DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF $\mathrm{U}(p,q)$ AND $\mathrm{Spin}_0(p,q)$

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Abstract. Let $p > q$ and let G be the group $\mathrm{U}(p,q)$ or $\mathrm{Spin}_0(p,q)$. Let $P = LN$ be the maximal parabolic subgroup of G with Levi subgroup $L \cong M \times U$ where

$$(M,U) = \begin{cases} (\mathrm{GL}_q(\mathbb{C}), \mathrm{U}(p-q)) & \text{if } G = \mathrm{U}(p,q) \\ (\mathrm{GL}_q^+(\mathbb{R}), \mathrm{Spin}(p-q)) & \text{if } G = \mathrm{Spin}_0(p,q). \end{cases}$$

Let χ be a 1 dimensional character of M and τ^μ an irreducible representation of U with highest weight μ. Let $\pi_{\chi,\mu}$ be the representation of P which is trivial on N and $\pi_{\chi,\mu}|_L = \chi \boxtimes \tau^\mu$. Let $I_{p,q}$ be the Harish-Chandra module of the induced representation $\mathrm{Ind}_P^G \pi_{\chi,\mu}$. In this paper, we shall determine (i) the reducibility of $I_{p,q}$, (ii) the K-types of all the irreducible subquotients of $I_{p,q}$ when it is reducible, where K is the maximal compact subgroup of G, (iii) the module diagram of $I_{p,q}$ (from which one can read off the composition structure), and (iv) the unitarity of $I_{p,q}$ and its subquotients.

Except in the cases $q = p-1$ and $q = 1$, $I_{p,q}$ is not K-multiplicity free.

1. Introduction

1.1. Let $p > q$ and let G be the group $\mathrm{U}(p,q)$ or $\mathrm{Spin}_0(p,q)$. Let $P = LN$ be the maximal parabolic subgroup of G with Levi subgroup $L \cong M \times U$ where

$$(M,U) = \begin{cases} (\mathrm{GL}_q(\mathbb{C}), \mathrm{U}(p-q)) & \text{if } G = \mathrm{U}(p,q) \\ (\mathrm{GL}_q^+(\mathbb{R}), \mathrm{Spin}(p-q)) & \text{if } G = \mathrm{Spin}_0(p,q). \end{cases}$$

Let χ be a one dimensional character of M and τ^μ an irreducible representation of U with highest weight μ. Let $\pi_{\chi,\mu}$ be the representation of P which is trivial on N and

$$\pi_{\chi,\mu}|_L = \chi \boxtimes \tau^\mu.$$

Consider the induced representation $\mathrm{Ind}_P^G \pi_{\chi,\mu}$ and let $I_{p,q}$ be its Harish-Chandra module. In this paper, we shall determine (i) the reducibility of $I_{p,q}$, (ii) the K structure of all the irreducible subquotients of $I_{p,q}$ when it is reducible, where K is a maximal compact subgroup of G, (iii) the module diagram of $I_{p,q}$ (from which one can read off the composition structure), and (iv) the unitarity of $I_{p,q}$ and its subquotients.

Note that $I_{p,q}$ is not K-multiplicity free except when $q = p-1$ and $q = 1$.

1.2. We introduce some notation. Let

$$\tilde{G} = \begin{cases} \mathrm{U}(p,p) & \text{if } G = \mathrm{U}(p,q) \\ \mathrm{Spin}_0(p,p) & \text{if } G = \mathrm{Spin}_0(p,q). \end{cases}$$

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Let $\mathfrak{g} = \mathrm{Lie(G)}$ and $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{\mathrm{G}})$, and let K and $\tilde{K}$ denote the maximal compact subgroups of G and $\tilde{G}$ respectively. In this paper we will call a $(\mathfrak{g}, K)$-module (resp. $(\tilde{\mathfrak{g}}, \tilde{K})$-module) an infinitesimal G-module (resp. infinitesimal $\tilde{G}$-module).

1.3. We now describe our main results. We first show that $I_{p,q}$ can be embedded into the Harish-Chandra module I_p (see §3.3 and §9.4) of a degenerate principal series representation of $\tilde{G}$. In addition I_p is K_1-admissible where

$$K_1 = \begin{cases} \mathrm{U}(p) \times 1 & \text{if } G = \mathrm{U}(p,q) \\ \mathrm{Spin}(p) \times 1 & \text{if } G = \mathrm{Spin}_0(p,q). \end{cases} \tag{1}$$

and I_p decomposes discretely when restricted to $(\mathfrak{g}, K)$ (see [Ko3]). Next we identify $I_{p,q}$ with its image in I_p. If W is an infinitesimal $\tilde{G}$-submodule of I_p, then it is K_1-admissible. By Proposition 1.6 in [Ko3], $W \cap I_{p,q}$ is a (possibly zero) infinitesimal G-submodule of $I_{p,q}$. Our main result states that the converse is also true. We first show that

Theorem. *Suppose $W_1 \subseteq W_2$ are infinitesimal $\tilde{G}$-submodules of I_p such that $R := W_2/W_1$ is an irreducible subquotient of I_p. Define*

$$R' := \frac{W_2 \cap I_{p,q}}{W_1 \cap I_{p,q}}.$$

Then R' is either zero or isomorphic to an irreducible subquotient of $I_{p,q}$. Moreover, all irreducible subquotients of $I_{p,q}$ are of this form.

Now the module structure of I_p is well known ([J1], [J2], [Le], [Mo], [S1], [S2], [Zh]). The above theorem together with structural results on I_p allows us to determine the module structure of $I_{p,q}$. In particular, we obtain the following corollary.

Corollary. *If W' is an infinitesimal G-submodule of $I_{p,q}$, then there exists an infinitesimal $\tilde{G}$-submodule W (not necessarily unique) of I_p such that $W' = W \cap I_{p,q}$.*

If we know the $\tilde{K}$-types contained in an infinitesimal $\tilde{G}$-submodule W of I_p, then it is relatively easy to determine the K-types of $W \cap I_{p,q}$. Thus we can obtain explicit description of the K-types which occur in each of the irreducible subquotients of $I_{p,q}$. Moreover, we show that the module diagram of $I_{p,q}$ can be identified with a spanning subgraph of the module diagram of I_p. This immediately gives the composition structure of $I_{p,q}$. These results are given in Theorem 7.3.1 for $G = \mathrm{U}(p,q)$, and in Theorem 12.2.1 for $G = \mathrm{Spin}_0(p,q)$.

Finally we also determine the unitarity of $I_{p,q}$ and all its irreducible subquotients. The results are contained in Theorem 8.1.1 for $G = \mathrm{U}(p,q)$ and in Theorem 13.1.1 for $G = \mathrm{Spin}_0(p,q)$.

1.4. The case $q = 1$ has been studied in detail by [Hi1], [Hi2], [Hi3] and [KG1]. When $q = 1$ and τ^μ is a one dimensional character, this is a special case in [HT] and [KG2]. By specializing to the case $q = 1$, we recover these results in the above papers.

1.5. We note that almost all previous successful results on the module structures of degenerate principal series representations treat the special cases where the representations are K-multiplicity free. In particular Hirai and Klimyk et al ([Hi1], [Hi2], [Hi3], [KG1], [KG2], [KG3]) study such problems using the Gelfand-Zetlin basis. Some examples of representations which are not K-multiplicity free are studied in the papers [Ho] and [P].

1.6. We will briefly describe our methods. Partly inspired by the works of Hirai ([Hi1]) and Klimyk and Gavrilik ([KG1]), we construct a basis $\mathcal{B}$ of I_p using the Gelfand-Zetlin basis of an irreducible representation of $\mathrm{U}(p)$ or $\mathrm{Spin}(p)$. This basis is compatible with the infinitesimal G-submodules of $I_{p,q}$ in the following sense: $\mathcal{B} \cap I_{p,q}$ is a basis of $I_{p,q}$, and for any infinitesimal G-submodule W of $I_{p,q}$, $\mathcal{B} \cap W$ is a basis of W. Moreover the Lie algebra action on $\mathcal{B}$ can be explicitly calculated.

Under the action of K_1 (see (1)), we have the decomposition $I_{p,q} = \sum_\lambda S_\lambda$, where each λ is a highest weight for K_1, and S_λ is the λ-isotypic component. With the aid of the basis $\mathcal{B}$, we show that every submodule of $I_{p,q}$ is the sum of a collection of isotypic components for K_1. Using structural results on I_p, we determine explicitly how $\mathfrak{g}_\mathbb{C}$ transforms each S_λ in $I_{p,q}$. These information allows us to deduce the module structure and unitarity of $I_{p,q}$ and its irreducible subquotients.

1.7. Although the main ideas of the proofs for both the cases $G = \mathrm{U}(p,q)$ and $G = \mathrm{Spin}_0(p,q)$ are similar, several special considerations lead to very different results in each case. We have therefore divided the paper in two parts. The first part treats $\mathrm{U}(p,q)$ and the second treats $\mathrm{Spin}_0(p,q)$. We will be brief in Part 2 and we will mainly point out the differences from Part 1.

1.8. Our proofs in this paper rely heavily on the fact that our representations are K_1-admissible (see (1)). Such representations are studied under a more general framework by T. Kobayashi (see Chapter 4 of [Ko1], and [Ko2], [Ko3]). Our results on the restriction of the "ladder type" representation of $\mathrm{Spin}_0(2p,2p)$ also overlaps with Example 3.4 in [Ko2]

1.9. We were informed by H. Matumoto that he had also studied the representation $I_{p,q}$, and had obtained results on its module structure.

1.10. We will show in a future paper ([LL]) that similar results hold for a family of degenerate principal series representation of $\mathrm{Sp}(p,q)$.

1.11. **Notation.** We introduce a notation for later use. Let G_1 be a reductive Lie group and let $P_1 = L_1 N_1$ be a parabolic subgroup of G_1 with Levi subgroup L_1. Let (π, U) be a representation of L_1. Then we extend π to a representation of P_1 by letting N_1 act trivially and we define the normalized induced representation

$$\mathrm{Ind}_{P_1}^{G_1} \pi := \{ f : G_1 \to U : f \text{ is } C^\infty, f(gp) = (\Delta(p^{-1}))^{\frac{1}{2}} \pi(p^{-1}) f(g), g \in G_1, p \in P_1 \}$$

where Δ is the modular function of P_1, and G_1 acts by left translation.

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Part 1. The degenerate principal series of $\mathrm{U}(p,q)$

2. The representations

2.1. Let $p > q$ and let $P = LN$ be the maximal parabolic subgroup of $\mathrm{U}(p,q)$ with Levi subgroup $L \cong \mathrm{GL}_q(\mathbb{C}) \times \mathrm{U}(p-q)$. For $s \in \mathbb{C}$ and $\sigma \in \mathbb{Z}$, let $\chi_{s,\sigma} : \mathrm{GL}_q(\mathbb{C}) \longrightarrow \mathbb{C}^\times$ be given by

$$\chi_{s,\sigma}(a) = |\det a|^s \left(\frac{\det a}{|\det a|} \right)^\sigma .$$

Let τ^μ_{p-q} be the irreducible representation of $\mathrm{U}(p-q)$ with highest weight μ. Let

$$\pi_{s,\sigma,\mu} = \chi_{s,\sigma} \boxtimes \tau^\mu_{p-q},$$

and consider the normalized induced representation $\mathrm{Ind}_P^{\mathrm{U}(p,q)} \pi_{s,\sigma,\mu}$ (cf. §1.11). Let $I_{p,q}(s,\sigma,\mu)$ be the Harish-Chandra module of $\mathrm{Ind}_P^{\mathrm{U}(p,q)} \pi_{s,\sigma,\mu}$. In this part, we shall determine the module structure and unitarity of $I_{p,q}(s,\sigma,\mu)$.

2.2. We define some notations. For each $r \geq 1$, let

$$\Lambda^+(r) = \{(\lambda_1, \lambda_2, ..., \lambda_r) \in \mathbb{Z}^r : \ \lambda_j \geq \lambda_{j+1}, \text{ for all } 1 \leq j \leq r-1\}.$$

Then $\Lambda^+(r)$ can be identified with the set of dominant weights of the unitary group $\mathrm{U}(r)$ in the usual way. For each $\lambda \in \Lambda^+(r)$, τ^λ_r (or simply τ^λ) shall denote a copy of the irreducible representation of $\mathrm{U}(r)$ with highest weight λ. We also let

$$\begin{aligned} \lambda^* &= (-\lambda_r, -\lambda_{r-1}, ..., -\lambda_1) \\ \mathbf{1}_r &= (1, 1, ..., 1) \in \Lambda^+(r) \\ \varepsilon_j &= (\overbrace{0, ..., 0, 1}^{j}, 0, ..., 0) \text{ for } 1 \leq j \leq p. \end{aligned}$$

Note that $\tau^{\lambda^*}_r$ is the dual representation of τ^λ_r and $\tau^{\mathbf{1}_r}_r$ is the determinant character of $\mathrm{U}(r)$.

2.3. By Frobenius reciprocity, the K-type $\tau^\lambda_p \boxtimes \tau^\eta_q$ occurs in $I_{p,q}(s,\sigma,\mu)$ with multiplicity

$$\dim \mathrm{Hom}_{\mathrm{U}(q)\times\mathrm{U}(p-q)}(\tau_q^{\eta^*+\sigma\mathbf{1}_q} \boxtimes \tau^\mu_{p-q}, \tau^\lambda_p). \tag{2}$$

2.4. The infinitesimal character of $I_{p,q}(s,\sigma,\mu)$ is given by

$$\begin{aligned} &(\tfrac{s+\sigma+q-1}{2}, \tfrac{s+\sigma+q-3}{2}, \dots, \tfrac{s+\sigma-(q-1)}{2}, \tfrac{-s+\sigma+q-1}{2}, \tfrac{-s+\sigma+q-3}{2}, \dots, \tfrac{-s+\sigma-(q-1)}{2}, \\ &\mu_1 + \tfrac{p-q-1}{2}, \mu_2 + \tfrac{p-q-3}{2}, \dots, \mu_{p-q} - \tfrac{p-q-1}{2}). \end{aligned} \tag{3}$$

Note that it is defined up to an action of the Weyl group S_{p+q}. Given s and σ, the infinitesimal character (3) determines the infinitesimal character of τ^μ_{p-q}. Since irreducible finite dimensional representations are uniquely determined by its infinitesimal character, we conclude that the infinitesimal character of the degenerate principal series in (3) determines τ^{μ}_{p-q} for fixed s and σ.

3. Restriction of the degenerate series of $\mathrm{U}(p,p)$ to $\mathrm{U}(p,q)\times\mathrm{U}(p-q)$

3.1. In this section, we shall consider a degenerate principal series representation $I_p(s,\sigma)$ of $\mathrm{U}(p,p)$. By restricting the action of $\mathrm{U}(p,p)$ to $\mathrm{U}(p,q)\times\mathrm{U}(p-q)$, we show that we can embed $I_{p,q}(s,\sigma,\mu)$ into $I_p(s,\sigma)$

3.2. Recall that $\mathrm{U}(p,p)$ is the isometry group of the following Hermitian form on $\mathbb{C}^{2p}$:

$$\langle z,w\rangle=\sum_{i=1}^{p} z_i\overline{w}_i-\sum_{j=p+1}^{2p} z_j\overline{w}_j \qquad z=(z_1,...,z_{2p}),\ w=(w_1,...,w_{2p})\in\mathbb{C}^{2p}.$$

Let $\{e_1,...,e_{2p}\}$ be the standard basis of $\mathbb{C}^{2p}$. Set $T=\{1,\dots,p\}\cup\{2p-q+1,\dots,2p\}$. Let V_T and V_T' be the span of $\{e_i : i\in T\}$ and $\{e_i : i\notin T\}$ respectively, so that $\mathbb{C}^{2p}=V_T\oplus V_T'$. Let

$$G=\{g\in\mathrm{U}(p,p):\ g|_{V_T'}=\mathrm{id}\},\quad H=\{g\in\mathrm{U}(p,p):\ g|_{V_T}=\mathrm{id}\}.$$

Then $G\cong\mathrm{U}(p,q)$ and $H\cong\mathrm{U}(p-q)$. In the language of Howe correspondences, we say that $G\times H$ is a compact reductive dual pair in $\mathrm{U}(p,p)$. From now on, we shall always identify $\mathrm{U}(p,q)$ and $\mathrm{U}(p-q)$ with G and H respectively.

3.3. We now define a degenerate principal series representation of $\mathrm{U}(p,p)$. Let $\tilde{P}$ be the stabilizer of the span of $\{e_j+e_{p+j} :\ 1\le j\le p\}$ in $\mathrm{U}(p,p)$. It is the maximal parabolic of $\mathrm{U}(p,p)$, and $\tilde{P}=\tilde{L}\tilde{N}$ where $\tilde{L}\cong\mathrm{GL}_p(\mathbb{C})$ is its Levi subgroup. Let $s\in\mathbb{C}$ and $\sigma\in\mathbb{Z}$, and let

$$\tilde{\chi}_{s,\sigma}(a)=|\det a|^s\left(\frac{\det a}{|\det a|}\right)^{\sigma}\qquad (a\in\mathrm{GL}_p(\mathbb{C})\cong\tilde{L}).$$

be a character of $\tilde{L}$. Let $I_p(s,\sigma)$ denote the Harish-Chandra module of $\mathrm{Ind}_{\tilde{P}}^{\mathrm{U}(p,p)}\tilde{\chi}_{s,\sigma}$ (cf. §1.11). Under the action of $\tilde{K}=\mathrm{U}(p)\times\mathrm{U}(p)$,

$$I_p(s,\sigma)=\sum_{\lambda\in\Lambda^+(p)} V_\lambda$$

where for each $\lambda\in\Lambda^+(p)$, $V_\lambda\cong\tau_p^\lambda\boxtimes\tau_p^{\lambda^*+\sigma\mathbf{1}_p}$. We will describe the module structure and unitarity of $I_p(s,\sigma)$ in §7.1 and §8. This is well known and can be found in [J1], [J2], [Le], [Mo], [S1], [S2] and [Zh].

3.4. For each $\mu\in\Lambda^+(p-q)$, let $I_p(s,\sigma)_\mu$ denote the τ^μ_{p-q}-isotypic part of $I_p(s,\sigma)$, that is, it is the image of the H-map

$$\mathrm{Hom}_H(\tau^\mu_{p-q},I_p(s,\sigma))\otimes\tau^\mu_{p-q}\to I_p(s,\sigma)$$

given by $h\otimes v\mapsto h(v)$. Note that since the actions of G and H commute with each other, $I_p(s,\sigma)_\mu$ is also an infinitesimal G-module.

Now H acts on $C^\infty(H)$ by left and right translation:

$$(l_g f)(h) = f(g^{-1}h), \quad (r_g f)(h) = f(hg).$$

For each $\mu \in \Lambda^+(p-q)$, let $C^\infty(H)_\mu$ denote the τ^μ_{p-q}-isotypic part of $C^\infty(H)$ with respect to the left translation. By the Peter-Weyl theorem,

$$C^\infty(H)_\mu \cong \tau^{\mu^*}_{p-q} \boxtimes \tau^\mu_{p-q}$$

with respect to the action $r \times l$ by $H \times H$.

We now fix $\mu \in \Lambda^+(p-q)$. For each $f \in I_p(s,\sigma)_{\mu^*}$, let $Af : G \to C^\infty(H)$ be given by

$$(Af(g))(h) = f(g,h).$$

Note that for each fixed $g \in G$, the map $f \to Af(g)$ from $I_p(s,\sigma)_{\mu^*}$ to $C^\infty(H)$ is H-equivariant (with respect to left translation), so that $Af(g) \in C^\infty(H)_{\mu^*}$. So $Af : G \to C^\infty(H)_{\mu^*}$. Let $\{\ell_1, ..., \ell_r\}$ be a basis of $\tau^{\mu^*}_{p-q}$. Then every element y in $C^\infty(H)_{\mu^*}$ can be expressed uniquely in the form $y = \sum_{j=1}^r v_j \boxtimes \ell_j$ where $v_j \in \tau^\mu_{p-q}$ for $1 \le j \le r$. It follows that

$$Af = \sum_{j=1}^r f_j \boxtimes \ell_j \tag{4}$$

where for $1 \le j \le r$, $f_j : G \to \tau^\mu_{p-q}$. Let $p_1 \in P$. Then we can write p_1 as $p_1 = (a_1, a_2)n_1$ where $a_1 \in \mathrm{GL}_q(\mathbb{C})$, $a_2 \in \mathrm{U}(p-q)$ and $n_1 \in N$. Now one can check that for $g \in G$ and $h \in H$,

$$(Af(gp_1))(h) = (\delta(p_1^{-1}))^{-\frac{1}{2}} \chi_{s,\sigma}(a_1^{-1})(\det a_2)^{-\sigma} r_{a_2^{-1}}(Af(g))(h).$$

where δ is the modular function of P. So we have

$$\sum_{j=1}^r f_j(gp_1) \boxtimes \ell_j = (\delta(p_1^{-1}))^{-\frac{1}{2}} \chi_{s,\sigma}(a_1^{-1})(\det a_2)^{-\sigma} \sum_{j=1}^r \tau^\mu_{p-q}(a_2^{-1})(f_j(g)) \boxtimes \ell_j.$$

It follows that for each $1 \le j \le r$,

$$f_j(gp) = (\delta(p_1^{-1}))^{-\frac{1}{2}} \chi_{s,\sigma}(a_1^{-1})(\det a_2)^{-\sigma} \tau^\mu_{p-q}(a_2^{-1})(f_j(g)) = \{(\chi_{s,\sigma} \boxtimes \tau^{\mu+\sigma\mathbf{1}_{p-q}}_{p-q})(p_1)\}(f_j(g)).$$

Hence $f_j \in I_{p,q}(s,\sigma,\mu+\sigma\mathbf{1}_{p-q})$ and (4) defines a map

$$A : I_p(s,\sigma)_{\mu^*} \to I_{p,q}(s,\sigma,\mu+\sigma\mathbf{1}_{p-q}) \boxtimes \tau^{\mu^*}. \tag{5}$$

Proposition 3.4.1. *The map A in (5) is an infinitesimal $G \times H$-module isomorphism.*

Proof. First we show that A is injective. Indeed, since $f \in I_p(s,\sigma)_{\mu^*} \subseteq I_p(s,\sigma)$, f is completely determined by its values on $\mathrm{U}(p) \times 1$. Next, by a straightforward application of Frobenius reciprocity (see (2)), we see that the spaces $I_p(s,\sigma)_{\mu^*}$ and $I_{p,q}(s,\sigma,\mu) \boxtimes \tau^{\mu^*}$ have the same $\mathrm{U}(p) \times \mathrm{U}(q) \times \mathrm{U}(p-q)$-types. Hence A is an infinitesimal isomorphism. $\square$

Let $G_1 \subseteq G_2$ be reductive Lie groups and let K_1 be the maximal compact subgroup of G_1. If V is an infinitesimal G_2 module such that it is K_1 admissible, then we shall abuse notation and denote the restriction of V to the complexified Lie algebra of G_1 and K_1 by $\mathrm{Res}^{G_2}_{G_1} V$.

Corollary 3.4.2. *For each $\mu \in \Lambda^+(p-q)$, let $\hat{\mu} = \mu^* + \sigma \mathbf{1}_{p-q}$. Then*

$$\mathrm{Res}^{\mathrm{U}(p,p)}_{\mathrm{U}(p,q)\times\mathrm{U}(p-q)} I_p(s,\sigma) = \sum_{\mu\in\Lambda^+(p-q)} I_{p,q}(s,\sigma,\mu) \boxtimes \tau^{\hat{\mu}}_{p-q}. \tag{6}$$

Note that if $q = 0$, then (6) gives the $\tilde{K}$-types of $I_p(s,\sigma)$.

3.5. **Restriction of $I_{p,q}(s,\sigma,\mu)$ to $\mathrm{U}(p,q-1)\times\mathrm{U}(1)$.** To prove the next lemma, we need the following branching rule (see Exercise 6.12 in [FH]):

$$\mathrm{Res}^{\mathrm{U}(p)}_{\mathrm{U}(p-1)\times\mathrm{U}(1)} \tau^\lambda_p = \sum_{\lambda'} \tau^{\lambda'}_{p-1} \boxtimes \det{}_1^{\sum_i \lambda_i - \sum_j \lambda'_j} \tag{7}$$

where the sum is taken over all $\lambda' = (\lambda'_1, \dots, \lambda'_{p-1}) \in \Lambda^+(p-1)$ such that

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \cdots \geq \lambda'_{p-1} \geq \lambda_p.$$

Proposition 3.5.1. *Let $\mu \in \Lambda^+(p-q)$. Then*

$$\mathrm{Res}^{\mathrm{U}(p,q)}_{\mathrm{U}(p,q-1)\times\mathrm{U}(1)} I_{p,q}(s,\sigma,\mu) = \sum_{\mu'} I_{p,q-1}(s,\sigma,\mu') \boxtimes \det{}_1^{\sum_i \mu_i - \sum_j \mu'_j - \sigma} \tag{8}$$

where the sum is taken over all $\mu' = (\mu'_1, \dots, \mu'_{p-q+1}) \in \Lambda^+(p-q+1)$ such that $\mu'_1 \geq \mu_1 \geq \mu'_2 \geq \ldots \geq \mu_{p-q} \geq \mu'_{p-q+1}$.

Proof. Note that $\mathrm{U}(p,q)\times\mathrm{U}(p-q)$ and $\mathrm{U}(p,q-1)\times\mathrm{U}(p-q+1)$ forms a see-saw dual pair in $\mathrm{U}(p,p)$. The corollary now follows from Corollary 3.4.2 by applying the theory of see-saw pairs and the branching rule in (7) to the above dual pairs. □

We shall ignore the action of $\mathrm{U}(1)$ on $I_{p,q}(s,\sigma,\mu)$ and write (8) as

$$I_{p,q}(s,\sigma,\mu) = \sum_{\mu'} I_{p,q-1}(s,\sigma,\mu').$$

Proposition 3.5.2. *If W is an infinitesimal $\mathrm{U}(p,q)$-submodule of $I_{p,q}(s,\sigma,\mu)$, then*

$$W = \sum_{\mu'} (W \cap I_{p,q-1}(s,\sigma,\mu'))$$

where the sum is taken over all $\mu' = (\mu'_1, \dots, \mu'_{p-q+1}) \in \Lambda^+(p-q+1)$ such that $\mu'_1 \geq \mu_1 \geq \mu'_2 \geq \ldots \geq \mu_{p-q} \geq \mu'_{p-q+1}$.

Proof. Let $w \in W$. By (8), we can write w as a finite sum $w = \sum_{\mu'} w_{\mu'}$ where for each μ' in the sum, $w_{\mu'} \in I_{p,q-1}(s,\sigma,\mu')$. Let $\mathcal{Z}_{q-1}$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{u}(p,q-1)_{\mathbb{C}})$. Now $\mathcal{Z}_{q-1}$ acts by a different infinitesimal character on each $w_{\mu'}$ (c.f. (3)). Since distinct characters are linearly independent, there exists $Z \in \mathcal{Z}_{q-1}$ such that $w_{\mu'} = Zw \in W$. This proves the proposition. □

4. A basis of $I_{p,q}(s, \sigma, \mu)$

4.1. In this section, we shall first construct a basis $\mathcal{B}$ of $I_p(s, \sigma)$ using the Gelfand-Zetlin basis of an irreducible representation of $\mathrm{U}(p)$. It turns out that the intersection of $\mathcal{B}$ with $I_{p,q}(s, \sigma, \mu)$ is a basis of $I_{p,q}(s, \sigma, \mu)$. Moreover, this basis is compatible with the infinitesimal G-submodules of $I_{p,q}(s, \sigma, \mu)$ in the sense that the intersection of any infinitesimal G-submodule W of $I_{p,q}(s, \sigma, \mu)$ with $\mathcal{B}$ is a basis of W.

Readers who are only interested in the main results may skip this section and next two sections and proceed directly to §7.

4.2. We shall first review the theory of Gelfand-Zetlin (GZ) basis. Consider the chain of subgroups of $\mathrm{U}(p)$:

$$\mathrm{U}(1) \subseteq \mathrm{U}(2) \subseteq \cdots \subseteq \mathrm{U}(p) \tag{9}$$

where for each $1 \leq r \leq p-1$, $\mathrm{U}(r)$ is identified with the subgroup of matrices

$$\left\{ \begin{pmatrix} X & 0 \\ 0 & I_{p-r} \end{pmatrix} \in M_p(\mathbb{C}) : X\overline{X}^t = I_r \right\}$$

of $\mathrm{U}(p)$. This induces the obvious embedding of their complexified Lie algebras

$$\mathfrak{gl}_1(\mathbb{C}) \subseteq \mathfrak{gl}_2(\mathbb{C}) \subseteq \cdots \subseteq \mathfrak{gl}_p(\mathbb{C}).$$

Let $E_{ij} \in \mathfrak{gl}_p(\mathbb{C})$ denote the matrix with 1 at the (i,j)-th entry and 0 elsewhere. Then for $1 \leq r \leq p$, $\mathfrak{gl}_r(\mathbb{C})$ is the span of $\{E_{ij} : 1 \leq i, j \leq r\}$.

Let $\lambda \in \Lambda^+(p)$. Recall that τ_p^λ denotes a copy of the irreducible representation of $\mathrm{U}(p)$ with highest weight λ. It has a Gelfand-Zetlin (GZ) basis with respect to the embedding (9). Each basis vector $[m_{kl}]$ is represented up to a scalar as a set of integers [GZ1]

$$[m_{kl}] = \begin{bmatrix} m_{1p} & & m_{2p} & & \cdots & & m_{p-1,p} & & m_{pp} \\ & m_{1,p-1} & & m_{2,p-1} & \cdots & m_{p-2,p-1} & & m_{p-1,p-1} & \\ & & & \cdots & \cdots & \cdots & & & \\ & & & & m_{11} & & & & \end{bmatrix} \tag{10}$$

Here $m_{ip} = \lambda_i$ for $i = 1, \ldots, p$ and m_{kl} are integers satisfying

$$m_{kl} \geq m_{k,l-1} \geq m_{k+1,l}. \tag{11}$$

We assume that if the above inequality is not satisfied by some $m_{k,l-1}$, then we set $[m_{kl}] = 0$. If $[m_{kl}] \neq 0$, then for $1 \leq r \leq p$, $(m_{1r}, m_{2r}, \ldots, m_{rr})$ is the highest weight of the $\mathfrak{gl}_r(\mathbb{C})$-module in which $[m_{kl}]$ lies in.

There are several normalizations of the GZ bases given in [GZ1], [GG], [Ca] and [Z]. In Part 1 of this paper, we will use the normalization given in [Z]. In [Z], the author defines a $\mathrm{U}(n)$-invariant Hermitian form on τ_p^λ and the normalized GZ basis forms an orthonormal basis with respect to the Hermitian form. Hence the GZ basis is uniquely determined up to a nonzero scalar depending on the invariant Hermitian form.

Let $v = [m_{kl}] \in \tau_p^\lambda$. For each $1 \leq i \leq r \leq p-1$, let v_{ir}^+ (resp. v_{ir}^-) be the GZ basis vector obtained from v by increasing (resp. decreasing) m_{ir} by 1 while leaving the rest of the m_{kl}'s unchanged. Again we have implicitly assumed that $v_{ir}^\pm$ is zero if

(11) is not satisfied. The following theorem about the normalized GZ basis is due to Gelfand, Zetlin and Graev (see [Z] and pp 667-669 [GG].)

Theorem 4.2.1. *Let v be a GZ basis vector in τ_p^λ. For $1 \leq r \leq p-1$,*

$$E_{r,r+1}v = \sum_{i=1}^{r} a_i v_{ir}^{+}, \quad E_{r+1,r}v = \sum_{i=1}^{r} b_i v_{ir}^{-}$$

where a_i and b_i are **nonzero** *complex numbers.*

The exact values of a_i and b_i are given in [Z]. For our purposes, it is sufficient to know that a_i (resp. b_i) is nonzero whenever v_{ir}^{+} (resp. v_{ir}^{-}) is nonzero.

4.3. We shall now describe a basis of $I_p(s,\sigma)$. We recall that under the action of $\tilde{K} = \mathrm{U}(p) \times \mathrm{U}(p)$,

$$I_p(s,\sigma) = \sum_{\lambda \in \Lambda^+(p)} V_\lambda$$

where for each λ, $V_\lambda \cong \tau_p^\lambda \boxtimes \tau_p^{\hat{\lambda}}$, and $\hat{\lambda} = \lambda^* + \sigma \mathbf{1}_p$. Hence V_λ has a basis consisting of vectors

$$[m_{kl}] \otimes [\hat{m}_{k'l'}]$$

where $[m_{kl}]$ and $[\hat{m}_{k'l'}]$ are normalized GZ basis of τ_p^λ and $\tau_p^{\hat{\lambda}}$ respectively. It follows that

$$\mathcal{B} = \bigcup_{\lambda \in \Lambda^+(p)} \left\{ [m_{kl}] \otimes [\hat{m}_{k'l'}] \ : \ (m_{kp}) = \lambda_k, (\hat{m}_{kp}) = \hat{\lambda}_k \right\}$$

is a basis of $I_p(s,\sigma)$. We note that each vector in $\mathcal{B}$ is only defined up to a scalar depending on V_λ.

Recall that $\mathfrak{u}(p,q)$ and $\mathfrak{u}(p-q)$ are the Lie algebras of $\mathrm{U}(p,q)$ and $\mathrm{U}(p-q)$ respectively. Let $\widetilde{\mathfrak{k}}$ be the Lie algebra of $\tilde{K} = \mathrm{U}(p) \times \mathrm{U}(p)$. We shall identify the complexified Lie algebra of $\mathrm{U}(p,p)$ with the Lie algebra $\mathfrak{gl}_{2p}(\mathbb{C})$ of $2p$ by $2p$ complex matrices with standard basis $\{E_{ij} : 1 \leq i,j \leq p\}$. Under this identification,

$$\begin{aligned} \widetilde{\mathfrak{k}}_{\mathbb{C}} &= \mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_p(\mathbb{C}) \\ \mathfrak{u}(p,q)_{\mathbb{C}} &= \mathrm{Span}\{E_{ij} : i,j \in T\} \\ \mathfrak{u}(p-q)_{\mathbb{C}} &= \mathrm{Span}\{E_{p+i,p+j} : 1 \leq i,j \leq p-q\}, \end{aligned} \tag{12}$$

where $T = \{1, 2, \ldots, p\} \cup \{2p-q+1, 2p-q+2, \ldots, 2p\}$ (see §3.2).

We now introduce a notation. Let $\lambda \in \Lambda^+(p)$ and $r < p$. If $u = [m_{kl}]$ is a GZ basis vector in τ_p^λ, then $d_r(u)$ shall denote the GZ basis vector for the group $\mathrm{U}(r)$ obtained by deleting the top $p-r$ rows from $[m_{kl}]$. The following proposition follows from Proposition 3.4.1.

Proposition 4.3.1. *Let $\mu \in \Lambda^+(p-q)$.*

(i) *Recall that $\hat{\mu} = \mu^* + \sigma \mathbf{1}_{p-q}$. The set*

$$\mathcal{B}_{\hat{\mu}} = \{[m_{k,l}] \otimes [\hat{m}_{k'l'}] : (\hat{m}_{1,p-q}, \hat{m}_{2,p-q}, \ldots, \hat{m}_{p-q,p-q}) = \hat{\mu}\}$$

is a basis of $I_p(s,\sigma)_{\hat{\mu}}$.

(ii) *Let v_0 be a fixed GZ basis vector in $\tau_{p-q}^{\hat{\mu}}$. Then*

$$\mathcal{B}(v_0) := \{[m_{kl}] \otimes [\hat{m}_{k'l'}] \in \mathcal{B} : d_{p-q}([\hat{m}_{k'l'}]) = v_0\}$$

is a basis of $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$.

4.4. From now on, we shall fix a GZ basis vector v_0 in $\tau_{p-q}^{\hat{\mu}}$ and identify $I_{p,q}(s,\sigma,\mu)$ with the subspace $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$ of $I_p(s,\sigma)_{\hat{\mu}}$. We shall show that the basis $\mathcal{B}(v_0)$ behaves well with respect to restrictions. Suppose that $\tau_{p-q+1}^{\mu'} \supset \tau_{p-q}^{\mu}$. Then clearly $\tau_{p-q+1}^{\hat{\mu}'} \supset \tau_{p-q}^{\hat{\mu}}$. There exists a unique (up to scalars) GZ basis vector $w_{\hat{\mu}'}$ in $\tau_{p-q+1}^{\hat{\mu}'}$ such that $d_{p-q}(w_{\hat{\mu}'}) = v_0$. By definition

$$\mathcal{B}(w_{\hat{\mu}'}) := \{[m_{kl}] \otimes [\hat{m}_{k'l'}] \in \mathcal{B} : d_{p-q+1}([\hat{m}_{k'l'}]) = w_{\hat{\mu}'}\} \tag{13}$$

and Proposition 4.3.1 states that (13) is a basis of $I_{p,q-1}(s,\sigma,\mu')$. Moreover $\mathcal{B}(v_0) = \bigcup_{\mu'} \mathcal{B}(w_{\hat{\mu}'})$ (disjointed union) where the union is taken over all $\mu' \in \Lambda^+(p-q+1)$ such that $\tau_{p-q+1}^{\mu'}$ contains τ_{p-q}^{μ}. Hence we have given an alternative proof of Proposition 3.5.1.

Proposition 4.4.1. *If W is an infinitesimal* $\mathrm{U}(p,q)$*-submodule of $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$, then $W \cap \mathcal{B}(v_0) = W \cap \mathcal{B}$ is a basis of W.*

Proof. We will prove this by induction on q. The case $q = 0$ is trivial because W is a representation of $\mathrm{U}(p)$. Suppose $q-1$ is true. By Proposition 3.5.2,

$$W = \sum_{\tau_{p-q+1}^{\mu'} \supset \tau_{p-q}^{\mu}} (W \cap I_{p,q-1}(s,\sigma,\mu')). \tag{14}$$

We have already shown that $I_{p,q-1}(s,\sigma,\mu')$ is spanned by $\mathcal{B}(w_{\hat{\mu}'})$. By induction hypothesis, $W \cap \mathcal{B}(w_{\hat{\mu}'})$ is a basis of $W \cap I_{p,q-1}(s,\sigma,\mu')$. Hence

$$W \cap \mathcal{B}(v_0) = \bigcup_{\tau_{p-q+1}^{\mu'} \supset \tau_{p-q}^{\mu}} (W \cap \mathcal{B}(w_{\hat{\mu}'}))$$

is a basis of W. $\square$

5. $\mathrm{U}(p)$-isotypic subspaces in $I_{p,q}(s,\sigma,\mu)$

5.1. As in the previous section, we let v_0 is a fixed GZ basis vector in $\tau_{p-q}^{\hat{\mu}}$ and identify $I_{p,q}(s,\sigma,\mu)$ with the subspace $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$ of $I_p(s,\sigma)$. Recall that under the action of $\tilde{K} = \mathrm{U}(p) \times \mathrm{U}(p)$,

$$I_p(s,\sigma) = \sum_{\lambda \in \Lambda^+(p)} V_\lambda$$

where for each λ, $V_\lambda \cong \tau_p^\lambda \boxtimes \tau_p^{\hat{\lambda}}$ and $\hat{\lambda} = \lambda^* + \sigma \mathbf{1}_p$. We now fix $\lambda \in \Lambda$. and define

$$\begin{aligned} J_{\hat{\lambda}} &:= \mathrm{Span}\{[\hat{m}_{ij}] \in \tau_p^{\hat{\lambda}} : d_{p-q}([\hat{m}_{ij}]) = v_0\} \\ S_\lambda &:= (I_{p,q}(s,\sigma,\mu) \boxtimes v_0) \cap V_\lambda \cong \tau_p^\lambda \boxtimes J_{\hat{\lambda}}. \end{aligned} \tag{15}$$

Note that $J_{\hat\lambda}$ is a $\mathrm{U}(q)$-submodule of $\tau_p^{\hat\lambda}$, and S_λ is the τ_p^λ-isotypic component in $I_{p,q}(s,\sigma,\mu)$.

Lemma 5.1.1. *The following statements are equivalent:*

(i) $S_\lambda \neq 0$.
(ii) $J_{\hat\lambda} \neq 0$.
(iii) $\tau_p^{\hat\lambda}$ *contains* $\tau_{p-q}^{\hat\mu}$.
(iv) τ_p^{λ} *contains* τ_{p-q}^{μ}.
(v) $\lambda_i \geq \mu_i \geq \lambda_{i+q}$ *for* $i = 1, \dots, p-q$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are clear. By the definition of the GZ basis, (iv) implies (v). It remains to show that (v) implies (iv). We will prove this by induction on q. If $q = 1$, then (v) is just the branching rule given in (7). Next suppose $q-1$ is true and $q \geq 2$. We formally define $\mu_i = \infty$ if $i \leq 0$ and $\mu_i = -\infty$ if $i > p-q$. Since $\lambda_i \geq \mu_i$ and $\mu_{i-q+1} \geq \lambda_{i+1}$, we have

$$\min(\lambda_i, \mu_{i-q+1}) \geq \max(\lambda_{i+1}, \mu_i).$$

Define $\lambda' = (\lambda'_1, \dots, \lambda'_{p-1})$ by

$$\lambda'_i = \begin{cases} \max(\lambda_{i+1}, \mu_i) & \text{if } \ i = 1, \dots, p-q \\ \min(\lambda_i, \mu_{i-q+1}) & \text{if } \ i = p-q+1, \dots, p-1. \end{cases}$$

Then $\lambda' \in \Lambda^+(p-1)$ and $\lambda'_i \geq \mu_i \geq \lambda'_{i+(q-1)}$ for $1 \leq i \leq p-q-1$. By induction hypothesis, $\tau_{p-1}^{\lambda'} \supset \tau_{p-q}^{\mu}$. Since $\lambda_i \geq \lambda'_i \geq \lambda_{i+1}$, then by (7) $\tau_p^\lambda \supset \tau_{p-1}^{\lambda'} \supset \tau_{p-q}^{\mu}$. □

5.2. The following two lemmas are vital to our later investigations.

Lemma 5.2.1. *Let W be an infinitesimal G-submodule of $I_{p,q}(s,\sigma,\mu)$. If $W \cap S_\lambda \neq 0$, then $S_\lambda \subseteq W$.*

Proof. Let $I_{p,q} = I_{p,q}(s,\sigma,\mu)$ and let v_0 be a fixed GZ basis vector of $\tau_{p-q}^{\hat\mu}$. By Proposition 4.3.1(ii), we identify $I_{p,q} \simeq I_{p,q} \boxtimes v_0$ in $I_p(s,\sigma)$ and $I_{p,q} \boxtimes v_0$ has a basis $\mathcal{B}(v_0)$. By Proposition 4.4.1, $W \cap \mathcal{B}(v_0)$ is a basis of W. Since $W \cap S_\lambda \neq 0$, W contains a vector of the form

$$w = u \boxtimes v$$

where u is the highest weight vector of τ_p^λ, and v is a GZ basis vector for $\tau_p^{\hat\lambda}$ such that $d_{p-q}(v) = v_0$.

Suppose $x = [m_{kl}]$ is a GZ basis vector of $\tau_p^{\hat\lambda}$. Let $j \geq p-q$, and let x_{ij}^+ (resp. x_{ij}^-) denote the GZ basis vector obtained from x by increasing (resp. decreasing) m_{ij} by 1 (cf. §4.2). Then it suffices to show that if $u \otimes x \in W$, then $u \otimes x_{ij}^\pm \in W$.

We refer to (12) and $E_{p+r-1,p+r}, E_{p+r,p+r-1} \in \mathfrak{u}(q)_\mathbb{C} \subset \mathfrak{u}(p,q)_\mathbb{C}$ for $r = p-q+2, \dots, p$. By Theorem 4.2.1

$$E_{p+r-1,p+r}x = \sum_{j=1}^{r} a_j x_{rj}^+ \quad \text{and} \quad E_{p+r-1,p+r}x = \sum_{j=1}^{r} b_j x_{rj}^-$$

where a_j and b_j are nonzero complex numbers. Note that the subscripts are shifted by p because $\mathfrak{u}(p)_\mathbb{C} = M_p(\mathbb{C})$ is embedded in the lower right corner of $M_{2p}(\mathbb{C})$. Thus

$u \boxtimes (E_{p+r-1,p+r}x)$ and $u \boxtimes (E_{p+r,p+r-1}x)$ lie in W. Since W is spanned by a subset of $\mathcal{B}(v_0)$, $u \boxtimes x^{\pm}_{rj} \in \mathcal{B}(v_0)$. This proves that $u \boxtimes x^{\pm}_{rj} \in W$. $\square$

For each $\mu \in \Lambda^+(p-q)$, let

$$\Lambda^+(p,\mu) = \{\lambda \in \Lambda^+(p) \ : \ \lambda_i \geq \mu_i \geq \lambda_{i+q}, \ \forall 1 \leq i \leq p-q\}. \tag{16}$$

Then by Lemma 5.1.1, $S_\lambda \neq 0$ for $\lambda \in \Lambda^+(p,\mu)$ and

$$I_{p,q}(s,\sigma,\mu) = \sum_{\lambda \in \Lambda^+(p,\mu)} S_\lambda.$$

The following lemma follows from Lemma 5.2.1

Lemma 5.2.2. *Let W be an infinitesimal $\mathrm{U}(p,q)$-submodule of $I_{p,q}(s,\sigma,\mu)$, and let*

$$\Lambda^+(W) = \{\lambda \in \Lambda^+(p,\mu) \ : \ S_\lambda \cap W \neq 0\}. \tag{17}$$

Then

$$W = \sum_{\lambda \in \Lambda^+(W)} S_\lambda.$$

Consequently if $W_1 \subseteq W_2$ are infinitesimal submodules of $I_{p,q}(s,\sigma,\mu)$, then

$$\mathrm{Res}^{\mathrm{U}(p,q)}_{\mathrm{U}(p)\times\mathrm{U}(q)} W_2/W_1 = \sum_{\lambda \in \Lambda^+(W_2)-\Lambda^+(W_1)} S_\lambda. \qquad \square$$

6. Transition coefficients.

6.1. In this section, we will study the action of the Lie algebra on S_λ.

First we recall the following fact about representations of $\mathrm{U}(p)$.

$$\mathbb{C}^p \otimes \tau_p^\lambda = \sum_{i=1}^p \tau_p^{\lambda+\varepsilon_i}, \quad (\mathbb{C}^p)^* \otimes \tau_p^\lambda = \sum_{i=1}^p \tau_p^{\lambda-\varepsilon_i}. \tag{18}$$

Note that if $\lambda \pm \varepsilon_i$ is not a dominant weight, then we set $\tau_p^{\lambda\pm\varepsilon_i} = 0$.

6.2. The Lie algebras of $\mathrm{U}(p,q)$ and $\mathrm{U}(p,p)$ have Cartan decompositions

$$\mathfrak{u}(p,q) = \mathfrak{k} \oplus \mathfrak{p} \ \text{ and } \ \mathfrak{u}(p,p) = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{p}},$$

where $\mathfrak{k} = \mathfrak{u}(p) \oplus \mathfrak{u}(q)$ and $\widetilde{\mathfrak{k}} = \mathfrak{u}(p) \oplus \mathfrak{u}(p)$. As $\mathfrak{k}$ and $\widetilde{\mathfrak{k}}$ modules, we have

$$\begin{array}{ll} \mathfrak{p}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^- & \widetilde{\mathfrak{p}}_\mathbb{C} = \widetilde{\mathfrak{p}}^+ \oplus \widetilde{\mathfrak{p}}^- \\ \mathfrak{p}^+ \cong \mathbb{C}^p \boxtimes (\mathbb{C}^q)^* & \mathfrak{p}^- \cong (\mathbb{C}^p)^* \boxtimes \mathbb{C}^q \\ \widetilde{\mathfrak{p}}^+ \cong \mathbb{C}^p \boxtimes (\mathbb{C}^p)^* & \widetilde{\mathfrak{p}}^- \cong (\mathbb{C}^p)^* \boxtimes \mathbb{C}^p. \end{array}$$

By Proposition 4.3.1(ii), we fix a GZ basis vector $v_0 \in \tau_{p-q}^{\hat{\mu}}$ and identify $I_{p,q}(s,\sigma,\mu)$ with the subspace $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$ of $I_p(s,\sigma)$. We now fix $\lambda \in \Lambda^+(p,\mu)$. Recall that V_λ is the $\mathrm{U}(p)\times\mathrm{U}(p)$-type of $I_p(s,\sigma)$ isomorphic to $\tau_p^\lambda \boxtimes \tau_p^{\hat{\lambda}}$, where $\hat{\lambda} = \lambda^* + \sigma\mathbf{1}_p$. Let $L : \widetilde{\mathfrak{p}}_\mathbb{C} \otimes V_\lambda \to I_p(s,\sigma)$ be the Lie algebra action on $I_p(s,\sigma)$. It is also a $\mathrm{U}(p)\times\mathrm{U}(p)$

map. Let $p_i : I_p(s,\sigma) \to V_{\lambda-\varepsilon_i}$ denote the projection map. We consider the following commutative diagram

$$\begin{array}{ccccccc} ((\mathbb{C}^p)^* \otimes \tau_p^{\lambda}) \boxtimes (\mathbb{C}^p \otimes \tau_p^{\hat{\lambda}}) & = & \widetilde{\mathfrak{p}}^- \otimes V_\lambda & \xrightarrow{L} & \sum_{j=1}^p V_{\lambda-\varepsilon_j} & \xrightarrow{p_i} & V_{\lambda-\varepsilon_i} \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ ((\mathbb{C}^p)^* \otimes \tau_p^{\lambda}) \boxtimes (\mathbb{C}^q \otimes J_{\hat{\lambda}}) & = & \mathfrak{p}^- \otimes S_\lambda & \xrightarrow{L} & \sum_{j=1}^p S_{\lambda-\varepsilon_j} & \xrightarrow{p_i} & S_{\lambda-\varepsilon_i} \end{array} \tag{19}$$

Define $T_{\lambda,\lambda-\varepsilon_i} = p_i \circ L$ and let $T'_{\lambda,\lambda-\varepsilon_i} : \mathfrak{p}^- \otimes S_\lambda \to S_{\lambda-\varepsilon_i}$ denote its restriction to $\mathfrak{p}^- \otimes S_\lambda$. By (18), $\widetilde{\mathfrak{p}}^- \otimes V_\lambda = \sum_{a,b=1}^p V_{ab}$ as a $\mathrm{U}(p) \times \mathrm{U}(p)$-module where $V_{ab} \simeq \tau_p^{\lambda-\varepsilon_a} \boxtimes \tau_p^{\hat{\lambda}+\varepsilon_b}$. Hence $T_{\lambda,\lambda-\varepsilon_i}$ is either zero or it is an isomorphism on V_{ii}.

Similarly we define

$$T_{\lambda,\lambda+\varepsilon_i} : \widetilde{\mathfrak{p}}^+ \otimes V_\lambda \to V_{\lambda+\varepsilon_i} \quad \text{and} \quad T'_{\lambda,\lambda+\varepsilon_i} : \mathfrak{p}^+ \otimes S_\lambda \to S_{\lambda+\varepsilon_i}$$

by replacing $\widetilde{\mathfrak{p}}^-$, $\mathfrak{p}^-$, $\lambda - \varepsilon_i$ in (19) by $\widetilde{\mathfrak{p}}^-$, $\mathfrak{p}^-$, $\lambda + \varepsilon_i$ respectively. We will state Prop. 5.15 of [Le].

Proposition 6.2.1. *Let $\alpha = -(s+p-\sigma)/2$, $\beta = -(s+p+\sigma)/2$ and $1 \le j \le p$.*

(i) *If $\lambda + \varepsilon_j \in \Lambda^+(p)$, then $T_{\lambda,\lambda+\varepsilon_j} \neq 0$ if and only if $\lambda_j \neq \alpha + j - 1$.*
(ii) *If $\lambda - \varepsilon_j \in \Lambda^+(p)$, then $T_{\lambda,\lambda-\varepsilon_j} \neq 0$ if and only if $\lambda_j \neq -\beta - p + j$.*

6.3. For each $1 \le i \le p$, let $\mathrm{pr}_i : \mathbb{C}^q \otimes J_{\hat{\lambda}} \to \tau_p^{\hat{\lambda}+\varepsilon_i}$ and $\mathrm{pr}_i^d : (\mathbb{C}^q)^* \otimes J_{\hat{\lambda}} \to \tau_p^{\hat{\lambda}-\varepsilon_i}$ be respectively the composition of the maps

$$\mathbb{C}^q \otimes J_{\hat{\lambda}} \hookrightarrow \mathbb{C}^p \otimes \tau_p^{\hat{\lambda}} \xrightarrow{\text{proj}} \tau_p^{\hat{\lambda}+\varepsilon_i} \tag{20}$$

$$(\mathbb{C}^q)^* \otimes J_{\hat{\lambda}} \hookrightarrow (\mathbb{C}^p)^* \otimes \tau_p^{\hat{\lambda}} \xrightarrow{\text{proj}} \tau_p^{\hat{\lambda}-\varepsilon_i}. \tag{21}$$

where proj is defined by using (18). We remark that the images of pr_i and pr_i^d lie in $J_{\hat{\lambda}+\varepsilon_i}$ and $J_{\hat{\lambda}-\varepsilon_i}$ respectively.

By (19), we see that $T'_{\lambda,\lambda-\varepsilon_i} \neq 0$ if and only if $\mathrm{pr}_{p-i+1} \neq 0$ and $T_{\lambda,\lambda-\varepsilon_i} \neq 0$. Similarly $T'_{\lambda,\lambda+\varepsilon_i} \neq 0$ if and only if $\mathrm{pr}^d_{p-i+1} \neq 0$ and $T_{\lambda,\lambda+\varepsilon_i} \neq 0$. Note that $(\lambda \hat{+} \epsilon_i) = \hat{\lambda} - \epsilon_{p+1-i}$.

Lemma 6.3.1. (i) *If $J_{\hat{\lambda}+\varepsilon_i} \neq 0$, then $\mathrm{pr}_i \neq 0$.*
(ii) *If $J_{\hat{\lambda}-\varepsilon_i} \neq 0$, then $\mathrm{pr}_i^d \neq 0$.*

We shall postpone the proof of this lemma to the end of this subsection. If U is a subspace of $I_{p,q}(s,\sigma,\mu)$, we let $\mathfrak{p}^{\pm}(U) = L(\mathfrak{p}^{\pm} \otimes U)$.

Proposition 6.3.2. *Let $\lambda \in \Lambda^+(p,\mu)$ so that S_λ is nonzero in $I_{p,q}(s,\sigma,\mu)$. Let W be the infinitesimal $\mathrm{U}(p,q)$-submodule generated by S_λ. Let $\alpha = -(s+p-\sigma)/2$, $\beta = -(s+p+\sigma)/2$ and $1 \le j \le p$.*

(i) *If $\lambda + \varepsilon_j \in \Lambda^+(p,\mu)$, then $S_{\lambda+\varepsilon_j} \subseteq W$ if and only if $\lambda_j \neq \alpha + j - 1$.*
(ii) *If $\lambda - \varepsilon_j \in \Lambda^+(p,\mu)$, then $S_{\lambda-\varepsilon_j} \subseteq W$ if and only if $\lambda_j \neq -\beta - p + j$.*

Proof. We shall prove (ii). The proof for (i) is similar.

First if $\lambda_j = -\beta - p + j$, then by [Le], the subspace

$$\mathcal{Y} = \sum_{\lambda'_j \geq \lambda_j} V_{\lambda'}$$

is an infinitesimal $\mathrm{U}(p,p)$-submodule of $I_p(s,\sigma)$, so that

$$\mathcal{Y} \cap (I_{p,q}(s,\sigma,\mu) \boxtimes v_0) = \sum_{\lambda'_j \geq \lambda_j} S_{\lambda'}$$

is an infinitesimal $\mathrm{U}(p,q)$-submodule of $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$. In particular, $S_{\lambda-\varepsilon_j} \cap W = 0$.

Conversely suppose $\lambda_j \neq -\beta - p + j$, then by Proposition 6.2.1 $T_{\lambda,\lambda-\varepsilon_j} \neq 0$. By Lemma 6.3.1, $\mathrm{pr}_{p-j+1} \neq 0$ and hence $T'_{\lambda,\lambda-\varepsilon_j} \neq 0$. From the definition of $T'_{\lambda,\lambda-\varepsilon_j}$, its image must lie in $W \cap S_{\lambda-\varepsilon_j}$. Therefore $W \cap S_{\lambda-\varepsilon_j} \neq 0$ and by Lemma 5.2.1, $S_{\lambda-\varepsilon_j} \subseteq W$. $\square$

6.4. We shall devote the rest of this section to the proof of Lemma 6.3.1. We write $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_p)$ and let $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_{p+1})$ be defined by $\widetilde{\lambda}_1 = \hat{\lambda}_1 + 1$ and $\widetilde{\lambda}_j = \hat{\lambda}_{j-1}$ for $j \geq 2$. We consider the irreducible representation $\tau_{p+1}^{\widetilde{\lambda}}$ of $\mathrm{U}(p+1)$ with highest weight $\widetilde{\lambda}$. Let $\mathcal{B}(\widetilde{\lambda})$ be the GZ basis of $\tau_{p+1}^{\widetilde{\lambda}}$. Each member $[m_{kl}]$ of $\mathcal{B}(\widetilde{\lambda})$ has $p+1$ rows, and $m_{k,p+1} = \widetilde{\lambda}_k$ for $1 \leq k \leq p+1$. We shall identify $\tau_p^{\hat{\lambda}}$ and $\tau_p^{\hat{\lambda}+\varepsilon_i}$ with the following subspaces of $\tau_{p+1}^{\widetilde{\lambda}}$:

$$\begin{aligned}
\tau_p^{\hat{\lambda}} &= \mathrm{Span}\{[m_{kl}] \in \mathcal{B}(\widetilde{\lambda}) \ : \ m_{kp} = \hat{\lambda}_k, k = 1, \ldots, p\} \\
\tau_p^{\hat{\lambda}+\varepsilon_i} &= \mathrm{Span}\{[m_{kl}] \in \mathcal{B}(\widetilde{\lambda}) \ : \ m_{ip} = \hat{\lambda}_i + 1, m_{kp} = \hat{\lambda}_k \text{ if } k \neq i\} \qquad (22) \\
J_{\hat{\lambda}} &= \mathrm{Span}\{[m_{kl}] \in \mathcal{B}(\widetilde{\lambda}) \ : \ m_{kp} = \hat{\lambda}_k, d_{p-q}([m_{kl}]) = v_0\} \\
J_{\hat{\lambda}+\varepsilon_i} &= \mathrm{Span}\{[m_{kl}] \in \mathcal{B}(\widetilde{\lambda}) \ : \ m_{kp} = \hat{\lambda}_k + \delta_{ki}, d_{p-q}([m_{kl}]) = v_0\}
\end{aligned}$$

Note that $\tau_p^{\hat{\lambda}}$ and $\tau_p^{\hat{\lambda}+\varepsilon_i}$ occurs in $\tau_{p+1}^{\widetilde{\lambda}}$ with multiplicity one.

Let $W_p \subseteq \mathfrak{gl}_{p+1}(\mathbb{C})$ denote the span of $\{E_{1,p+1}, E_{2,p+1}, \ldots, E_{p,p+1}\}$. Then $W_p \simeq \mathbb{C}^p$ as representations of $\mathrm{U}(p)$, so that $W_p \otimes \tau_p^{\hat{\lambda}} \simeq \sum_i \tau^{\hat{\lambda}+\varepsilon_i}$. Let $L : W_p \otimes \tau_p^{\hat{\lambda}} \to \tau_{p+1}^{\widetilde{\lambda}}$ be the $\mathrm{U}(p)$-map defined by the Lie algebra action: $L(X \otimes w) = Xw$. Let $\tilde{v}$ denote the highest weight vector of $\tau_p^{\hat{\lambda}}$. Then $\tilde{v} = [m_{kl}]$ where $m_{pl} = \hat{\lambda}_p$ for all $l \leq p$. By Theorem 4.2.1

$$L(E_{p,p+1} \otimes \tilde{v}) = E_{p,p+1}\tilde{v} = \sum_{i=1}^{p} a_i \tilde{v}_{ip}^{+} \qquad (23)$$

where $a_i \neq 0$ and $\tilde{v}_{ip}^{+}$ is the GZ basis vector obtained from $\tilde{v}_{ip}^{+}$ by increasing m_{ip} by 1. This shows that the image of L has a nontrivial component in $\tau_p^{\hat{\lambda}+\varepsilon_i}$. This implies that L is injective on $\tau^{\hat{\lambda}+\varepsilon_i} \subseteq W_p \otimes \tau_p^{\hat{\lambda}}$ and hence L is an injection.

Proof of Lemma 6.3.1. We will only prove (i) as the proof of (ii) is similar.

Using the above identification, $\mathbb{C}^q$ is the span of $\{E_{p-q+1,p+1},\dots,E_{p,p+1}\}$ in W_p. If we replace $\tilde{v}=[m_{ij}]$ in (23) by a GZ basis vector in $J_{\hat{\lambda}}$, then $\mathrm{pr}_i(E_{p,p+1}\otimes\tilde{v})=a_i\tilde{v}^+_{ip}$. It remains to show that we can choose $\tilde{v}$ such that $\tilde{v}^+_{ip}$ is still a GZ basis vector, that is, $\tilde{v}^+_{ip}\neq 0$. Indeed $J_{\hat{\lambda}+\varepsilon_i}\subseteq\tau^{\hat{\lambda}+\varepsilon_i}$ is nonzero implies that $m_{i-1,p}>m_{ip}$. Pick a nonzero GZ basis vector $v_1=[m'_{kl}]\in J_{\hat{\lambda}}$. Suppose

$$m'_{ip}=m'_{i-1,p-1}=\ldots=m'_{i-r,p-r}<m'_{i-r-1,p-r-1}\le\ldots\le m'_{i-q,p-q}=\hat{\mu}_i.$$

This implies that $m'_{i-r-1,p-r}\ge m'_{i-r-2,p-r-1}\ge\ldots m'_{i-1,p}>m'_{ip}=m'_{i-r,p-r}$. Let $\tilde{v}$ be the (nonzero) GZ basis vector obtained from v_1 by increasing $m_{i,p-1},\dots,m_{i-r,p-r}$ each by 1. It is now easy to see that $\tilde{v}^+_{ip}\neq 0$. This proves Part (i) of the lemma. □

6.5. We could replace (23) by the Clebsch-Gordan formula of $\mathrm{U}(p)$ (see Chapter 5 [KV]) and give an alternative proof of Lemma 6.3.1. In Part 2, we will apply a similar argument to prove a statement analogous to Lemma 6.3.1 for the group $\mathrm{Spin}(p)$. However, the Clebsch-Gordan formula for $\mathrm{Spin}(p)$ is not available.

7. Module structure

7.1. In this section, we shall determine the reducibility of $I_{p,q}(s,\sigma,\mu)$, and describe its module structure when it is reducible. We shall first review the results of [Le] on the structure of $I_p(s,\sigma)$. Let $\alpha=-(s+p-\sigma)/2$ and $\beta=-(s+p+\sigma)/2$. Then $I_p(s,\sigma)$ is reducible if and only if α and β are integers, or equivalently, s is an integer and $s+p\equiv\sigma(\mathrm{mod}\ 2)$. Let

$$c_x=\max(\alpha,-\beta-p)\quad\text{and}\quad c_y=\min(\alpha,-\beta-p).$$

If $I_p(s,\sigma)$ is reducible, then its irreducible subquotients are of the form $R_{a(r,t)}$ where r and t are nonnegative integers such that $p-c_x+c_y\le r+t\le p$ and as a $\mathrm{U}(p)\times\mathrm{U}(p)$-module,

$$R_{a(r,t)}=\sum_{\lambda\in\Lambda^+(R_{a(r,t)})}\tau_p^{\lambda}\boxtimes\tau_p^{\hat{\lambda}}$$

where

$$\Lambda^+(R_{a(r,t)})=\left\{\lambda\in\Lambda^+(p):\ \lambda_r\ge c_x+r\ge\lambda_{r+1},\ \lambda_{p-t}\ge c_y+p-t\ge\lambda_{p-t+1}\right\}. \tag{24}$$

7.2. The detailed module structure of $I_p(s,\sigma)$ can best be described by a directed simple group $\mathcal{G}=\mathcal{G}(I_p(s,\sigma))$, called the *module diagram* of $I_p(s,\sigma)$ (see §7 of [Le]). The vertex set of $\mathcal{G}$ is the set of all irreducible subquotients $R_{a(r,t)}$ in $I_p(s,\sigma)$. There is a directed edge from the node R_1 to the node R_2 if and only if there are submodules U and V of $I_p(s,\sigma)$ such that $V\subseteq U$ and there is a nonsplit exact sequence of infinitesimal $\mathrm{U}(p,p)$-modules

$$0\to R_2\to U/V\to R_1\to 0.$$

We shall also arrange the nodes in $\mathcal{G}$ in such a way that all the edges are directed downward. Then one can recover the lattice of submodules of $I_p(s,\sigma)$ from the graph $\mathcal{G}$. Note that if we reverse the direction of the edges of $\mathcal{G}$, we obtain the module diagram for the dual representation of $I_p(s,\sigma)$. Now $I_p(-s,-\sigma)$ is isomorphic to the dual representation of $I_p(s,\sigma)$. So we only need to describe $\mathcal{G}$ for $s \leq 0$ (or equivalently, $\alpha+\beta \geq -p$).

If $s \leq -p$ (or equivalently, $\alpha+\beta \geq 0$), then $\mathcal{G}$ is given in Figure 1. If $-p+1 \leq s \leq 0$, then $\mathcal{G}$ can be obtained by deleting the lowest $s+p$ rows from the graph in Figure 1.

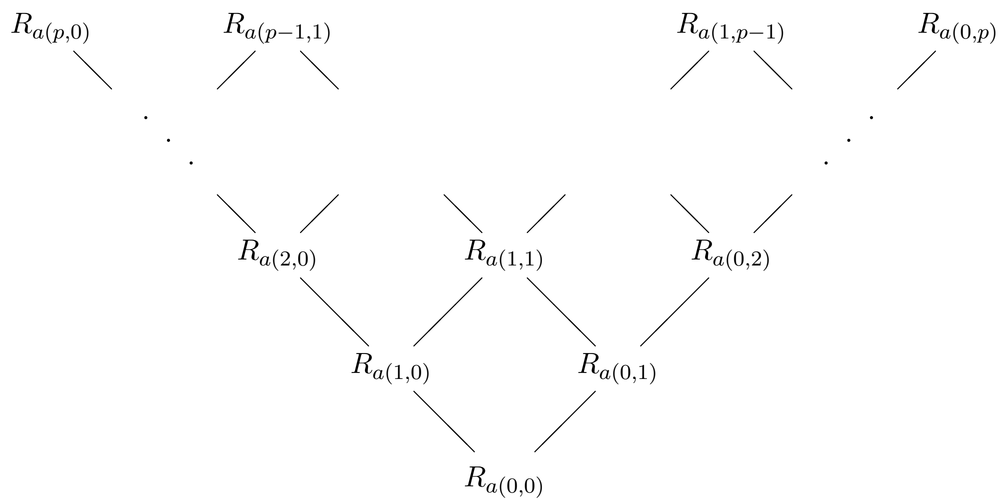

Figure 1

7.3. As in the previous section, we shall identify $I_{p,q}(s,\sigma,\mu)$ with the subspace $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$ of $I_p(s,\sigma)$, where v_0 is a fixed GZ-basis vector in $\tau_{p-q}^{\hat{\mu}}$. Suppose $R_{a(r,t)}$ is an irreducible subquotient in $I_p(s,\sigma)$, and $W_1 \subset W_2$ are infinitesimal $\mathrm{U}(p,p)$-submodules of $I_p(s,\sigma)$ such that $R_{a(r,t)} = W_2/W_1$. We define

$$R'_{a(r,t)} = (W_2 \cap I)/(W_1 \cap I),$$

where $I = I_{p,q}(s,\sigma,\mu)$. There is a canonical injection $R'_{a(r,t)} \hookrightarrow R_{a(r,t)}$. Note that as a representation of $\mathrm{U}(p) \times \mathrm{U}(q)$,

$$R'_{a(r,t)} = \sum_{\lambda \in \Lambda^+(R'_{a(r,t)})} S_\lambda$$

where (cf. (16) and (24))

$$\Lambda^+(R'_{a(r,t)}) := \Lambda^+(R_{a(r,t)}) \cap \Lambda^+(p,\mu).$$

Note that $I_{p,q}(-s,-\sigma,\mu^*)$ is isomorphic to the dual representation of $I_{p,q}(s,\sigma,\mu)$. Thus if s is real, we may assume that $s \leq 0$. We are now ready to state our first main theorem.

Theorem 7.3.1. *Let $s \in \mathbb{C}$, $\mu \in \Lambda^+(p-q)$, $\alpha = -(s+p-\sigma)/2$ and $\beta = -(s+p+\sigma)/2$.*

(A) (a) *If $s \notin \mathbb{R}$, then $I_{p,q}(s,\sigma,\mu)$ is irreducible.*

(b) *If $s \leq 0$, then $I_{p,q}(s,\sigma,\mu)$ is irreducible if and only if either one of the following conditions holds:*

(i) *Both α and β are non-integers.*

(ii) *α and β are integers, $2q \le p$ and there exist i, j such that $q+1 \le i, j \le p-q$ and*

$$\mu_i = \mu_{i-1} = \ldots = \mu_{i-q+1} = \alpha + i$$
$$\mu_j = \mu_{j-1} = \ldots = \mu_{j-q+1} = -\beta - p + j + 1.$$

(B) *Suppose $s \le 0$ and $I_{p,q}(s,\sigma,\mu)$ is reducible. In this case, $\alpha, \beta \in \mathbb{Z}$, so that $I_p(s,\sigma)$ is also reducible.*
 (a) *If $R_{a(r,t)}$ is an irreducible subquotient in $I_p(s,\sigma)$ and $R'_{a(r,t)} \neq 0$, then $R'_{a(r,t)}$ is an irreducible subquotient of $I_{p,q}(s,\sigma,\mu)$.*
 (b) *All irreducible subquotients of $I_{p,q}(s,\sigma,\mu)$ are of the form $R'_{a(r,t)}$.*
 (c) *Let $\mathcal{G}$ and $\mathcal{G}'$ be the module diagrams of $I_p(s,\sigma)$ and $I_{p,q}(s,\sigma,\mu)$ respectively, then $\mathcal{G}'$ is the subgraph of $\mathcal{G}$ obtained by removing the set of vertices*

$$V' = \{R_{a(r,t)} \ : \ R'_{a(r,t)} = 0\}$$

and removing all edges connected to the vertices in V'. In other words, $\mathcal{G}'$ is a spanning subgraph of $\mathcal{G}$.

We shall postpone the proof of this theorem to §7.7. A consequence of Part (B)(c) of the theorem is the following corollary which we will prove in §7.8.

Corollary 7.3.2. *Suppose W' is an infinitesimal $\mathrm{U}(p,q)$-submodule of $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$, then there exists an infinitesimal $\mathrm{U}(p,p)$-submodule W of $I_p(s,\sigma)$ such that $W' = W \cap (I_{p,q}(s,\sigma,\mu) \boxtimes v_0)$.*

7.4. We shall describe a method of deciding if $R'_{a(r,t)} \neq 0$. Define

$$S_i = \{z \in \mathbb{Z} : \mu_i \le z \le \mu_{i-q}\} \qquad S_{1i} = \{z \in \mathbb{Z} : z \le c_y + i - 1\}$$
$$S_{2i} = \{z \in \mathbb{Z} : c_y + i \le z \le c_x + i - 1\} \qquad S_{3i} = \{z \in \mathbb{Z} : c_x + i \le z\}$$

for $i = 1, \ldots, p$. Here $\mu_i = \infty$ if $i \le 0$ and $\mu_i = -\infty$ if $i > p-q$. Note that $S_{1i} \cup S_{2i} \cup S_{3i} = \mathbb{Z}$. Define $L_0 = \{3\}$, $L_{p+1} = \{1\}$ and $L_i := \{j : S_i \cap S_{ji} \text{ nonempty}\}$ if $1 \le i \le p$. Hence L_i is a subset of $\{1,2,3\}$. Next we define

$$e_j = \min\{i : j \in L_i\} \quad f_j = \max\{i : j \in L_i\}.$$

Clearly $e_3 = 0$, $f_1 = p+1$, $e_3 \le e_2 \le e_1$, $f_3 \le f_2 \le f_1$, $f_3 \ge e_2 - 1$ and $f_2 \ge e_1 - 1$.

Lemma 7.4.1. *The following conditions are equivalent:*
 (i) *$R'_{a(r,t)} \neq 0$.*
 (ii) *$r + t \ge p - c_x + c_y$. $3 \in L_r$, $1 \in L_{p-t+1}$. If $r < p-t$, then $2 \in L_{r+1}$, $2 \in L_{p-t}$ and $1 \in L_{p-t+1}$.*
 (iii) *$r + t \ge p - c_x + c_y$, $r \le f_3$ and $p - t \ge e_1 - 1$. If $r < p-t$, then $e_2 - 1 \le r < p - t \le f_2$.*
 (iv) *$r + t \ge p - c_x + c_y$, $e_2 - 1 \le r \le f_3$ and $e_1 - 1 \le p - t \le f_2$.*

Proof. The condition $r + t \ge p - c_x + c_y$ in (ii) to (iv) is to ensure that $R_{a(r,t)} \neq 0$. Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) are clear from the definitions of L_i, e_i and f_i. To prove (iii) $\Leftrightarrow$ (iv), we just have to consider the cases $r = p-t$ and $r < p-t$ separately.

Finally we show (ii) $\Rightarrow$ (i). Define $\lambda \in \Lambda^+(p)$ by

$$\lambda_j = \begin{cases} \max(c_x + r, \mu_j) & \text{if } 1 \leq j \leq r \\ \max(c_y + p - t, \mu_j) & \text{if } r+1 \leq j \leq p-t \\ \min(c_y + p - t, \mu_{j-q}) & \text{if } p-t+1 \leq j \leq p. \end{cases}$$

Then $S_\lambda \neq 0$ and $S_\lambda \subseteq R'_{a(r,t)}$. $\square$

7.5. We now describe how the module diagram $\mathcal{G}'$ of $I_{p,q}(s, \sigma, \mu)$ can be obtained from that of $I_p(s, \sigma)$. For each $1 \leq j \leq p$, let

$$l_j = \{R_{a(j,t)} \in \mathcal{G}\} \text{ and } r_j = \{R_{a(r,p-j)} \in \mathcal{G}\}$$

Then l_j and r_j form two "straight lines" in $\mathcal{G}$ passing through $R_{(j,p-j)}$. The case $s \leq -p$ is illustrated in Figure 2.

Starting with the module diagram $\mathcal{G}$ of $I_p(s, \sigma)$, we first discard all the vertices on l_j for $j < e_2 - 1$ and for $j > f_3$. Next we discard all the vertices on r_j for $j < e_1 - 1$ and for $j > f_2$. We also remove the edges connected to the discarded vertices. Rename the remaining vertices $R_{a(r,t)}$ by $R'_{a(r,t)}$. By Lemma 7.4.1(iv) and Theorem 7.3.1(B)(c), the resulting subgraph $\mathcal{G}'$ is the module diagram for $I_{p,q}(s, \sigma, \mu)$. We illustrate this in Figure 3 where the shaded area is $\mathcal{G}'$.

7.6. We study an interesting special case. If $s \in \mathbb{Z}$, $s \leq -p$ and $s + \sigma = -p$, then $I_p(s, \sigma)$ contains a unique finite dimensional subrepresentation of $\mathrm{U}(p,p)$, namely $R_{a(0,0)}$ (cf. §7.1). $R_{a(0,0)}$ has highest weight $\gamma = (\alpha, \alpha, \ldots, \alpha, 0, 0, \ldots, 0) \in \Lambda^+(2p)$ where there are p copies of α's. Using the branching rules in Exercise 6.12 in [FH] and the Littlewood-Richardson numbers, we can show that

$$\mathrm{Res}^{\mathrm{U}(p,p)}_{\mathrm{U}(p,q)\times \mathrm{U}(p-q)} R_{a(0,0)} = \sum_{\eta,\mu} \tau^{\eta}_{p,q} \boxtimes \tau^{\mu}_{p-q} \tag{25}$$

where the sum is taken over all highest weight $\eta = (\eta_1, \ldots, \eta_{p+q}) \in \Lambda^+(p+q)$ and $\mu = (\mu_1, \ldots, \mu_q) \in \Lambda^+(p-q)$ satisfying $\eta_i \geq 0$, $\mu_j \geq 0$ and

$$\eta_1 = \eta_2 = \ldots = \eta_q = \alpha$$
$$\eta_{q+i} + \mu_i = \alpha \text{ for } i = 1, \ldots, p-q.$$

One can verify that the (25) is in agreement with Theorem 7.3.1.

7.7. Proof of Theorem 7.3.1. We first prove the "only if" part of (A)(b). Suppose $\alpha, \beta \in \mathbb{Z}$ and $I_{p,q}(s, \sigma, \mu)$ is irreducible. We assume that $s \leq -1$, so that for any $1 \leq j \leq p$, $\alpha + j - 1 \geq -\beta + p - j$. First we note that $\alpha + q - 1 < \mu_q$ and $-(\beta + q - 1) < \mu_{p-2q+1}$. Let $i = \min\{t : \alpha + t - 1 \geq \mu_t\}$. Then $i \geq q$. We claim that $\mu_i = \alpha + i - 1$. Otherwise, since $\alpha + i - 2 < \mu_{i-1}$, we have

$$\mu_i < \alpha + i - 1 = (\alpha + i - 2) + 1 \leq \mu_{i-1} \leq \mu_{i-q}.$$

Then we can choose λ and η in $\Lambda^+(p, \mu)$ such that $\lambda_i = \alpha + i - 1$ and $\eta_i = \mu_{i-q}$. Then λ and η belong to different constituents which implies that $I_{p,q}(s, \sigma, \mu)$ is reducible. Thus we must have $\mu_i = \alpha + i - 1$. We also claim that $\mu_i = \mu_{i-q}$, for otherwise we can again find two elements of $\Lambda^+(p,\mu)$ belonging to two different constituents. Similarly let $j=\min\{t:\ -\beta+n-t>\mu_t\}$. Then similar arguments show that $\mu_j=\mu_{j-q}$. This proves the "only if" part of the statement.

Next we prove (A)(a) and the "if" part of (A)(b). First we assume that α and β are not integers and shall show that $I_{p,q}(s,\sigma,\mu)$ is irreducible. The proof of this is similar to that of Theorem 6.2 of [Le]. Suppose that $\lambda,\lambda'\in\Lambda^+(p,\mu)$, that is, S_λ and $S_{\lambda'}$ are nonzero isotypic components for U(p) in $I_{p,q}(s,\sigma,\mu)$. It suffices to prove that the infinitesimal U(p,q)-submodule W of $I_{p,q}(s,\sigma,\mu)$ generated by S_λ contains $S_{\lambda'}$. To do this, we construct a sequence $(\eta^{(m)})_{m=0}^n$ in $\mathbb{Z}^n$ inductively as follows. Let $1\le i_1<i_2<\cdots<i_a\le p$ and $p\ge j_1>j_2>\cdots>j_b\ge 1$ be such that $\lambda_{i_u}\le\lambda'_{i_u}$ and $\lambda_{j_v}>\lambda'_{j_v}$, $\forall 1\le u\le a$, $1\le v\le b$, and $\{i_1,...,i_a\}\cup\{j_1,...,j_b\}=\{1,2,...,p\}$. Set $\eta^{(0)}=\lambda$. For $1\le m\le a$, we set

$$\eta_k^{(m)}=\begin{cases}\lambda'_{i_m} & k=i_m\\ \eta_k^{(m-1)} & k\ne i_m\end{cases}$$

and for $a+1\le m\le p$,

$$\eta_k^{(m)}=\begin{cases}\lambda'_{j_{m-a}} & k=j_{m-a}\\ \eta_k^{(m-1)} & k\ne j_{m-a}.\end{cases}$$

It is clear that $\eta^{(m)}\in\Lambda^+(p)$. By Lemma 5.1.1, $\lambda_i\ge\mu_i\ge\lambda_{i+q}$ and $\lambda'_i\ge\mu_i\ge\lambda'_{i+q}$ for $i=1,2,...,p-q$. Since for each m the components of $\eta^{(m)}$ are taken from these numbers, we clearly have

$$\eta_i^{(m)}\ge\mu_i\ge\eta_{i+q}^{(m)}\qquad\forall i=1,2,...,p-q.$$

Thus $\eta^{(m)}\in\Lambda^+(p,\mu)$. We now claim that W contains $S_{\eta^{(1)}}$. First we note that $\eta^{(1)}=\lambda+k_1\varepsilon_{i_1}$ for some nonnegative integer k_1 and $\lambda+t\varepsilon_{i_1}\in\Lambda^+(p,\mu)$ for $1\le t\le k_1-1$. By Proposition 6.3.2, since $-\lambda_{i_1}\ne\alpha+i_1-1$, $S_{\lambda+\varepsilon_{i_1}}\subseteq W$. Similarly, since $-\lambda_{i_1}\ne\alpha+t+i_1-1$ for $1\le t\le k_1-1$, $S_{\lambda+2\varepsilon_{i_1}},...,S_{\lambda+k_1\varepsilon_{i_1}}=S_{\eta^{(1)}}$ are contained in W. Next we note that $\eta^{(2)}=\eta^{(1)}+k_2\varepsilon_{i_2}$ for some nonnegative integer k_2. It is now clear that if we proceed with the above arguments along the sequence $(\eta^{(m)})$, we will obtain $S_{\eta^{(n)}}=S_{\lambda'}\subseteq W$.

Similar arguments also prove (B)(a).

If condition (A)(b)(ii) holds, then by Lemma 7.4.1 one can check that $R'_{a(r,t)}\ne 0$ if and only if $(r,t)=(i,p-j)$. Hence $I_{p,q}(s,\sigma,\mu)=R'_{a(i,p-j)}$ is irreducible by b(i).

Part (B)(b) is true because the U(p)-isotypic components S_λ contained in all the $R'_{a(r,t)}$ exhaust all the U(p)-isotypic components in $I_{p,q}(s,\sigma,\mu)$.

Finally we prove (B)(c). Let $R_1=R_{a(r,t)}$, $R_2=R_{a(u,v)}$, $R'_1=R'_{a(r,t)}$ and $R'_2=R'_{a(u,v)}$. Suppose that there is a directed edge in $\mathcal{G}$ from R_1 to R_2 and R'_1 and R'_2 are nonzero. We need to show that there is a directed edge from R'_1 to R'_2 in $\mathcal{G}'$.

To do this, we need to show that there exist $\lambda \in \Lambda^+(R_1')$ and $\eta \in \Lambda^+(R_2')$ such that $\mathfrak{p}_{\mathbb{C}}(S_\lambda) \cap S_\eta \neq 0$. From Figure 1, we see that $R_1 \to R_2$ if and only if $(u,v) = (r-1,t)$ or $(u,v) = (r,t-1)$. We shall only prove the case $(u,v) = (r-1,t)$ as the other case is similar. Let $A_r = -(\beta + p - r)$ and $B_r = \alpha + r - 1$. Since both $R'_{a(r,t)}$ and $R'_{a(r-1,t)}$ are nonzero, $[\mu_r, \mu_{r-q}] \cap [A_r, B_r] \neq \emptyset$ and $[\mu_r, \mu_{r-q}] \cap [B_r, \infty) \neq \emptyset$. In particular, $B_r, B_{r+1} \in [\mu_r, \mu_{r-q}]$. Thus there exist $\lambda \in R'_{a(r,t)}$ and $\eta \in R'_{a(r-1,t)}$ such that $\lambda_r = B_r + 1$, $\eta_r = B_r$ and $\lambda_j = \eta_j$ for all $j \neq s$. By the proof of Proposition 6.3.2, $\mathfrak{p}^-(S_\lambda) \cap S_\eta \neq 0$. This completes the proof. □

7.8. Proof of Corollary 7.3.2. If the corollary holds for W_1' and W_2' in $I_{p,q}$, then it holds for $W_1' + W_2'$. Hence we may assume that W' generated by a single vector.

Let

$$R_{a(r_1,t_1)} \to R_{a(r_2,t_2)} \to \ldots \to R_{a(r_k,t_k)}$$

be a directed path in the module diagram $\mathcal{G}$. Suppose $R'_{a(r_1,t_1)}$ and $R'_{a(r_k,t_k)}$ are nonzero, then from the description of $\mathcal{G}'$ in §7.5, $R'_{a(r_i,t_i)} \neq 0$ for all $i = 1, \ldots, k$. Then Corollary 7.3.2 follows from the general theory of module diagrams. □

8. Unitarity

8.1. In this section, we shall determine the unitarity of $I_{p,q}(s,\sigma,\mu)$ and its subquotients. Recall that we can identify $I_{p,q}(s,\sigma,\mu) \simeq I_{p,q}(s,\sigma,\mu) \boxtimes v_0$ in $I_p(s,\sigma)$, where v_0 is a fixed GZ basis vector in $\tau^{\hat{\mu}}_{p-q}$. Thus if $I_p(s,\sigma)$ is unitarizable, then so is $I_{p,q}(s,\sigma,\mu)$. So there are two obvious families of unitarizable representations: unitary induction and the restriction of the complementary series of $\mathrm{U}(p,p)$. Similarly, if $I_p(s,\sigma)$ is reducible and $R_{a(r,t)}$ is a unitary subquotient in $I_p(s,\sigma)$ such that $R'_{a(r,t)} \neq 0$, then $R'_{a(r,t)}$ is also unitary. We shall determine which other representations $I_{p,q}(s,\sigma,\mu)$ or their subquotients are unitary.

Theorem 8.1.1. *Let $s \in \mathbb{C}$, $\sigma \in \mathbb{Z}$, $\mu \in \Lambda^+(p-q)$, $\alpha = -(s+p-\sigma)/2$ and $\beta = -(s+p+\sigma)/2$.*

- (A) (Unitary induction) *If $\mathrm{Re}(s) = 0$, then $I_{p,q}(s,\sigma,\mu)$ is unitarizable.*
- (B) (Restriction of the complementary series of $\mathrm{U}(p,p)$) *If $\sigma \equiv p+1 \pmod 2$ and $|s| < 1$, then $I_{p,q}(s,\sigma,\mu)$ is unitarizable.*
- (C) (Other unitarizable representations) *Let $s < 0$. Suppose that the following conditions are satisfied:*
  - (i) *$p \geq 2q$;*
  - (ii) *there exists an integer m such that*

  $$\mu_j = m \qquad (a \leq j \leq b)$$

  where $b - a \geq q - 1$.
  - (iii) $\begin{cases} \alpha < -a - q + m + 2 \\ \beta < b - p - m + 1. \end{cases}$

  Then $I_{p,q}(s,\sigma,\mu)$ is unitarizable.

(D) *If $I_{p,q}(s,\sigma,\mu)$ is unitarizable, then it must be one of the representations described in parts (A), (B) and (C) (and their duals).*

(E) (Unitary subquotients) *Suppose that $\alpha, \beta \in \mathbb{Z}$ and $\alpha+\beta \geq -p+1$ (equivalently, $s \in \mathbb{Z}$, $s \leq -1$ and $s+p \equiv \sigma (\mathrm{mod}\ 2)$). In this case, $I_p(s,\sigma)$ is reducible.*
  - (a) (Restriction of the unitarizable subquotients in $I_p(s,\sigma)$)
    - (1) *If $0 \leq j \leq n$ and $R'_{a(j,n-j)} \neq 0$, then $R'_{a(j,n-j)}$ is unitarizable.*
    - (2) *If $-p \leq s \leq -1$, $r+t = s+p$ and $R'_{a(r,t)} \neq 0$, then $R'_{a(r,t)}$ is unitarizable.*
  - (b) *(*Other unitarizable subquotients*) Assume that $p+s < r+t < p$.*
    - (1) *Suppose $R'_{a(r,t)}$ is nonzero and satisfies the following conditions:*
      - (i) *$t \geq q$.*
      - (ii) *$\mu_j = \alpha + r$ for $r+1 \leq j \leq p-t$.*

      Then $R'_{a(r,t)}$ is a unitarizable subquotient in $I_{p,q}(s,\sigma,\mu)$.
    - (2) *Suppose $R'_{a(r,t)}$ is nonzero and satisfies the following conditions:*
      - (i) *$r \geq q$.*
      - (ii) *$\mu_{j-q} = -(\beta+t)$ for $r+1 \leq j \leq p-t$.*

      Then $R'_{a(r,t)}$ is a unitarizable subquotient in $I_{p,q}(s,\sigma,\mu)$.
    - (3) *Suppose $R'_{a(r,t)}$ is nonzero and satisfies the following conditions:*
      - (i) *$r \geq q$ and $t \geq q$.*
      - (ii) *There exists m such that $-(\beta+t) \leq m \leq \alpha+r$ and $\mu_j = m$ for $s+1-q \leq j \leq p-t$.*

      Then $R'_{a(r,t)}$ is a unitarizable subquotient in $I_{p,q}(s,\sigma,\mu)$.
  - (c) *If $R'_{a(r,t)}$ is unitarizable then it belongs to one of the cases given in Parts (a) or (b).*

If the unitarizable subquotients of $I_{p,q}(s,\sigma,\mu)$ given in Parts (E)(b) of the above theorem occur, then they correspond to the vertices located at either the left, the right or the upper corner of the module diagram. In the examples given in Figure 4(a)-(c), each of these unitarizable subquotients are enclosed by a rectangle.

If $\sigma = 0$ and τ^{μ}_{p-q} is the trivial representation of $\mathrm{U}(p-q)$, then Part (C) of the theorem implies the following:

Corollary 8.1.2. *If $p \geq 2q$ and s is real and*

$$|s| < p - 2q + 2,$$

then $I_{p,q}(s,0,0)$ is irreducible and unitarizable.

Note that the length of the complementary series given in the above corollary increases with $p-2q$. This phenomenon of long complementary series is well known for classical groups of real rank one. The case of $\mathrm{SO}_0(p,q)$ with $q > 1$ was first determined by J-S Li [Li] (c.f. Corollary 13.1.2).

8.2. We now briefly describe the main ideas of the proof for Theorem 8.1.1. We assume that both α and β are real and R is an irreducible subquotient of $I_p(s,\sigma)$. Suppose R gives rise to an irreducible subquotient R' in $I_{p,q}(s,\sigma,\mu)$. We will first construct a (not necessary positive definite) $\mathfrak{u}(p,p)$-invariant Hermitian form $\langle .,. \rangle$ on R. Then the restriction of $\langle .,. \rangle$ to R' is clearly $\mathfrak{u}(p,q)$-invariant and we will see that it is nontrivial on R'. Since R' is irreducible the Hermitian form is uniquely determined up to a nonzero scalar. The Hermitian form is either positive or negative definite on a $\tilde{K}$-type V_λ and hence also on S_λ. Finally we determine whether the signatures are the same on those V_λ whose restrictions to S_λ are nontrivial.

8.3. We shall devote the remaining part of this section to the proof of Theorem 8.1.1. Assume that both α and β are real. Let R be an irreducible subquotient of $I_p(s,\sigma)$. If $\alpha,\beta \notin \mathbb{Z}$, then $R \cong I_p(s,\sigma)$, and if $\alpha,\beta \in \mathbb{Z}$, then $R = R_{a(r,t)}$ for some r and t. We first describe a non-degenerate $\mathrm{U}(p,p)$ invariant Hermitian form on R (see §9 of [Le]). Recall that $\Lambda^+(R)$ denotes the set of $\mathrm{U}(p)$ highest weights which occur in R, so that

$$R = \sum_{\lambda \in \Lambda^+(R)} V_\lambda,$$

where for each λ, $V_\lambda \cong \tau_p^\lambda \boxtimes \tau_p^{\hat{\lambda}}$ as a representation of $\mathrm{U}(p) \times \mathrm{U}(p)$. We now fix an element $\eta \in \Lambda^+(R)$. For $1 \le j \le p$, let h_j be the function on $\mathbb{Z}$ given by

$$h_j(m) = \begin{cases} (-1)^{|m|} \prod_{a=0}^{|m|-1} \frac{\alpha-\eta_j+j+a}{\beta+\eta_j+p-j-a} & m<0 \\ 1 & m=0 \\ (-1)^m \prod_{b=1}^{m} \frac{\beta+\eta_j+p-j+b}{\alpha-\eta_j+j-b} & m>0. \end{cases}$$

For each $\lambda \in \Lambda^+(R)$, let

$$c_\lambda = h_1(m_1) h_2(m_2) \cdots h_p(m_p), \tag{26}$$

where $m_j = \lambda_j - \eta_j$ for $1 \le j \le p$, and define the Hermitian form $\langle .,. \rangle_\lambda$ on V_λ by

$$\langle f_1, f_2 \rangle_\lambda = c_\lambda \int_{\mathrm{U}(p)\times\mathrm{U}(p)} f_1(k)\overline{f_2(k)}\, dk \qquad (f_1, f_2 \in V_\lambda).$$

Let $\langle .,. \rangle$ be the sum of all $\langle .,. \rangle_\lambda$. Then

$$(\alpha - \lambda_j + j - 1)c_{\lambda+\varepsilon_j} + (\beta + \lambda_j + p - j + 1)c_\lambda = 0. \tag{27}$$

for all $1 \le j \le p$ and $\lambda \in \Lambda^+(R)$ such that $\lambda, \lambda+\varepsilon_j \in \Lambda^+(R)$. This equation is equivalent to the condition that the Lie algebra of $\mathrm{U}(p,p)$ acts on $I_p(s,\sigma)$ by skew-hermitian operators. Consequently the form $\langle .,. \rangle$ is $\mathrm{U}(p,p)$ invariant.

Lemma 8.3.1. *Let*

$$R' = \begin{cases} I_{p,q}(s,\sigma,\mu) & \text{if } R = I_p(s,\sigma) \textit{ is irreducible}, \\ R'_{a(r,t)} & \text{if } I_p(s,\sigma) \text{ is reducible and } R = R_{a(r,t)}. \end{cases}$$

Let $\Lambda^+(R')$ be the set of $\mathrm{U}(p)$ highest weights which occurs in R'. Then R' is unitarizable if and only if all the numbers $\{c_\lambda\}_{\lambda\in\Lambda^+(R')}$ are of the same sign.

Proof. Let $\mathcal{Y} = \oplus_{\lambda \in \Lambda^+(R')} V_\lambda \subseteq R$. Then

$$R' \subseteq \mathcal{Y}.$$

Let $\langle .,. \rangle_{\mathcal{Y}}$ be the restriction of the form $\langle .,. \rangle$ to $\mathcal{Y}$. If all the numbers $\{c_\lambda\}_{\lambda \in \Lambda^+(R')}$ are of the same sign, then by multiplying $\langle .,. \rangle_{\mathcal{Y}}$ by -1 if necessary, we may assume that $\langle .,. \rangle_{\mathcal{Y}}$ is positive definite. Thus its restriction to R' is a $\mathrm{U}(p,q)$-invariant inner product. Conversely suppose that there exist λ and λ' in $\Lambda^+(R')$ such that $c_\lambda > 0$ and $c_{\lambda'} < 0$. Then for any nonzero functions $f_1 \in S_\lambda$ and $f_2 \in S_{\lambda'}$, $\langle f_1, f_1 \rangle_{\mathcal{Y}} > 0$ and $\langle f_2, f_2 \rangle_{\mathcal{Y}} < 0$. Let $(.,.)_{R'}$ is the restriction of $\langle .,. \rangle_{\mathcal{Y}}$ to R'. Then it is neither positive definite nor negative definite. Since R' is irreducible, any $\mathrm{U}(p,q)$ invariant Hermitian form on R' must be a multiple of $(.,.)_{R'}$. Hence R' has no $\mathrm{U}(p,q)$ invariant inner product. $\square$

8.4. Proof of Theorem 8.1.1(C). Note that under the given conditions, every element λ of $\Lambda^+(p,\mu)$ are such that

$$\lambda_j \begin{cases} \geq \mu_j & 1 \leq j \leq a+q-1 \\ = m & a+q \leq j \leq b \\ \leq \mu_{j-q} & b+1 \leq j \leq p \end{cases}$$

Take $\eta = (\mu_1, ..., \mu_{a-1}, m, ..., m, \mu_{b+1}, ..., \mu_{p-q})$, and for each $\lambda \in \Lambda^+(p,\mu)$, define c_λ by (26). Then one can check that $c_\lambda > 0$ for all $\lambda \in \Lambda^+(p,\mu)$. Thus by Lemma 8.3.1, $I_{p,q}(s,\sigma,\mu)$ is unitarizable. $\square$

8.5. We need the following lemma to prove Part (D) of the theorem.

Lemma 8.5.1. *Suppose that $s < 0$ and $I_{p,q}(s,\sigma,\mu)$ is irreducible. If for some j, there is an integer m and $\lambda \in \Lambda^+(p,\mu)$ such that*

(i) $-\beta - p + j - 1 < m < \alpha + j - 1$.
(ii) $\lambda_j = m$.
(iii) $\lambda + \varepsilon_j \in \Lambda^+(p,\mu)$.

Then $I_{p,q}(s,\sigma,\mu)$ is not unitarizable.

Proof. Under the given conditions, $\alpha - \lambda_j + j - 1 > 0$ and $\beta + \lambda_j + p - j + 1 > 0$, so that $-c_\lambda / c_{\lambda+\varepsilon_j} > 0$. $\square$

Remark. Note that if α and β in the previous lemma are integers, and m is either equal to $\alpha + j - 1$ or $-\beta - p + j - 1$, then $I_{p,q}(s,\sigma,\mu)$ is reducible and not completely reducible. Consequently $I_{p,q}(s,\sigma,\mu)$ is not unitarizable in this case.

8.6. Proof of Theorem 8.1.1(D). First we consider the case when s is not real and $\mathrm{Re}(s) \neq 0$. Note that for any $\eta \in \Lambda^+(p-1)$, $I_{p,1}(s,\sigma,\eta)$ is $\mathrm{U}(p) \times \mathrm{U}(1)$ multiplicity free. So the methods used in §9 of [Le] can be used to show that $I_{p,1}(s,\sigma,\eta)$ is not unitarizable. Now for any $\mu \in \Lambda^+(p-q)$, there exists $\eta \in \Lambda^+(p-1)$ such that $I_{p,1}(s,\sigma,\eta)$ is embedded into $I_{p,q}(s,\sigma,\mu)$. Hence $I_{p,q}(s,\sigma,\mu)$ is not unitarizable.

We now assume that $I_{p,q}(s,\sigma,\mu)$ is unitarizable and does not belong to the cases described in Parts (A), (B) and (C). Then s is real, and either $\sigma \equiv p (\mathrm{mod}\ 2)$ or

$|s| > 1$. If $p < 2q$, then $\{\lambda_{p-q+1} : \lambda \in \Lambda^+(p,\mu)\} = \mathbb{Z}$. So by Lemma 8.5.1 with $j = p - q + 1$, $I_{p,q}(s,\sigma,\mu)$ is not unitarizable. Thus $p \geq 2q$.

If $I_{p,q}(s,\sigma,\mu)$ is reducible and not unitarily induced, then by Theorem 7.3.1, $I_{p,q}(s,\sigma,\mu)$ is not completely reducible, and hence not unitarizable. So $I_{p,q}(s,\sigma,\mu)$ is irreducible.

By duality, we may assume that $s < 0$, or equivalently, $\alpha+\beta+p > 0$. For $1 \leq j \leq p$, let $A_j = \alpha + j - 1$ and $B_j = -\beta - p + j - 1$. Recall that in the case $-1 < s < 0$, we are assuming $\sigma \equiv p (\text{mod } 2)$, so that the midpoint of the interval $[B_j, A_j]$ which is given by $(\sigma - p)/2 + (j - 1)$ is an integer. Thus $[A_j]$ is always contained in $[B_j, A_j]$. Here $[x]$ denotes the greatest integer less than or equal to x.

We claim that $A_q < \mu_q$. Otherwise by applying Lemma 8.5.1 with $m = [A_q]$, $I_{p,q}(s,\sigma,\mu)$ is not unitarizable. We now consider two cases.

Case 1: Suppose there exists $q + 1 \leq i \leq p - q$ such that $A_i \geq \mu_i$. Let j be the smallest integer i such that $A_i \geq \mu_i$. Then since $A_j = A_{j-1} + 1 < \mu_{j-1} + 1$, $[A_j] \leq \mu_{j-1} \leq \mu_{j-q}$. If A_j is an integer and either $A_j < \mu_{j-1}$ or $\mu_{j-1} < \mu_{j-q}$, then $I_{p,q}(s,\sigma,\mu)$ is reducible. On the other hand, if A_j is not an integer and not all of $[A_j]$, μ_{j-1} and μ_{j-q} are equal, then applying Lemma 8.5.1 with $m = [A_j]$ contradicts unitarity. Thus we must have $[A_j] = \mu_{j-1} = \mu_{j-q}$.

If (B_j, A_j) contains exactly one element, then we are done. If (B_j, A_j) contains k integers with $k \geq 2$ and $\mu_j < \mu_{j-q}$, then again by applying Lemma 8.5.1 with $m = [A_j] - 1$, $I_{p,q}(s,\sigma,\mu)$ is not unitarizable. So $\mu_j = \mu_{j-q}$. By the same reasoning, we must have $\mu_j = \mu_{j+1} = \cdots = \mu_{j+k-1} = [A_j]$. In particular, $j + k - 1 \leq p - q$.

Case 2: Suppose that $A_{p-q} < \mu_{p-q}$. Then $A_{p-q+1} < \mu_{p-q} + 1 \leq \mu_{p-2q+1} + 1$, which together with Lemma 8.5.1 imply that $[A_{p-q+1}] = \mu_{p-q} = \mu_{p-2q+1}$. Moreover, (B_{p-q+1}, A_{p-q+1}) contains exactly one integer.

8.7. Proof of Theorem 8.1.1(E)(b). We shall only prove (i). Recall that $\Gamma^+(R')$ is the set of all $\lambda \in \Lambda(R_{a(r,t)})$ such that $S_\lambda \neq 0$. First we note that $\Lambda^+(R')$ is nonempty as

$$(\mu_1, \mu_2, ..., \mu_{p-q}, \mu_{p-q}, ..., \mu_{p-q}) \in \Lambda^+(R').$$

We need to show that the numbers $\{c_\lambda\}_{\lambda \in S}$ have the same sign. Note that since $\alpha + \beta \leq r + t$, we have $-\beta + j - 1 \geq \alpha - p + j$ for $1 \leq j \leq p$, so for $\lambda \in \Lambda^+(p)$,

(i) $\lambda_j \geq -\beta + j$ or $\lambda \leq \alpha - p + j - 2$ implies that $c_\lambda / c_{\lambda+\varepsilon_j} > 0$.

(ii) $\alpha - p + j \leq \lambda_j \leq -\beta + j - 2$ implies that $c_\lambda / c_{\lambda+\varepsilon_j} < 0$.

Now if $\lambda \in S$, then $\mu_j \leq \lambda_j \leq -\beta + s$ for $s + 1 \leq j \leq p - t$ and $\lambda_{p-t+1} \leq \alpha - t$. By (b), $\lambda_j = -\beta = s$ for $s + 1 \leq j \leq p - t$. In other words, for $s + 1 \leq j \leq p - t$ there is no transitions of the form $\lambda \to \lambda + \varepsilon_j$ in $R'_{a(r,t)}$. On the other hand, $\lambda_s \geq -\beta + s$, so that $\lambda_j \geq -\beta + s \geq -\beta + j$ for $1 \leq j \leq s$, and $\lambda_{p-t+1} \leq \alpha + t$, so that $\lambda_j \leq \alpha + t \leq \alpha - p + j$ for all $p - t + 1 \leq j \leq p$. Thus $c_\lambda / c_{\lambda+\varepsilon_j} > 0$ for all $\lambda \in \Lambda^+(R')$ such that $\lambda + \varepsilon_j \in \Lambda^+(R')$. □

8.8. Proof of Theorem 8.1.1(E)(c) Suppose $R'_{a(r,t)}$ is unitarizable and does not belong to any of the cases given in Part (E)(a) of the theorem. First we note that if

$\lambda \in \Lambda^+(R'_{a(r,t)})$, then $-(\beta+t) \le \lambda_j \le \alpha + r$ for $s+1 \le j \le p-t$. If $\mu_{r+1} > \alpha + r$, then $R'_{a(r,t)} = 0$. So we must have $\mu_{r+1} \le \alpha + r$. It is clear that to have unitarity, there are only three possibilities: $\mu_{r+1} = \alpha + r$, $\mu_{r+1} = \mu_{r+1-q}$ or $\mu_{r+1-q} = -(\beta+t)$. If $\mu_{r+1} = \alpha+r$, then $\mu_j = \alpha+r$ for $r+1 \le j \le p-t$. The other cases are similar. $\square$

Part 2. **The degenerate principal series of** $\mathrm{Spin}_0(p,q)$

In this part, we shall use the methods in Part 1 to study a similar family of degenerate principal series representations of $\mathrm{Spin}_0(p,q)$. Since the ideas and proofs are very similar to Part 1, we will only state the main results and omit most of the proofs.

9. The representations

9.1. Recall that $\mathrm{SO}(p,p)$ is the group of $2p \times 2p$ real matrices of determinant 1 which fixes the symmetric form on $V = \mathbb{R}^{2p}$:

$$\langle (x_1, ..., x_{2p}), (y_1, ..., y_{2p}) \rangle = x_1y_1 + \cdots + x_py_p - (x_{p+1}y_{p+1} + \cdots + x_{2p}y_{2p}).$$

Let $\{e_1, ..., e_{2p}\}$ be the standard basis of $\mathbb{R}^{2p}$. Let $q < p$ and set $T = \{1, \ldots, p\} \cup \{2p-q+1, \ldots, 2p\}$. Let V_T and V'_T be the span of $\{e_i : i \in T\}$ and $\{e_i : i \notin T\}$ respectively. Hence $V = V_T \oplus V'_T$. We shall identify $\mathrm{SO}(p,q)$ and $\mathrm{SO}(p-q)$ with the following subgroups of $\mathrm{SO}(p,p)$:

$$\begin{aligned} \mathrm{SO}(p,q) &\cong \{g \in \mathrm{SO}(p,p) : \ g|_{V'_T} = \mathrm{id}\}, \\ \mathrm{SO}(p-q) &\cong \{g \in \mathrm{SO}(p,p) : \ g|_{V_T} = \mathrm{id}\}. \end{aligned}$$

Let $\mathrm{SO}_0(p,q)$ denote the connected component of $\mathrm{SO}(p,q)$. $\mathrm{SO}_0(p,q)$ exhibits a double cover $\mathrm{Spin}_0(p,q)$. Set $\tilde{G} = \mathrm{Spin}_0(p,p)$, $G = \mathrm{Spin}_0(p,q)$ and $H = \mathrm{Spin}(p-q)$. Note that $\tilde{G}$ contains

$$G \times_{\mathbb{Z}/2\mathbb{Z}} H := (G \times H)/\{(x,x) : x \in \mathbb{Z}/2\mathbb{Z}\}.$$

9.2. Let $\mathfrak{gl}_{2p}(\mathbb{C})$ be the Lie algebra consisting of all $2p$ by $2p$ complex matrices. Then the complexified Lie algebra $\mathfrak{so}(p,p)_{\mathbb{C}}$ of $\tilde{G}$ can be identified with the subalgebra of skew symmetric matrices in $\mathfrak{gl}_{2p}(\mathbb{C})$. Let $E_{ij} = -E_{ji} \in \mathfrak{so}(p,p)_{\mathbb{C}}$ denote the matrix which is 1 at the (i,j)-th entry, -1 at (j,i)-th entry and 0 elsewhere. Then the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ of G and H are given by

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(p,q)_{\mathbb{C}} &= \mathrm{Span}\{E_{ij} : i,j \in T\} \qquad (28) \\ \mathfrak{h}_{\mathbb{C}} = \mathfrak{so}(p-q)_{\mathbb{C}} &= \mathrm{Span}\{E_{ij} : p+1 \le i,j \le 2p-q\}. \end{aligned}$$

9.3. Let P_0 be the stabilizer of the span of $\{e_1 + e_{p+1}, \ldots, e_p + e_{2p}\}$ in $\mathrm{SO}_0(p,p)$. It is a maximal parabolic subgroup with Levi subgroup $\mathrm{GL}_p^+(\mathbb{R})$. The intersection $P_0 \cap \mathrm{SO}_0(p,q)$ is a maximal parabolic subgroup of $\mathrm{SO}_0(p,q)$ with Levi subgroup $\mathrm{GL}_q^+(\mathbb{R}) \times \mathrm{SO}(p-q)$. Let P_1 denote the double cover of $P_0 \cap \mathrm{SO}_0(p,q)$ in G. Similarly let $\tilde{P}$ denote the double cover of P_0 in $\tilde{G}$. The Levi subgroups of P_1 and $\tilde{P}$ are respectively

$$\begin{aligned} \mathrm{SL}_q(\mathbb{R}) \times (\mathbb{R}^* \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(p-q)) &\simeq \mathrm{GL}_q^+(\mathbb{R}) \times \mathrm{Spin}(p-q) \ \text{ and} \\ \mathrm{SL}_q(\mathbb{R}) \times \mathbb{R}^* &\simeq \mathrm{GL}_p^+(\mathbb{R}) \times \{\pm 1\}. \end{aligned}$$

The modular function of P_1 (resp. $\tilde{P}$) is $\delta(xy) = (\det x)^{p-1}$ where $x \in \mathrm{GL}_q^+(\mathbb{R})$, and $y \in \mathrm{Spin}(p-q)$ (resp. $y \in \{\pm 1\}$).

9.4. For $s \in \mathbb{C}$ and $\sigma \in \{0,1\}$, let $\chi_{s,\sigma}$ denote the 1 dimensional character of the Levi subgroup of $\tilde{P}$ defined by

$$\chi_{s,\sigma}(xy) = (\det x)^s y^\sigma$$

where $x \in \mathrm{GL}_p^+(\mathbb{R}), y \in \{\pm 1\}$. Let $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}} \chi_{s,\sigma}$ denote the corresponding (normalized) induced representation of $\tilde{G}$ (see §1.11 for its definition). It will descend to a representation of $\mathrm{SO}_0(p,p)$ if and only if $\sigma = 0$. Let $I_p(s,\sigma)$ denote its Harish-Chandra module. The module structure and unitarity of $I_p(s,\sigma)$ has been determined by Johnson ([J1]) and Sahi ([S2]).

9.5. For $r \in \mathbb{Z}^+$ and $\sigma \in \{0,1\}$, define

$$\begin{aligned} \Lambda^\sigma(2r+1) &= \{(\lambda_1, \lambda_2, \ldots, \lambda_r) : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0,\ \lambda_i - \tfrac{\sigma}{2} \in \mathbb{Z}\ \forall i\} \\ \Lambda^\sigma(2r) &= \{(\lambda_1, \lambda_2, \ldots, \lambda_r) : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{r-1} \geq |\lambda_r|,\ \lambda_i - \tfrac{\sigma}{2} \in \mathbb{Z}\ \forall i\} \end{aligned}$$

Hence $\Lambda^0(p)$ (resp. $\Lambda^1(p)$) is the set of highest weights of irreducible representations of $\mathrm{Spin}(p)$ which descend (resp. do not descend) to representations of $\mathrm{SO}(p)$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda^\sigma(p)$, let τ_p^λ be the irreducible representation of $\mathrm{Spin}(p)$ with highest weight λ. We remark that τ_p^λ is a self dual representation.

9.6. Let $L_1 = \mathrm{GL}_q^+(\mathbb{R}) \times \mathrm{Spin}(p-q)$ denote the Levi subgroup of $P_1 \subseteq G$. For $s \in \mathbb{C}$ and $x \in \mathrm{GL}_q^+(\mathbb{R})$, let

$$\chi_s(x) = (\det x)^s.$$

Let $\mu \in \Lambda^\sigma(p-q)$ and let

$$\pi_{s,\sigma,\mu} = \chi_s \boxtimes \tau_{p-q}^\mu.$$

Let $\mathrm{Ind}_{P_1}^G \pi_{s,\sigma,\mu}$ denote the corresponding (normalized) induced representation of G (cf. 1.11). It will descend to a representation of $\mathrm{SO}_0(p,q)$ if and only if τ_{p-q}^μ descends to a representation of $\mathrm{SO}(p-q)$. Let $I_{p,q}(s,\sigma,\mu)$ denote its Harish-Chandra module. The purpose of Part 2 is to determine the module structure and unitarity of $I_{p,q}(s,\sigma,\mu)$.

9.7. Let $r_1 = [(p-q)/2]$. The infinitesimal character of $I_{p,q}(s,\sigma,\mu)$ is given by

$$(s + \tfrac{q-1}{2}, s + \tfrac{q-3}{2}, \ldots, s - \tfrac{q-1}{2}, \mu_1 + \tfrac{p-q-2}{2}, \mu_2 + \tfrac{p-q-4}{2}, \ldots, \mu_{r_1} + \tfrac{p-q}{2} - r_1) \tag{29}$$

and it is defined up to an action of the Weyl group. We also note that $I_{p,q}(-s,\sigma,\mu)$ is the dual representation of $I_{p,q}(s,\sigma,\mu)$.

9.8. Let

$$\tilde{K} = \mathrm{Spin}(p) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(p), \quad K = \mathrm{Spin}(p) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(q).$$

Then $\tilde{K}$ and K are the maximal compact subgroups of $\tilde{G} = \mathrm{Spin}_0(p,p)$ and $G = \mathrm{Spin}_0(p,q)$ respectively. The following are obtained by straightforward applications of Frobenius reciprocity:

(i) Under the action of $\tilde{K}$,

$$I_p(s,\sigma) = \sum_{\lambda \in \Lambda^\sigma(p)} \tau_p^\lambda \boxtimes \tau_p^\lambda. \tag{30}$$

(ii) The K-type $\tau_p^\lambda \boxtimes \tau_q^\eta$ occurs in $I_{p,q}(s,\sigma,\mu)$ with multiplicity

$$\dim \mathrm{Hom}_{\mathrm{Spin}(q)\times\mathrm{Spin}(p-q)}(\tau_q^\eta \boxtimes \tau_{p-q}^\mu, \tau_p^\lambda). \tag{31}$$

Note that the multiplicity is nonzero only if $\lambda \in \Lambda^\sigma(p)$ and $\eta \in \Lambda^\sigma(q)$.

9.9. The following proposition relates $I_p(s,\sigma)$ with $I_{p,q}(s,\sigma,\mu)$ and it is analogous to Propositions 3.4.1 in Part 1.

Proposition 9.9.1. *Let $s \in \mathbb{C}$ and $\sigma \in \{0,1\}$. Then we have*

$$\mathrm{Res}^{\tilde{G}}_{G\times H} I_p(s,\sigma) = \sum_{\mu \in \Lambda^\sigma(p-q)} I_{p,q}(s,\sigma,\mu) \boxtimes \tau_{p-q}^\mu. \tag{32}$$

Note that each $I_{p,q}(s,\sigma,\mu)$ on the right hand side of (32) has a distinct infinitesimal character.

Proposition 9.9.1 is the starting point of our investigation on the submodules of $I_{p,q}(s,\sigma,\mu)$ and their unitarity. In §10 we will define a basis of $I_{p,q}(s,\sigma,\mu)$ using the Gelfand-Zeltin basis. This sets up the notation so that we can compute the transition maps $T_{\lambda,\lambda'}$ in §11. The arguments are parallel to that of Part 1 and we will omit most of the details. Readers who are only interested in the statements of the main results may skip the next two sections and proceed to §12 and §13 directly. We choose to keep §10 and 11 for two reasons: Firstly if p is odd, $T_{\lambda,\lambda}$ may not be zero. This differs from Part 1 and it affects the final theorem on the unitarily. We feel that this case should be treated in detail. The second reason is that the transition coefficients carry more information than the reducibility of $I_{p,q}(s,\sigma,\mu)$ and it is useful to write them down explicitly.

10. A basis of $I_{p,q}(s,\sigma,\mu)$

10.1. In this section, we shall construct a basis of $I_{p,q} := I_{p,q}(s,\sigma,\mu)$ using the Gelfand-Zetlin (GZ) basis for an irreducible representation of $\mathrm{Spin}(p)$.

We identify the real Lie algebra $\mathfrak{so}(p)$ of $\mathrm{Spin}(p)$ as the subspace of skew symmetric p by p real matrices. Suppose $i \neq j$ and we let $E_{ij} = -E_{ji} \in \mathfrak{so}(p)$ denote the matrix which is 1 at the (i,j)-th entry, -1 at (j,i)-th entry and 0 elsewhere. Let $\mathfrak{so}(t)$ be the span of $\{E_{ij} : 1 \leq i,j \leq t\}$. We assume that the inclusion $\mathrm{Spin}(t) \subseteq \mathrm{Spin}(t+1)$ induces the obvious embedding of matrices $\mathfrak{so}(t) \subseteq \mathfrak{so}(t+1)$.

Let τ_p^λ be an irreducible representation of $\mathrm{Spin}(p)$ with highest weight $\lambda \in \Lambda^\sigma(p)$. With respect to the embedding $\mathrm{Spin}(t) \subseteq \mathrm{Spin}(t+1)$ above, τ_p^λ has a Gelfand-Zetlin

(GZ) basis where each vector $[m_{kl}]$ is represented up to scalars where $(m_{1l} \geq m_{2l} \geq \ldots \geq m_{[l/2],l}) \in \Lambda^\sigma(l)$ [GZ2]. If $p = 2r+1$ is odd then

$$[m_{kl}] = \begin{bmatrix} m_{1p} & m_{2p} & \cdots & m_{r-1,p} & m_{rp} \\ m_{1,p-1} & m_{2,p-1} & \cdots & m_{r-1,p-1} & m_{r,p-1} \\ m_{1,p-2} & m_{2,p-2} & \cdots & m_{r-2,p-2} & m_{r-1,p-2} \\ & \cdots & \cdots & \cdots & \\ & & m_{13} & & \\ & & m_{12} & & \end{bmatrix} \tag{33}$$

If $p = 2r$ is even then

$$[m_{kl}] = \begin{bmatrix} m_{1p} & m_{2p} & \cdots & m_{r-1,p} & m_{rp} \\ m_{1,p-1} & m_{2,p-1} & \cdots & m_{r-2,p-1} & m_{r-1,p-1} \\ m_{1,p-2} & m_{2,p-2} & \cdots & m_{r-2,p-2} & m_{r-1,p-2} \\ & \cdots & \cdots & \cdots & \\ & & m_{13} & & \\ & & m_{12} & & \end{bmatrix} \tag{34}$$

In addition m_{kl} satisfies

$$m_{kl} \geq m_{k,l-1} \geq m_{k+1,l}, \quad m_{l-1,2l-1} \geq |m_{l,2l}|, \quad m_{l,2l+1} \geq |m_{l,2l}|. \tag{35}$$

An explicit formula for Lie algebra action of $E_{t+1,t}$ on a normalized GZ basis vector is given in [GZ2].

10.2. By (30), the set $\mathcal{B}$ consisting of all vectors of the form

$$[m_{kl}] \otimes [m_{k'l'}]$$

where $[m_{kl}]$ and $[m_{k'l'}]$ are normalized GZ basis vectors of τ_p^λ and $\lambda \in \Lambda^\sigma(p)$ is a basis of $I_p(s,\sigma)$.

10.3. We now fix $\mu \in \Lambda^\sigma(p-q)$ and consider $I_{p,q} = I_{p,q}(s,\sigma,\mu)$. We also fix a GZ basis vector v_0 in τ^μ_{p-q}. By (32), $I_{p,q} \boxtimes \tau^\mu_{p-q} \subseteq I_p(s,\sigma)$ so that we may identify $I_{p,q}$ with the subspace $I_{p,q} \boxtimes v_0$ of $I_p(s,\sigma)$. If $u = [m_{kl}]$ is a GZ basis vector in an irreducible representation of $\mathrm{Spin}(p)$, then $d_{p-q}(u)$ will denote the GZ basis vector for the group $\mathrm{Spin}(q)$ obtained by deleting the top $p-q$ rows from $[m_{kl}]$. We now define

$$\mathcal{B}(v_0) := \{[m_{kl}] \otimes [m_{k'l'}] \in \mathcal{B} : d_{p-q}([m_{k'l'}]) = v_0\}.$$

Then $\mathcal{B}(v_0)$ forms a basis of $I_{p,q}(s,\sigma,\mu) \boxtimes v_0$. Furthermore we define

$$\begin{aligned} J_\lambda &:= \mathrm{Span}\{[m_{ij}] \in \tau_p^\lambda : d_{p-q}([m_{ij}]) = v_0\} \\ S_\lambda &:= \tau_p^\lambda \boxtimes J_\lambda. \end{aligned} \tag{36}$$

Hence $I_{p,q} = \sum_\lambda S_\lambda$. The following lemma can be deduced easily from the GZ bases and we will need it in §12.2.

Lemma 10.3.1. *The following statements are equivalent:*

(i) $S_\lambda \neq 0$.
(ii) $\tau_p = \tau_p^\lambda$ *contains* $\tau_{p-q} = \tau^\mu_{p-q}$.

(iii) $\mu_i - \lambda_j \in \mathbb{Z}$ *and* $|\lambda_i| \geq |\mu_i| \geq |\lambda_{i+q}|$ *for* $i = 1, \ldots, [\frac{p-q}{2}]$. *Here we formally define* $\lambda_{i+q} = 0$ *if* $i + q > r$.

10.4. With the help of Proposition 9.9.1 and the basis $\mathcal{B}(v_0)$ of $I_{p,q}(s,\sigma,\mu)$ constructed in §10.3, we can deduce statements analogous to Propositions 3.5.1, 3.5.2 and 4.4.1 in Part 1. We will leave the precise formulation for the reader.

Let W be a submodule of $I_{p,q}$. Define

$$\Lambda(W) = \{\lambda \in \Lambda^\sigma(p) : S_\lambda \cap W \neq 0\} \tag{37}$$

Then under the action of K, $W = \sum_{\lambda\in\Lambda(W)} S_\lambda$. The following lemma is the main result of this section (cf. Lemma 5.2.1)

Lemma 10.4.1. *If $W_1 \subseteq W_2$ be infinitesimal G-submodules of $I_{p,q}$, then as a representation of K,*

$$W_2/W_1 = \sum_{\lambda\in\Lambda(W_1)-\Lambda(W_2)} S_\lambda. \quad \square$$

11. Transition coefficients.

11.1. In this section, we determine how the Lie algebra of $\mathrm{Spin}(p,q)$ transforms the $\mathrm{Spin}(p)$-isotypic components S_λ in $I_{p,q}(s,\sigma,\mu)$.

11.2. Let $\lambda \in \Lambda^\sigma(p)$. Recall that

$$\mathbb{C}^p \otimes \tau_p^\lambda = \sum_{j=1}^p \tau_p^{\lambda+\varepsilon_j} + \sum_{j=1}^p \tau_p^{\lambda-\varepsilon_j} + \gamma_1 \tau_p^\lambda \tag{38}$$

where $\gamma_1 = 0, 1$ and it is 1 if and only if $p = 2r+1$ is odd and $\lambda_r \neq 0$. Note that if $\lambda \pm \varepsilon_j$ is not a dominant weight, then we set $\tau_p^{\lambda\pm\varepsilon_j} = 0$.

11.3. Let

$$\begin{aligned} \mathfrak{so}(p,p)_\mathbb{C} &= \mathfrak{so}(p)_\mathbb{C} \oplus \mathfrak{so}(p)_\mathbb{C} \oplus \widetilde{\mathfrak{p}}_\mathbb{C} \\ \mathfrak{so}(p,q)_\mathbb{C} &= \mathfrak{so}(p)_\mathbb{C} \oplus \mathfrak{so}(q)_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C} \end{aligned}$$

be the Cartan decompositions. We remark that

$$\widetilde{\mathfrak{p}}_\mathbb{C} \simeq \mathbb{C}^p \boxtimes \mathbb{C}^p, \quad \mathfrak{p}_\mathbb{C} \simeq \mathbb{C}^p \boxtimes \mathbb{C}^q \tag{39}$$

as representations of $\tilde{K}$ and K respectively. Fix a $\tilde{K}$-type $V_\lambda = \tau_p^\lambda \boxtimes \tau_p^\lambda$ of $I_p(s,\sigma)$ and consider the $\tilde{K}$-module map $L : \mathfrak{p}_\mathbb{C} \otimes V_\lambda \to I_p(s,\sigma)$ given by

$$L(X \otimes u) = X.u \quad (X \in \mathfrak{p}_\mathbb{C}, u \in V_\lambda).$$

Let $p_{\lambda'} : I_p(s,\sigma) \to V_{\lambda'}$ denote the projection map onto the $\tilde{K}$-type $V_{\lambda'}$. We define $T_{\lambda,\lambda'} = p_{\lambda'} \circ L$ and let $T'_{\lambda,\lambda'} : \mathfrak{p}_\mathbb{C} \otimes S_\lambda \to S_{\lambda'}$ denote its restriction to $\mathfrak{p}_\mathbb{C} \otimes S_\lambda$. $T'_{\lambda,\lambda'}$ essentially describes the Lie algebra action of $\mathfrak{p}$ on S_λ. Since $T_{\lambda,\lambda'}$ is a $\tilde{K}$ homomorphism, it is easy to see by (38) that $T_{\lambda,\lambda'} \neq 0$ if and only if λ' is one of the following:

(i) $\lambda' = \lambda + \varepsilon_j$ or $\lambda - \varepsilon_j$ and it is a dominant weight.
(ii) $p = 2r+1$ is odd, $\lambda_r > 0$ and $\lambda' = \lambda$.

Proposition 11.3.1. ([J1]) *Let* $\alpha = -s - (p-1)/2$.

(i) $T_{\lambda,\lambda+\varepsilon_j} = 0$ *if and only if* $\lambda_j = \alpha + j - 1 = -s - \frac{p-1}{2} + j - 1$.

(ii) $T_{\lambda,\lambda-\varepsilon_j} = 0$ *if and only if* $\lambda_j = -\alpha + j - p + 1 = s + j - \frac{p-1}{2}$.

(iii) *If* $p = 2r+1$ *is odd, then* $T_{\lambda,\lambda} = 0$ *if and only if* $s\lambda_r = 0$.

The next lemma deduces $T'_{\lambda,\lambda'}$ from $T_{\lambda,\lambda'}$.

Lemma 11.3.2. *Suppose* $\lambda' = \lambda + \epsilon_j$ *or* $\lambda - \epsilon_i$ *and suppose* S_λ *and* $S_{(\lambda')}$ *are nonzero. Then* $T_{\lambda,\lambda'} \neq 0$ *if and only if* $T'_{\lambda,\lambda'} \neq 0$.

Proof. We will give a sketch. First we modify (19) using (39) and (38). The proof then reduces to a lemma similar to Lemma 6.3.1. ☐

Finally combining the above lemma with Proposition 11.3.1 allows us to deduce the following proposition (cf. Proposition 6.3.2).

Proposition 11.3.3. *Suppose the* τ_p^λ*-isotypic component* S_λ *in* $I_{p,q}(s,\sigma,\mu)$ *is nonzero, and* W *is the infinitesimal* G*-submodule generated by* S_λ. *Let* $1 \leq j \leq r$.

(i) *If* $S_{\lambda+\varepsilon_j} \neq 0$, *then* W *contains* $S_{\lambda+\varepsilon_j}$ *if and only if* $\lambda_j \neq \alpha + j - 1$.

(ii) *If* $S_{\lambda-\varepsilon_j} \neq 0$, *then* W *contains* $S_{\lambda-\varepsilon_j}$ *if and only if* $\lambda_j \neq -\alpha - p + j + 1$. ☐

11.4. We note that $T_{\lambda,\lambda}$ and $T'_{\lambda,\lambda}$ may be nonzero when p is odd. This is a major difference from Part 1. It has no effect on the module structure of $I_{p,q}(s,\sigma,\mu)$ but we will see in §13.3 that it severely affect the unitarity of $I_{p,q}(s,\sigma,\mu)$.

Lemma 11.4.1. *Suppose* $p = 2r+1$ *is odd and* $S_\lambda \neq 0$. *Suppose* $s\lambda_r \neq 0$ *so that* $T_{\lambda,\lambda} \neq 0$ *by Proposition 11.3.1(iii). Then* $T'_{\lambda,\lambda} = 0$ *if and only if* $q \leq r$ *and* $\mu_{1+r-q} = 0$.

Proof. We will only give a sketch. Let pr denote the composite of the following map:

$$\mathrm{pr} : \mathbb{C}^q \otimes J_\lambda \hookrightarrow \mathbb{C}^p \otimes \tau_p^\lambda \stackrel{\mathrm{proj}}{\to} \tau_p^\lambda$$

where proj denote the canonical projection in (38) and J_λ was defined in (36).

The first half of the proof is similar to Lemmas 6.3.1 and (19) by reducing the statement to the fact that $T'_{\lambda,\lambda} = 0$ if and only if $\mathrm{pr} = 0$.

To determine if $\mathrm{pr} = 0$, we modify the proof of Lemma 6.3.1 in the following manner: First we interpret pr as the Lie algebra action of $\mathfrak{so}(p+1)_\mathbb{C}$ on some irreducible representation τ_{p+1} containing τ_p^λ (with multiplicity one). One can show that $\mathrm{pr} = 0$ if and only if $E_{p,p-1}J_\lambda$ in τ_{p+1} has a trivial projection onto the subspace τ_p^λ. By the explicit action of $E_{p,p-1}$ in [GZ2], the latter condition holds if and only if $m_{r-1,p-1} = 0$ for all GZ basis vector $[m_{kl}]$ in J_λ. The last condition is equivalent to $\mu_{1+r-q} = 0$. ☐

12. Module structures

12.1. In this section, we will determine the reducibility of $I_{p,q}(s,\sigma,\mu)$ and describe all its irreducible subquotients when it is reducible. First we review the result of [J1] on the structure of $I_p(s,\sigma)$. Let $\alpha = -s - \frac{p-1}{2}$. Then $I_p(s,\sigma)$ is irreducible if and only if either one of the following conditions holds:

(i) $s + \frac{p-1+\sigma}{2} = -\alpha + \frac{\sigma}{2} \notin \mathbb{Z}$

(ii) p is odd and $|s| = \frac{\sigma}{2}$.

This is an immediate consequence of Proposition 11.3.1.

Next we shall describe the subquotients of $I_p(s,\sigma)$ when it is reducible. Since $I_p(-s,\sigma)$ is isomorphic to the dual module of $I_p(s,\sigma)$, it is sufficient to consider the case when $s \le 0$. Let $t = \max(0, -\alpha + \frac{\sigma}{2})$, then there exists a filtration of submodules

$$0 = W_{t-1} \subsetneq W_t \subsetneq W_{t+1} \subsetneq \ldots \subsetneq W_r = I_p(s,\sigma)$$

such that for each i, the quotient $R_i := W_i/W_{i-1}$ has K-types given by,

$$R_i = \sum_{\lambda} \tau_p^\lambda \boxtimes \tau_p^\lambda$$

where the sum is taken over all $\lambda \in \Lambda^\sigma(p)$ such that $|\lambda_i| \ge \alpha + i \ge |\lambda_{i+1}|$. R_i is irreducible except when $p = 2r$ is even and $i = r$. If $p = 2r$ is even and $i = r$, then $W_r = W_r^+ + W_r^-$ for some submodules $W_r^\pm$ of $I_p(s,\sigma)$ containing W_{r-1}. Define $R_r^\pm := W_r^\pm/W_{r-1}$. Then $R_r = R_r^+ \oplus R_r^-$, and R_r^+ (resp. R_r^-) is irreducible and it has K-types

$$\sum_{\lambda} \tau_p^\lambda \boxtimes \tau_p^\lambda$$

where the sum is taken over all $\lambda \in \Lambda^\sigma(p)$ such that $\lambda_r \ge \alpha + r$ (resp. $-\lambda_r \ge \alpha + r$). The module diagram $\mathcal{G}$ of I_p is as follows:

FIGURE 4 $p = 2r+1$ $p = 2r$

R_0 is nonzero if and only if $\alpha \ge \frac{\sigma}{2}$. In this case R_0 is a finite dimensional representation of $\tilde{G} = \mathrm{Spin}_0(p,p)$ with highest weight $(\alpha, \alpha, \ldots, \alpha) \in \Lambda^\sigma(2p)$.

12.2. We define

$$\begin{aligned} R_i' &:= (W_i \cap I_{p,q})/(W_{i-1} \cap I_{p,q}) \subset R_i \\ (R_r^\pm)' &:= (W_r^\pm \cap I_{p,q})/(W_{r-1} \cap I_{p,q}) \subset R_r^\pm. \end{aligned} \tag{40}$$

These are infinitesimal G-submodules. By Lemma 10.3.1 $R_i' \ne 0$ if and only if

$$|\mu_{i+1}| \le \alpha + i \le |\mu_{i-q}|. \tag{41}$$

Here we assume that $\mu_i = \infty$ if $i < 0$ and $\mu_i = 0$ if $i > r_1$. $(R_r^+)' \ne 0$ if and only if $(R_r^-)' \ne 0$. The following theorem describes the structure of $I_{p,q}(s,\sigma,\mu)$. Its proof is similar to that of Theorem 7.3.1 and we will omit it. Note that we have omitted the case when $s > 0$. This is because $I_{p,q}(-s,\sigma,\mu)$ is isomorphic to the dual module of $I_{p,q}(s,\sigma,\mu)$.

Theorem 12.2.1. *Suppose $s \in \mathbb{C}$ and let $I_{p,q} = I_{p,q}(s,\sigma,\mu)$ where $\mu \in \Lambda^\sigma(p-q)$.*

(A) (a) *If $s + \frac{p-1+\sigma}{2} \notin \mathbb{Z}$, then $I_{p,q}$ is irreducible.*

(b) *If $s + \frac{p-1+\sigma}{2} \in \mathbb{Z}$ and $s \le 0$, then $I_{p,q}$ is irreducible if and only if either the one of the following is true:*

(i) *$p = 2r+1$ is odd and $(s,\sigma) = (0,0)$ or $(-\frac{1}{2}, 1)$.*

(ii) *There exists* i *such that* $q \leq i \leq [\frac{p-q}{2}]$ *and*

$$|\mu_i| = |\mu_{i-1}| = \ldots = |\mu_{i-q+1}| = -s - \frac{p-1}{2} + i \geq 0.$$

(B) *Suppose* $s \leq 0$ *and* $I_{p,q}$ *is reducible.*

(i) *Then* R_i' *(resp* $(R_r^{\pm})'$*) is either zero or it is an irreducible subquotient of* $I_{p,q}$*.*

(ii) *Every irreducible subquotient of a reducible* $I_{p,q}$ *is of the form* R_i' *or* $(R_r^{\pm})'$*.*

(iii) *Let* $\mathcal{G}'$ *and* $\mathcal{G}'$ *be the module diagrams of* $I_{p,q}$ *and* $I_p(s,\sigma)$ *respectively. Then* $\mathcal{G}'$ *is a spanning subgraph of* $\mathcal{G}$*. It is obtained by deleting those vertices* R_i *and* $R_r^{\pm}$ *from* $\mathcal{G}$ *such that* $R_i' = 0$ *and* $(R_r^{\pm})' = 0$*.*

Note that in Part (A)(a) and (A)(b)(i) of the above theorem, $I_p(s,\sigma)$ is irreducible.

12.3. We will now describe how to obtain the module digram $\mathcal{G}'$ of $I_{p,q}(s,\sigma,\mu)$ from the module diagram $\mathcal{G}$ of $I_p(s,\sigma)$.

Let

$$s_1 = \min\{i : |\mu_{i+1}| \leq \alpha + i\}, \quad s_2 = \max\{i : \alpha + i \leq |\mu_{i-q}|\}.$$

By (41), $R_j' \neq 0$ if and only if $s_1 \leq j \leq s_2$. Delete the vertex R_j (or $R_j^{\pm}$) from $\mathcal{G}$ if $j < s_1$ and $j > s_2$. Remove the edges connected to the deleted vertices. Rename the remaining vertices R_j by R_j' and $R_r^{\pm}$ by $(R_r^{\pm})'$. Then the resulting graph is the module diagram $\mathcal{G}'$ of $I_{p,q}$. $\mathcal{G}'$ is a connected subgraph of $\mathcal{G}$. It will contain $(R_r^{+})'$ if and only if it contains $(R_r^{-})'$. An argument similar to §7.8 proves the following corollary.

Corollary 12.3.1. *If* W' *is an infinitesimal* G*-submodule of* $I_{p,q}(s,\sigma,\mu)$*, then there exists an infinitesimal* $\tilde{G}$*-submodule* W *of* $I_p(s,\sigma)$ *such that*

$$W' = W \cap (I_{p,q}(s,\sigma,\mu) \boxtimes v_0).$$

13. Unitarity

13.1. In this section, we shall determine which of the representation $I_{p,q}(s,\sigma,\mu)$ or its irreducible subquotients are unitarizable.

Theorem 13.1.1. *Let* $s \in \mathbb{C}$*,* $\sigma \in \{0,1\}$*,* $\mu \in \Lambda^\sigma(p)$*,* $\alpha = -s - (p-1)/2$ *and* $r = [p/2]$*.*

(A) (Unitarity of $I_{p,q}(s,\sigma,\mu)$)

(a) (Unitary induction) *If* $\mathrm{Re}(s) = 0$*, then* $I_{p,q}(s,\sigma,\mu)$ *is unitarizable.*

(b) *Let* $p = 2r$ *be even.*

(i) (Restriction of the complementary series of $\mathrm{Spin}_0(p,p)$) *If* $s \in (-\frac{1}{2},\frac{1}{2})$*, then* $I_{p,q}(s,\sigma,\mu)$ *is unitarizable.*

(ii) *If* $\sigma = 0$*,* $q \leq r$ *and there exists* $q+1 \leq a \leq r$ *such that* $\mu_{a-q} = 0$*, and* $|s| < \frac{p+3}{2} - a$*, then* $I_{p,q}(s,0,\mu)$ *is unitarizable.*

(c) *If* $\sigma = 0$*,* $p = 2r+1$ *is odd,* $q \leq r$*, and there exists* $q+1 \leq b \leq r+1$ *such that* $\mu_{b-q} = 0$*, and* $|s| < \frac{p+3}{2} - b$*, then* $I_{p,q}(s,0,\mu)$ *is unitarizable.*

(d) *If* $I_{p,q}(s,\sigma,\mu)$ *does not belong one any the cases in Parts (a), (b) and (c), then it is not unitarizable.*

(B) (Unitary subquotients) *Suppose that* $\alpha \in \frac{\sigma}{2} + \mathbb{Z}$ *and* $\alpha \geq -\frac{p-1}{2}$ *(equivalently* $s \leq 0$ *and* $s + \frac{p-1+\sigma}{2} \in \mathbb{Z}$*) so that* $I_p(s,\sigma)$ *is reducible.*

(a) (Restriction of the unitarizable subquotients in $I_p(s,\sigma)$)

(i) *If* $p = 2r$ *is even, and* $(R_r^+)'$ *and* $(R_r^-)'$ *are nonzero, then they are unitarizable.*

(ii) *If* $\sigma = 0$, $-(r-1) \leq \alpha \leq 0$, *and* $R'_{-\alpha} \neq 0$, *then it is unitarizable.*

(b) (New unitary subquotients) *If* $q \leq r$, i *is an integer such that* $q \leq i \leq r$ *and* $\sigma = \mu_{i-q+1} = 0$, *then* R'_i *is unitarizable.*

(c) *(a) and (b) give all the irreducible subquotients of* $I_{p,q}(s,\sigma,\mu)$ *which are unitarizable.*

If the unitary subquotients given in Part (B)(b) occur in some $I_{p,q}(s,\sigma,\mu)$, then it must correspond to the highest vertex in the module diagram of $I_{p,q}(s,\sigma,\mu)$.

Corollary 13.1.2. ([Li]) *If* $q \leq [p/2]$ *and* $|s| < (p+1)/2 - q$, *then* $I_{p,q}(s,0,0)$ *is unitarizable.*

13.2. The proof of Theorem 13.1.1 follows the same strategy as of Theorem 8.1.1 outlined in §8.2. If p is even, the argument is similar to the proof of Theorem 8.1.1 given in §8.3. We will omit it. If p is odd, we need to given special consideration to the fact that $T'_{\lambda,\lambda}$ is not always zero and we will sketch the proof in the next subsection.

13.3. **Sketch of Proof of Theorem 13.1.1 when** $p = 2r+1$ **is odd.** Let R' denote an irreducible subquotient of $I_{p,q}(s,\sigma,\mu)$. As in the $\mathrm{U}(p,q)$ case, it is easy to see that R' is unitarizable only if $s \in \mathbb{R}$ or $\mathrm{Re}(s) = 0$. If $\mathrm{Re}(s) = 0$, $I_{p,q}(s,\sigma,\mu)$ is unitary induced.

Next we will assume that $s \in \mathbb{R} - \{0\}$. Suppose $T' := T'_{\lambda,\lambda} \neq 0$ for some λ. By Lemma 11.4.1, this implies that $\lambda_r > 0$ and, $\mu_{1+r-q} > 0$ if $r \geq q$. We may choose a nonzero $w = v_\lambda \boxtimes [m_{kl}] \in S_\lambda$ where $m_{p-1,r-1} > 0$ and v_λ is the highest weight vector of τ_p^λ. Let R'' denote the $\mathfrak{so}(p,1)$-submodule in R' generate by v. By Theorem 12.2.1, R'' is an irreducible subquotient of some degenerate principal series of $I_{p,1}(s,\sigma,\mu'')$ where $\mu''_{r-1} = m_{p-1,r-1} > 0$. We claim that R'' is not unitarizable so $R' \supseteq R''$ is not unitarizable. The claim follows from Theorem 13 of [KG1]. Alternatively let $\mathfrak{so}(p,1) = \mathfrak{k}'' \oplus \mathfrak{p}''$ denote the Cartan decomposition. By Equation (28) of [KG1], there exists $X \in \mathfrak{p}'' \subseteq \mathfrak{p}$ and $v \in S_\lambda \cap R''$ such that $T'(X \otimes v) = v$. Therefore for any Hermitian form $(.,.)$ on R'', $(Xv,v) + (v,Xv) = 2(v,v) \neq 0$ and this implies that R'' is not unitarizable.

Suppose $s \in \mathbb{R} - \{0\}$ and R' is unitarizable, then $T'_{\lambda,\lambda} = 0$ and the above discussion implies that either one of the following condition holds:

(a) If $S_\lambda \neq 0$, then $\lambda_r = 0$.
(b) $r \geq q$ and $\mu_{1+r-q} = 0$.

Next we assume (a) or (b) above and suppose R' embeds into the irreducible subquotient R of $I_p(s,\sigma)$. Using an argument similar to those used in §9 of [Le], we can construct a Hermitian product on R satisfying

$$\langle Xu, v\rangle + \langle u, Xv\rangle = 0 \tag{42}$$

for all $X \in \tilde{\mathfrak{p}}$, $u \in V_\lambda$ and $v \in V_\eta$ such that $\lambda \neq \eta$. Note that this Hermitian product is not $\mathfrak{so}(p,p)$-invariant because (42) does not necessary hold for $u, v \in V_\lambda$. On the other hand the restriction of $\langle\ ,\ \rangle$ to R' is $\mathfrak{so}(p,q)$-invariant because $T'_{\lambda,\lambda} = 0$.

The proof now proceeds by checking the signature of $\langle\ ,\ \rangle$ on R'. This portion of the proof is similar to that of Theorem 8.1.1 given in §8.3 and we will leave the details to the reader.

References

[Ca] R. W. Carter, Raising and lowering operators for $\mathfrak{sl}_n$, with applications to orthogonal bases of $\mathfrak{sl}_n$-modules, *in* "The Arcata Conference on Representations of Finite Groups," Proc. Sympos. Pure Math., Vol. 47, Part 2, pp. 351-366, Amer. Math. Soc., Providence, RI, 1987.

[FH] W. Fulton and J. Harris, "Representation Theory, A First Course," Graduate Texts in Mathematics, Vol. 129, Springer-Verlag, New York, 1991.

[GG] I. M. Gelfand and M. I. Graev, Finite dimensional irreducible representations of the unitary and the full linear groups and related special functions, *in* "Gelfand, Izrail M. Collected Papers," Vol. II, pp. 662-692, Springer-Verlag, Berlin-New York, 1988.

[GZ1] I. M. Gelfand and M. L. Tsetlin, Finite dimensional representations of the group of unimodular matrices, *in* "Gelfand, Izrail M. Collected Papers," Vol. II, pp. 653-656, Springer-Verlag, Berlin-New York, 1988.

[GZ2] I. M. Gelfand and M. L. Tsetlin, Finite dimensional representations of the group of orthogonal matrices, *in* "Gelfand, Izrail M. Collected Papers," Vol. II. Springer-Verlag, Berlin-New York, (1988). pp 657 - 661.

[Hi1] T. Hirai, On infinitesimal operators of irreducible representations of the Lorentz group of n-th order. Proc. Japan Acad. 38 1962 83–87.

[Hi2] T. Hirai, On irreducible representations of the Lorentz group of $n-$ th order. Proc. Japan Acad. 38 1962 258–262.

[Hi3] T. Hirai, Classification and the characters of irreducible representations of SU(p, 1). Proc. Japan Acad. 42 1966 907–912.

[Ho] R. Howe, K-type structure in the principal series of GL_3 I, *in* "Analysis on homogeneous spaces and representation theory of Lie groups," Okayama–Kyoto (1997), 77–98, Adv. Stud. Pure Math., 26, Math. Soc. Japan, Tokyo, 2000.

[HT] R. Howe and E. Tan, Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations, *Bull. Amer. Math. Soc.* **28** (1993) 1-74.

[J1] K. Johnson, Degenerate principal series and compact groups, *Math. Ann.* **287** (1990), 703-718.

[J2] K. Johnson, Degenerate principal series on tube type domains, *Contemp. Math.* **138** (1992), 175-187.

[KG1] A. U. Klimyk and A. M. Gavrilik, The representations of the groups $\mathrm{U}(n, 1)$ and $\mathrm{SO}_0(n, 1)$, preprint ITP-76-39E, Instituent for Theoretical Physics Kiev, USSR,1976.

[KG2] A. U. Klimyk and B. Gruber, Matrix elements for infinitesimal operators of the groups $\mathrm{U}(p+q)$ and $\mathrm{U}(p,\, q)$ in a $\mathrm{U}(p)\times\mathrm{U}(q)$ basis I, II, J. Math. Phys. 20 (1979), no. 10, 1995–2010, 2011–2013.

[KG3] A. U. Klimyk and B. Gruber, Structure and matrix elements of the degenerate series representations of $\mathrm{U}(p+q)$ and $\mathrm{U}(p,\, q)$ in a $\mathrm{U}(p) \times \mathrm{U}(q)$ basis, J. Math. Phys. 23 (1982), no. 8, 1399–1408.

[KV] N. Ja. Vilenkin and A. U. Klimyk, "Representation of Lie groups and special functions, Recent advances," Translated from the Russian manuscript by V. A. Groza and A. A. Groza, Mathematics and its Applications, 316. Kluwer Academic Publishers Group, Dordrecht, 1995.

[Ko1] T. Kobayashi, Singular unitary representations and discrete series for indefinite Stiefel manifolds $\mathrm{U}(p,q;F)/\mathrm{U}(p-m,q;F)$, Mem. Amer. Math. Soc. 95 (1992), no. 462, vi+106 pp.

[Ko2] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups II - micro-local analysis and asymptotic K-support, Ann. of Math. (2) **147** (1998), no. 3, 709–729.

[Ko3] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups III - restriction of Harish-Chandra modules and associated varieties. *Invent. Math.* **131** (1998), no. 2, 229–256.

[Le] S. T. Lee, On some degenerate principal series representations of $\mathrm{U(n,n)}$, *J. Funct. Anal.* **126** (1994), 305-366.

[LL] S. T. Lee and H. Y. Loke, Degenerate principal series of $\mathrm{Sp}(p,q)$. *Preprint.*

[Li] J-S Li, Degenerate principal series of classical groups: the phenomenon of long complementary series, *preprint.*

[Mo] V. F. Molchanov, Maximal degenerate series representations of the universal covering of the group $\mathrm{SU}(n,n)$, in "Lie groups and Lie algebras", Math. Appl. **433**, 313-336, Kluwer Academic Publishers Group, Dordrecht, (1998).

[P] A. Pasquale, Maximal degenerate representations of $\mathrm{SL}(n+1,\mathbb{H})$, J. Lie Theory 9 (1999), no. 2, 369–382.

[S1] S. Sahi, Unitary representations on the Shilov boundary of a symmetric tube domain, *in* "Representations of Groups and Algebras," *Contemp. Math.* **145** (1993), 275-286, Amer. Math. Soc., Providence.

[S2] S. Sahi, Jordan algebras and degenerate principal series, *J. reine angew Math.* **462** (1995), 1-18.

[Z] D. P. Zhelobenko, "Compact Lie Groups and Their Representations," Translations of Mathematical Monographs, Vol. 40, Amer. Math. Soc., Providence, RI, 1973.

[Zh] G. Zhang, Jordan algebras and generalized principal series representation, *Math. Ann.* **302**(1995), 773-786.

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