \beta_1 < \beta_2 < \beta + 1 \). Write \( \beta = \gamma + n \), and \( \beta_i = \gamma_i + n_i \), \( i = 1, 2 \), and consider two cases.
Case 1. $\beta_1 < \beta_2 < \beta$. By Lemma 3 and the inductive hypothesis,
\[
T_{\beta+1}(e_{\beta_1}) \cup T_{\beta+1}(e_{\beta_2}) = T_{\beta+1}(e_{\beta_1}) \cup [\gamma_1 + 2n_1 + 2, \gamma + 2n] \cup T_{\beta+1}(e_{\beta_2}) \cup [\gamma_2 + 2n_2 + 2, \gamma + 2n + 1] = T_{\beta}(e_{\beta_1}) \cup T_{\beta}(e_{\beta_2}) \cup [\gamma + 2n, \gamma + 2n + 1] = T_{\beta+1}(e_{\beta_1} \lor e_{\beta_2}),
\]
by definition of $T_{\beta+1}$, since $e_{\beta_1} \lor e_{\beta_2} \neq e_{\beta}$ by part (b) of Lemma 2.

Case 2. $\beta_1 < \beta_2 = \beta$. In this case,
\[
T_{\beta+1}(x) \cup T_{\beta+1}(y) = \bigcap_{e_{\beta} \in U_{\beta}} [T_{\beta}(x) \cup T_{\beta}(z)] \cup [\gamma + 2n, \gamma + 2n + 1].
\]
Note that by part (b) of Lemma 2, $x \lor e_{\beta} = e_{\xi}$ for some $\xi \leq \beta_1$. Hence, $x \lor e_{\beta} \in U_{\beta+1} \setminus \{e_{\beta}\} = U_{\beta}$. Thus, it suffices to show that
\[
\bigcap_{e_{\beta} \in U_{\beta}} [T_{\beta}(x) \cup T_{\beta}(z)] = T_{\beta}(x \lor e_{\beta}) = T_{\beta}(x \lor y).
\]
Since $e_{\beta} < x \lor e_{\beta} \in U_{\beta}$, $\bigcap_{e_{\beta} \in U_{\beta}} T_{\beta}(z) \subseteq T_{\beta}(x \lor e_{\beta})$. By the inductive hypothesis, $T_{\beta}(x) \subseteq T_{\beta}(x \lor e_{\beta})$. It follows that $\bigcap_{e_{\beta} \in U_{\beta}} [T_{\beta}(x) \cup T_{\beta}(z)] \subseteq T_{\beta}(x \lor e_{\beta})$. On the other hand, if $e_{\beta} < z \in U_{\beta}$, then $x \lor e_{\beta} \leq x \lor z \in U_{\beta}$. By the inductive hypothesis, $T_{\beta}(x \lor e_{\beta}) \subseteq T_{\beta}(x \lor z) = T_{\beta}(x) \cup T_{\beta}(z)$. Therefore, $T_{\beta}(x \lor e_{\beta}) \subseteq \bigcap_{e_{\beta} \in U_{\beta}} [T_{\beta}(x) \cup T_{\beta}(z)]$.

Suppose that $\beta$ is a limit ordinal and the Proposition holds for all $\beta' < \beta$. Let $x, y \in U_{\beta}$. We may assume that $x = e_{\beta_1}$ and $y = e_{\beta_2}$ for some $\beta_1 < \beta_2 < \beta$. Let $\beta_i = \gamma_i + n_i$, $i = 1, 2$. Using Lemma 3 and the inductive hypothesis,
\[
T_{\beta}(x) \cup T_{\beta}(y) = T_{\beta+1}(e_{\beta_1}) \cup T_{\beta+1}(e_{\beta_2}) \cup \gamma_1 + 2n_1 + 2, \beta) = T_{\beta+1}(e_{\gamma_1}) \cup T_{\beta+1}(e_{\gamma_2}) \cup \gamma_2 + 2n_2 + 2, \beta) = T_{\beta+1}(e_{\gamma_1} \lor e_{\gamma_2}) \cup \gamma_2 + 2n_2 + 2, \beta).
\]
By (b) of Lemma 2, $e_{\beta_1} \lor e_{\beta_2} = e_{\eta}$ for some $\eta \leq \beta_1$. By Lemma 3,
\[
T_{\beta+1}(e_{\beta_1} \lor e_{\beta_2}) = T_{\eta+1}(e_{\eta}) \cup \gamma_\eta + 2n_\eta + 2, \gamma_2 + 2n_2 + 2),
\]
and $T_{\beta+1}(e_{\beta_1} \lor e_{\beta_2}) = T_{\eta+1}(e_{\eta}) \cup \gamma_\eta + 2n_\eta + 2, \beta)$,
where $\eta = \gamma_\eta + n_\eta$. Combining the three preceding equations gives $T_{\beta}(x) \cup T_{\beta}(y) = T_{\beta}(x \lor y)$. \hfill \Box
Proof of Theorem 1. Since $L = U_{\beta_0}$, Theorem 1 follows immediately from Lemma 4 and Proposition 5 by taking $\beta = \beta_0$ in each instance. \hfill \Box

2. Good Lorentz Functions

A Lorentz sequence is a non-increasing sequence $(w(n))_{n=1}^\infty$ of positive numbers such that $w(1) = 1$, $\lim_n w(n) = 0$ and $\sum_{n=1}^\infty w(n) = \infty$. A Lorentz sequence is C-submultiplicative if $S(mn) \leq CS(m)S(n)$ for all $m, n \in \mathbb{N}$, where $S(n) = \sum_{k=1}^n w(k)$. In [4, §2], an infinite sequence of 1-submultiplicative Lorentz sequences is constructed so that the maxima of any two incomparable finite subsets are incomparable (see [4, Proposition 2.6]). For our purpose, we require an infinite sequence of C-submultiplicative Lorentz sequences so that the supremum of any (finite or infinite) subset remains a C-submultiplicative Lorentz function, and that the suprema of any two incomparable (finite or infinite) subsets are incomparable (Proposition 10). This is done by tweaking the arguments in [4, §2]. Following [4], we will find it more convenient to work with functions defined on real intervals. If $2 \leq N < \infty$, a good Lorentz function (GLF) on $(0, N)$ is a function $w : (0, N] \to (0, \infty)$ such that

(1) $w(x) = 1$, $x \in (0, 2]$,
(2) $w$ is nonincreasing, and
(3) If $1 \leq x, y \leq xy \leq N$, then $\int_0^x w \leq \int_0^y w$.

A GLF on $(0, \infty)$ (or simply a GLF) is a function $w : (0, \infty) \to (0, \infty)$ such that $w|_{(0, N]}$ is a GLF on $(0, N]$ for any $N \geq 2$, $\lim_{x \to \infty} w(x) = 0$ and $\int_0^\infty w = \infty$. It is an easy exercise to verify that if $w$ is a GLF, then $(w(n))_{n=1}^\infty$ is a 4-submultiplicative Lorentz sequence.

If $(u_i)$ is a finite or infinite sequence of real-valued functions with pairwise disjoint domains, let $\oplus_i u_i$ denote the set theoretic union. The constant 1 function with domain $I$ is denoted by $1_I$. We now recall the relevant facts from [4]. Note that the quantity $S(x)$ there corresponds to $\int_0^x w$ in our notation.

Lemma 6. [4, Lemma 2.2] Let $w$ be a GLF on $(0, N]$, $N \geq 2$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $w \oplus \varepsilon 1_{(N, N^2]}$ is a GLF on $(0, N^2]$.

Repeated applications of Lemma 6 yield

Lemma 7. Let $G$ be a finite set of GLF’s on $(0, N]$, $N \geq 2$. For any $N' > N$ and any $\varepsilon > 0$, there is a function $v : (N, N'] \to (0, \infty)$ such that $w \oplus v$ is a GLF on $(0, N']$ for all $w \in G$, $v(x) \leq \varepsilon$, $x \in (N, N']$, and $\int_N^{N'} v < \varepsilon$.

On the other hand, the proof of [4, Lemma 2.4] allows us to obtain GLF extensions with large total weight.

Lemma 8. Let $G$ be a finite set of GLF’s on $(0, N]$, $N \geq 2$ and set $K = \min_{w \in G} \int_0^N w$. For any $\varepsilon > 0$, there is a function $v : (N, N'] \to (0, \infty)$, $N' > N$, such that
Proposition 10. There exists an infinite sequence \((0, \epsilon)\), \(N \geq 2\). For any \(K < \infty\) and any \(\epsilon > 0\), there is a function \(v : (N, N') \rightarrow (0, \infty)\), \(N' > N\), such that

1. \(v(x) \leq \epsilon, x \in (N, N']\),
2. \(\int_{N}^{N'} v \geq \frac{K}{2}\).

We may repeat the preceding lemma to obtain

Lemma 9. Let \(G\) be a finite set of GLF’s on \((0, N]\), \(N \geq 2\). For any \(K < \infty\) and any \(\epsilon > 0\), there is a function \(v : (N, N') \rightarrow (0, \infty)\), \(N' > N\), such that

1. \(\forall w \in G, w \oplus v \text{ is a GLF on } (0, N']\),
2. \(v(x) \leq \epsilon, x \in (N, N']\),
3. \(\int_{N}^{N'} v \geq \frac{K}{2}\).

Proposition 10. There exists an infinite sequence \((w_p)_{p=1}^\infty\) of GLF’s on \((0, \infty)\) such that for every nonempty \(M \subseteq \mathbb{N}\) and every \(p' \not\in M\),

1. \(w_M = \sup_{p \in M} w_p \text{ is a GLF on } (0, \infty)\),
2. \(\sup_n \int_0^n w_{p'} = \infty\).

Proof. The desired family of incomparable GLF’s is constructed by defining its elements inductively on successive intervals. On each of the segments, each of the \(w_p\)’s is chosen to be either “high” or “low”.

Let \((p_i, q_i)_{i=1}^\infty\) be an enumeration of \(\{(p, q) : p < q, p, q \in \mathbb{N}\}\), and fix a positive sequence \((\varepsilon_i)\) decreasing to 0. For all \(p \in \mathbb{N}\), define \(w_p^0 : (0, 2] \rightarrow (0, \infty)\) by \(w_p^0(x) = 1\). Set \(G_0 = \{w_p^0 : p \in \mathbb{N}\}\).

Assume that for some \(i \in \mathbb{N}\), functions \(w_p^j : (N_{j-1}, N_j] \rightarrow (0, \infty)\), \(0 \leq j < i\) \((N_{-1} = 0, N_0 = 2)\), \(p \in \mathbb{N}\), have been defined so that \(G_{i-1} = \{w_p^0 \oplus \cdots \oplus w_r^{i-1} : r_0, \ldots, r_{i-1} \in \mathbb{N}\}\) is a finite set of GLF’s on \((0, N_{i-1}]\) and that \(\{w_p^j : p \in \mathbb{N}\}\) is a totally ordered set of functions (in the pointwise order) for each \(j \in [0, i]\). Set \(K_{i-1} = \int_{N_{i-1}}^{N_{i-1}} \max G_{i-1}\), where by max \(G_{i-1}\) we mean the pointwise maximum of the set of functions \(G_{i-1}\). By Lemma 9, choose a function \(w_p^{i-1}\) on \((N_{i-1}, N_i]\), \(N_i > N_{i-1}\), such that \(\forall w \oplus v\) is a GLF on \((0, N_i]\) for all \(w \in G_{i-1}\), that \(w_p^i(x) \leq \varepsilon_i\) for all \(x \in (N_{i-1}, N_i]\) and that \(\int_{N_{i-1}}^{N_i} w_p^i \geq q_i K_{i-1}\). On the other hand, by Lemma 7, there exists \(v\) on \((N_{i-1}, N_i]\) such that \(w \oplus v\) is a GLF on \((0, N_i]\) for all \(w \in G_{i-1}\), that \(v(x) \leq w_p^{i-1}(N_{i-1})\) for all \(x \in (N_{i-1}, N_i]\) and that \(\int_{N_{i-1}}^{N_i} v \leq 1\). Define \(w_p^i = v\) for all \(p \neq p_i\). Note that \(G_i = \{w \oplus w_p^i : w \in G_{i-1}, p \in \mathbb{N}\}\) is a finite set of GLF’s on \((0, N_i]\). Obviously, the set \(\{w_p^i : p \in \mathbb{N}\}\) is totally ordered. This completes the inductive construction. Define \(w_p = \oplus_i w_p^i\), \(p \in \mathbb{N}\). Observe that \(K_0 = 2\) and \(K_i \geq K_{i-1} + q_i K_{i-1} \geq 3K_{i-1}\). Hence \(K_i \rightarrow \infty\). Thus

\[N_i \geq N_i - N_{i-1} \geq \int_{N_{i-1}}^{N_i} w_p^i \geq q_i K_{i-1} \rightarrow \infty.\]
Hence $w_p$ is defined on $(0, \infty)$ for all $p \in \mathbb{N}$. If $\emptyset \neq M \subseteq \mathbb{N}$, let $w_M = \sup_{p \in M} w_p$. We claim that $w_M$ is a GLF on $(0, \infty)$. By definition, $w_M|_{[0,N_i]} \in G_i$ for all $i \in \mathbb{N}$. Thus $w_M$ is a GLF on $(0, N_i]$ for all $i \in \mathbb{N}$. Also note that $w_M(x) \leq \varepsilon_i$ for all $x \in (N_{i-1}, N_i]$. Therefore, $\lim_{x \to \infty} w_M(x) = 0$. Furthermore, since $w_M = w_{p_i}$ on $(N_{i-1}, N_i]$ if $p_i \in M$,

$$
\int_0^\infty w_M > \sup_{\{i:p_i \in M\}} \int_{N_{i-1}}^{N_i} w_{p_i} \geq \sup_{\{i:p_i \in M\}} q_i K_{i-1}.
$$

Because of the enumeration, $p_i \in M$ holds for infinitely many $i$. It follows that $\int_0^\infty w_M = \infty$. This shows that $w_M$ is a GLF on $(0, \infty)$.

Finally, note that for all $i$ such that $p_i \notin M$, $\int_{N_{i-1}}^{N_i} w_M \leq 1$ by construction. In particular, if $p' \notin M$, then for all $i$ such that $p_i = p'$,

$$
\int_0^{N_i} w_{p'} \leq \int_0^{N_{i-1}} w_M + \int_{N_{i-1}}^{N_i} w_{p'} \leq \int_0^{N_{i-1}} \max G_{i-1} + \max_{p \in M} \int_{N_{i-1}}^{N_i} w_p \leq K_{i-1} + 1.
$$

On the other hand, for all such $i$,

$$
\int_0^{N_i} w_{p'} \geq \int_{N_{i-1}}^{N_i} w_{p'} = \int_{N_{i-1}}^{N_i} w_{p_i} \geq q_i K_{i-1}.
$$

Hence

$$
\sup_n \int_0^n w_{p'} = \infty.
$$

Given a Lorentz sequence $(w(n))_{n=1}^\infty$ and $1 \leq p < \infty$, the Lorentz sequence space $d(w,p)$ consists of all real sequences $(a_n)$ such that $\sum a_n^p w_n < \infty$, where $(a_n^p)$ denotes the non-increasing rearrangement of $(|a_n|)$.

**Corollary 11.** Let $(w_p)_{p=1}^\infty$ be as above. For every $M \subseteq \mathbb{N}$, and $p \notin M$, the unit vector basis of $d(w,p,1)$ does not dominate that of $d(w,p,1)$.

**Proof.** Let $(v_i)$ and $(u_i)$ denote the respective unit vector bases of $d(w,p,1)$ and $d(w,1)$. According to Proposition 10, for any $K < \infty$, there exists $N \in \mathbb{N}$ such that $\int_0^{N+1} w_p \geq K \int_0^{N+1} w_M$. Then

$$
\|\sum_{i=1}^N v_i\| = \sum_{i=1}^N w_p(i) \geq \int_1^{N+1} w_p = \int_0^{N+1} w_p - 1 \\
\geq K \int_0^{N+1} w_M - 1 \geq K \sum_{i=1}^N w_M(i) - 1 = K \|\sum_{i=1}^N u_i\| - 1.
$$

The result follows since $K$ is arbitrary.  \(\square\)
3. Countable Semilattices of Spreading Models

In this section, we show that every countable semilattice without an infinite increasing sequence is order isomorphic to some $\mathbf{S}P_w(X)$. If $(x_i)$ and $(y_i)$ are sequences in the Banach spaces $X$ and $Y$ respectively, let $(x_i, y_i)$ denote the sequence $(z_i) = (x_i, y_i)$ in the direct sum $X \oplus Y$. The $\ell^p$-sum of an infinite sequence $(X_j)$ of Banach spaces is denoted by $(\sum_{j=1}^{\infty} \oplus X_j)_p$. We omit the easy proof of the next lemma.

Lemma 12. Let $w_1 = (w_1(n))$ and $w_2 = (w_2(n))$ be Lorentz sequences. Then $w = w_1 \lor w_2 = (w_1(n) \lor w_2(n))$ is a Lorentz sequence. Moreover, if $(u_{1n})$ and $(u_{2n})$ are the respective unit vector bases of $d(w_1, 1)$ and $d(w_2, 1)$, then $(u_{1n}) \oplus (u_{2n})$ is equivalent to $(u_n)$, the unit vector basis of $d(w, 1)$.

Lemma 13 ([4, Lemma 3.6]). Let $X = (\sum_{j=1}^{\infty} \oplus X_j)_p$, where $1 \leq p < \infty$ and each $X_j$ is an infinite-dimensional Banach space, and let $(\tilde{x}_i)$ be a spreading model generated by a normalized weakly null sequence in $X$. Then there exist non-negative $(c_j)_{j=0}^{\infty}$ with $\sum_{j=0}^{\infty} c_j^p = 1$ and normalized spreading models $(\tilde{x}_i^j)$ in $X_j$ generated by weakly null sequences such that for all scalars $(a_i)$,

$$\| \sum_i a_i \tilde{x}_i^j \| = \left[ \sum_j c_j^p \| \sum_i a_i \tilde{x}_i^j \|^p + c_0^{p} \| \sum_i |a_i|^p \right]^{1/p}. \quad (1)$$

Remark. If $p = 1$, the final term on the right of equation (1) may be omitted, i.e., $c_0 = 0$. In fact, according to the proof of Lemma 13 in [4, Lemma 3.6], the spreading model $(\tilde{x}_i)$ is generated by a weakly null sequence $(x_i)$ in $X$ in such a way that $c_0 = \lim ||x_i - P_i(x_i)||$, where $P_i(x_i) = (x_i^1, x_i^2, \ldots, x_i^1, 0, 0, 0, \ldots)$. However, since $\ell^1$ has the Schur property (weakly null sequences are norm null), it is easy to see that $\lim ||x_i - P_i(x_i)|| = 0$ for any weakly null sequence $(x_i)$ in $(\sum_{j=1}^{\infty} \oplus X_j)_1$.

The following is the crucial property of Lorentz sequence spaces that we require. It can be deduced from the arguments in [1, §4]:

Theorem 14. [1] Let $w = (w(n))$ be a $C$-submultiplicative Lorentz sequence and $(u_n)$ be the unit vector basis of $d(w, 1)$. For any $\varepsilon > 0$, every normalized block basis in $d(w, 1)$ has a subsequence $(x_n)$ such that either

(a) $(x_n)$ is equivalent to the unit vector basis of $\ell^1$, or
(b) there exists $c > 0$ such that for all $(a_n) \in c_0$, 

$$c \| \sum a_n u_n \| \leq \| \sum a_n x_n \| \leq (C + \varepsilon) \| \sum a_n u_n \|. \quad (2)$$

In particular, if $(\tilde{x}_n)$ is a spreading model generated by a normalized weakly null sequence, then $(\tilde{x}_n)$ satisfies (2) in place of $(x_n)$.

Theorem 15. Given a countable semilattice $L$ with no infinite increasing sequence, there is a Banach space $X_L$ such that $\mathbf{S}P_w(X_L)$ is order isomorphic to $L$. 


Proof. By Theorem 1, there exists a countable set \( V \) and an injective map \( T : L \to 2^V \setminus \{ \emptyset \} \) such that \( T(e \vee f) = T(e) \cup T(f) \) for all \( e, f \in L \). Since \( V \) is countable, by Proposition 10 (and Corollary 11), there is a family \((w_e)_{e \in V}\) of 4-submultiplicative GLF’s such that for each non-empty subset \( M \) of \( V \), \( w_M = \sup_{w \in M} w_v \) is again a (4-submultiplicative) GLF. Moreover, if \( p \notin M \), the unit vector basis of \( d(w_M, 1) \) does not dominate that of \( d(w_p, 1) \). Set \( X_L = (\bigoplus_{e \in L} d(w_{Te}, 1))_1 \). For any \( e \in L \), let \((u_e)\) be the unit vector basis of \( d(w_{Te}, 1) \). \((u_e)\) may be regarded in an obvious way as a normalized weakly null sequence in \( X_L \) which generates a spreading model equivalent to itself. Thus \([u_e]\), the equivalence class containing \((u_e)\), is an element of \( SP_w(X_L) \).

Define a map \( \Theta : L \to SP_w(X_L) \) by \( \Theta e = [\{u_e]\] \). We will show that \( \Theta \) is a bijection such that \( \Theta e_1 \leq \Theta e_2 \) if and only if \( e_1 \leq e_2 \). Hence \( SP_w(X_L) \) is order isomorphic to \( L \).

We first show that \( \Theta \) is onto. Let \([\tilde{x}_i]\) be an element in \( SP_w(X_L) \). By Lemma 13 and the subsequent Remark, there exist a non-negative sequence \((c_e)_{e \in L} \) with \( \sum c_e = 1 \) and normalized spreading models \((\tilde{x}_e)\) in \( d(w_{Te}, 1) \) such that
\[
\| \sum_i a_i \tilde{x}_i \| = \sum_{e \in L} c_e \sum_i a_i \tilde{x}_e. \tag{3}
\]
Since each \( w_{Te} \) is 4-submultiplicative, according to Theorem 14, for each \( e \in L \), there exists \( b_e > 0 \) such that
\[
b_e \sum_i a_i u_e^{f_i} \leq \| a_i \tilde{x}_e \| \leq 5 \sum_i a_i u_e^{f_i}. \tag{4}
\]
Let \( I = \{ e \in L : c_e > 0 \} \). If \( I \) is infinite, write its elements in a sequence \((e_i)_{i=1}^\infty \). Since the sequence \((\vee_{i=1}^n e_i)_{n=1}^\infty \) has no strictly increasing infinite subsequence, there is a finite subset \( J \) of \( I \) such that \( \vee_{e \in J} e \geq e' \) for all \( e' \in I \). If \( I \) is finite, take \( J = I \). Let \( f = \vee_{e \in J} e \). We claim that \((\tilde{x}_i)\) is equivalent to \((u_i)\). Observe that \( e \leq f \) for all \( e \in I \). Hence \( T e \subseteq T f \) and thus \( w_{Te} \leq w_{Tf} \). Therefore, \((u_i)\) is 1-dominated by \((u_i)\). By (3) and (4),
\[
\| \sum_i a_i \tilde{x}_i \| = \sum_{e \in L} c_e \sum_i a_i \tilde{x}_e \leq 5 \sum_{e \in L} c_e \sum_i a_i u_e^{f_i} \leq 5 \sum_{e \in L} c_e \sum_i a_i u_e^{f_i} = 5 \| a_i u_i^{f_i} \|.
\]
On the other hand, by Lemma 12, \( \bigoplus_{e \in J} (u_e) \) is equivalent to \((u_i)\). Using (3) and (4) again,
\[
\| \sum_i a_i \tilde{x}_i \| \geq \| \sum_{e \in J} c_e u_e^{f_i} \| \geq \| \sum_{e \in J} c_e u_e^{f_i} \| \geq \min_{e \in J} \{ c_e b_e \} \sum_i a_i u_i^{f_i} \geq K \| a_i u_i^{f_i} \| \text{ for some } K > 0.
\]
This shows that $\tilde{x}_i$ is equivalent to $(u^f_i)$. Hence $\Theta f = [(u^f_i)] = [(\tilde{x}_i)]$.

Next we show that

$$e_1 \leq e_2 \iff \Theta e_1 \leq \Theta e_2.$$  

If $e_1 \leq e_2$, then $T e_1 \subseteq T e_2$ and hence $w_{Te_1} \leq w_{Te_2}$. It follows that $[(u^e_1)] \leq [(u^e_2)]$. On the other hand, if $e_1 \not\leq e_2$, then $T (e_1) \not\subseteq T (e_2)$. Choose $p \in T (e_1) \setminus T (e_2)$. By Corollary 11, $(u^{e_2}_i)$ does not dominate $(v_i)$, the unit vector basis of $d (w_p, 1)$. But obviously $(u^{e_1}_i)$ dominates $(v_i)$. Hence $[(u^e_1)] \not\leq [(u^e_2)]$. Note that (5) also implies that $\Theta$ is injective. Hence $\Theta : L \to SP_w (X_L)$ is an order isomorphism. □

Remark. The example given here is non-reflexive. Given a countable semilattice $L$ without an infinite increasing sequence, the $\ell^p$ $(1 < p < \infty)$ version of the space defined above, i.e., $X_p = \bigoplus_{e \in L} d (w_{Te}, p)$, which is a reflexive space, has the property that $SP_w (X_p)$ is order isomorphic to the semilattice $\hat{L} = \{a\} \cup L$, $a > e$ for all $e \in L$. We do not know how to obtain a reflexive example for general semilattices. In fact, according to the authors of [4], it is not known if there is a reflexive space $X$ such that $SP_w (X)$ is order isomorphic to $((\{1, 2\}, \{1\}, \{2\}, \subseteq)$.

References


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