DUALITY FOR SPHERICAL REPRESENTATIONS IN EXCEPTIONAL THETA CORRESPONDENCES

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Abstract. We study the exceptional theta correspondence for real groups obtained by restricting the minimal representation of the split exceptional group of the type $\mathbf{E}_n$, to a split dual pair where one member is the exceptional group of the type $\mathbf{G}_2$. We prove that the correspondence gives a bijection between spherical representations if $n = 6, 7$, and a slightly weaker statement if $n = 8$.

1. Introduction

Let H be the group of real points of a split, simply connected algebraic group of the type $\mathbf{E}_n$, and let K_H be a maximal compact subgroup of H. The group H contains a split dual pair $G \times G'$ where one group is of the type $\mathbf{G}_2$, while the other is a simply connected group of the type $\mathbf{A}_2$, $\mathbf{C}_3$ and $\mathbf{F}_4$, respectively, see [LS15]. We shall not fix G or G' to be of the type $\mathbf{G}_2$ in advance. The actual choice will depend on the nature of arguments.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of G and G', respectively, and let K and K' be the maximal compact subgroups of G and G', respectively. Let $\mathbf{V}$ be the Harish-Chandra module of the minimal representation of H. Let V be an irreducible $(\mathfrak{g}, K)$-module and V' be an irreducible $(\mathfrak{g}', K')$-module. We say that V and V' correspond if $V \otimes V'$ is a quotient of $\mathbf{V}$. It is known that the infinitesimal character of V determines the infinitesimal character of V' by [HPS] and [Li]. We call this a *weak duality.*

Let V be an irreducible $(\mathfrak{g}, K)$-module. Following [Ho], there is a $(\mathfrak{g}', K')$-module $\Theta(V)$ such that

$$\mathbf{V}/\cap_\phi \ker\phi \simeq \Theta(V) \otimes V$$

where the intersection is taken over all $(\mathfrak{g}, K)$-module homomorphisms $\phi\colon \mathbf{V} \to V$. The aim in the theory of correspondences is to prove that $\Theta(V)$ is a finite length $(\mathfrak{g}', K')$-module with a unique irreducible quotient V', and then conversely, that $\Theta(V')$ is a finite length $(\mathfrak{g}, K)$-module with V as a unique irreducible quotient. We call this a *strong duality.* The goal of this paper is to establish the strong duality for spherical representations. The minimal representation $\mathbf{V}$ is itself spherical, and let v_0 be a non-zero spherical vector in $\mathbf{V}$. The strong duality for spherical representations follows from the identities

$$U(\mathfrak{g}) \cdot v_0 = \mathbf{V}^{K'} \text{ and } U(\mathfrak{g}') \cdot v_0 = \mathbf{V}^K$$

where $U(\mathfrak{g})$ and $U(\mathfrak{g}')$ are the enveloping algebras. In this paper we prove both of these identities for the dual pairs in $\mathbf{E}_6$ and $\mathbf{E}_7$, and thus we establish the strong duality

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for spherical representations in these two cases, but only one for the dual pair in $\mathbf{E}_8$. Note that only one identity is sufficient to establish a multiplicity one statement, more precisely, if V and V' are spherical, then $V \otimes V'$ can appear as a quotient of $\mathbf{V}$ with multiplicity at most one.

We now briefly sketch our arguments. Assume that G is the smaller member of the dual pair. Let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Let $S(\mathfrak{p})$ be the symmetric algebra generated by the vector space $\mathfrak{p}$. We have an isomorphism $U(\mathfrak{g}) \cong S(\mathfrak{p}) \otimes U(\mathfrak{k})$ of vector spaces. Since G is smaller, there exists $0 < p < 2$ such that the restriction to G of every matrix coefficient of $\mathbf{V}$ is contained in $L^p(G)$. This allows us to prove that a generic spherical representation V of G is a quotient of $\mathbf{V}$ by integrating the matrix coefficients of V against the matrix coefficients of $\mathbf{V}$. As a consequence, the map $X \mapsto X \cdot v_0$ is an injection of $S(\mathfrak{p})$ into $\mathbf{V}^{K'}$. In fact, this is true if we replace $\mathbf{V}$ by any non-trivial, spherical, unitary representation of H, since the matrix coefficients of any such representation decay faster than the matrix coefficients of $\mathbf{V}$, as proved in [LS06]. The minimality of $\mathbf{V}$ will assure that this map is a bijection. More precisely, let $\mathfrak{k}_H \oplus \mathfrak{p}_H$ be the Cartan decomposition of $\mathfrak{h}$, and let ω be the highest weight of the adjoint action of $\mathfrak{k}_H$ on $\mathfrak{p}_H$. Then the K_H-types of $\mathbf{V}$ are, see [BK],

$$\mathbf{V} = \bigoplus_{n=0}^{\infty} V(n\omega).$$

We prove that the dimension of $S^n(\mathfrak{p})$, the n-the symmetric power of $\mathfrak{p}$, is equal to the dimension of $V(n\omega)^{K'}$, the space of K'-fixed vectors in $V(n\omega)$. It follows at once that the map $X \mapsto X \cdot v_0$ is a bijection of $S(\mathfrak{p})$ and $\mathbf{V}^{K'}$. In particular, $U(\mathfrak{g}) \cdot v_0 = \mathbf{V}^{K'}$. Working from the other side, so G' is the larger member of the dual pair, we can show that every spherical representation V' of G', that weakly corresponds to a spherical representation V of G, is a quotient $U(\mathfrak{g}') \cdot v_0$. This again reduces the problem of verifying $U(\mathfrak{g}') \cdot v_0 = \mathbf{V}^{K'}$ to a branching problem, i.e. determining the dimension of $V(n\omega)^{K'}$.

The last two sections are devoted to the branching problems. We now assume that G' is of the type $\mathbf{G}_2$. It turns out that computing the dimension of $V(n\omega)^K$ is surprisingly easy and follows from some, more or less known, branching rules. On the other hand, computing the dimension of $V(n\omega)^{K'}$ is progressively more difficult, since K' is the same in all three cases. We do not know how to compute the dimension of $V(n\omega)^{K'}$ if H is of the type $\mathbf{E}_8$. In the other two cases the branching is derived from a branching rule for certain quaternionic representations [Lo00], using a see-saw dual pair argument.

We need to add a small caveat. If H is of the type $\mathbf{E}_6$ then the strong duality holds only in the context of slightly larger groups. More precisely, let $\tilde{H}$ be the semi-direct product of H and its group of outer automorphisms $\mathbb{Z}/2\mathbb{Z}$. If $G \times G'$ is the dual pair in H such that $G \cong \mathrm{SL}_3(\mathbb{R})$, then the centralizer of G' in $\tilde{H}$ is $\tilde{G} \cong \mathrm{SL}_3(\mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z}$. We prove that the strong duality holds between spherical $(\mathfrak{g}, \tilde{K})$-modules and spherical $(\mathfrak{g}', K')$-modules. A similar situation holds for classical dual pairs $\mathrm{Sp}_{2n}(\mathbb{R}) \times \mathrm{O}(p,q)$, where the strong duality fails if $\mathrm{O}(p,q)$ is replaced by $\mathrm{SO}(p,q)$.

2. Geometry of exceptional groups

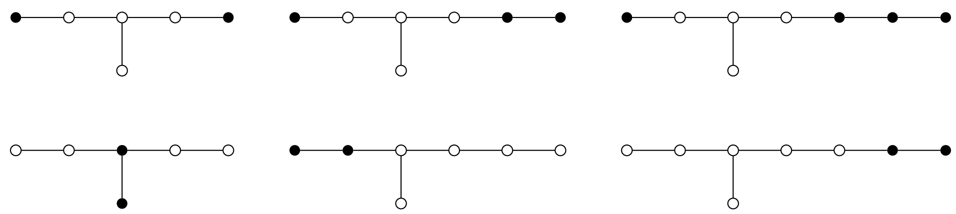
2.1. Let $\mathfrak{h}$ be a split simple real Lie algebra of the type $\mathbf{E}_n$. Fix a maximal split Cartan subalgebra $\mathfrak{a}_H \subset \mathfrak{h}$. Let $\Phi(\mathfrak{h}, \mathfrak{a}_H)$ be the root system arising from the adjoint action of $\mathfrak{a}_H$ on $\mathfrak{h}$. Let H be the group of real points of the simply connected Chevalley group attached to $\mathfrak{h}$. Let K_H be the maximal compact subgroup of H given as the group of fixed points of a Chevalley involution of G, i.e. one that acts as -1 on $\mathfrak{a}_H$. Fix a set of positive roots $\Phi^+(\mathfrak{h}, \mathfrak{a}_H)$. Then any element in $\mathfrak{a}_H$ can be uniquely written as a sum $\sum_i t_i\omega_i^\vee$ where $t_i \in \mathbb{R}$ and $\omega_1^\vee, \ldots, \omega_n^\vee \in \mathfrak{a}_H$ are the fundamental co-weights. The dominant cone $\mathfrak{a}_H^+ \subset \mathfrak{a}_H$ is the set of all $\sum_i t_i\omega_i^\vee$ where $t_i \geq 0$. Let $A_H = \exp(\mathfrak{a}_H) \subset H$. Then H has a Cartan decomposition

$$H = KA_H^+K$$

where $A_H^+ = \exp(\mathfrak{a}_H^+)$.

2.2. We shall now construct the dual pair $G \times G' \subset H$ in two ways. In the first construction G' will be of the type $\mathbf{G}_2$. In the second construction G will of the type $\mathbf{G}_2$. In each of the two constructions G will come equipped with a Cartan decomposition $G = KA^+K$ such that $K \subset K_H$ and $A^+ \subset A_H^+$. These inclusions will be a tool to understand integrality properties of matrix coefficients of H when restricted to G.

Let $\mathfrak{a} \subset \mathfrak{a}_H$ be the subalgebra spanned by the fundamental co-characters $\omega_i^\vee$ corresponding to the black points in the following marked Dynkin diagrams:



Consider the first row of the marked diagrams. By inspection, one checks that the adjoint action of $\mathfrak{a}$ on the Lie algebra of $\mathfrak{h}$ gives rise to a restricted root system $\Phi(\mathfrak{h}, \mathfrak{a})$ of the type $\mathbf{A}_2$, $\mathbf{C}_3$ and $\mathbf{F}_4$, respectively. The long root spaces are one-dimensional while the short root spaces are 8-dimensional. (If the restricted root system is of the type $\mathbf{A}_2$ then all roots are considered short.) Let C be the derived group of the centralizer of $\mathfrak{a}$ in H. Its Dynkin diagram is obtained by removing the black vertices. Thus C is a simply connected Chevalley group of the type $\mathbf{D}_4$ in each of the three cases. The Chevalley involution of H restricts to a Chevalley involution of C. The automorphism of order 3 of the root system $\mathbf{D}_4$ can be lifted to an automorphism of C commuting with the Chevalley involution. Let G' be the group of fixed points in C of that automorphism. It is a Chevalley group of the type $\mathbf{G}_2$. Let G be the centralizer of G' in H. It is well known that G is split, simply connected, of the type $\mathbf{A}_2$, $\mathbf{C}_3$ and $\mathbf{F}_4$, respectively. In fact, this is easy to see on the level of Lie algebras, since the group G' fixes a line in each short root space for the restricted root system. The maximal split torus in $\mathfrak{g}$ is $\mathfrak{a}$. The restriction to G of the Chevalley involution of H is a Chevalley involution of G since it acts as -1 on $\mathfrak{a}$. Hence the set of fixed points in G is a maximal compact subgroup K such that $K \subset K_H$. The root system $\Phi(\mathfrak{g}, \mathfrak{a})$ is equal to the restricted root system $\Phi(\mathfrak{h}, \mathfrak{a})$, and the choice of positive roots $\Phi^+(\mathfrak{h}, \mathfrak{a})$ determines a choice of positive roots $\Phi^+(\mathfrak{g}, \mathfrak{a})$. In particular, all $\omega_i^\vee$'s spanning $\mathfrak{a}$ belong to the dominant cone $\mathfrak{a}^+$ and it is easy to check that they are in fact precisely the fundamental co-weights for $\Phi^+(\mathfrak{g}, \mathfrak{a})$. Hence $A^+ \subset A_H^+$.

Now consider the second row of the marked diagrams. In this case the rank of $\mathfrak{a}$ is two. By inspection, one checks that the adjoint action of $\mathfrak{a}$ on the Lie algebra of $\mathfrak{h}$ gives rise to a restricted root system $\Phi(\mathfrak{h}, \mathfrak{a})$ of the type $\mathbf{G}_2$ in each case. The long root spaces are one-dimensional while the short root spaces are 9, 15 and 27-dimensional, respectively. Let C be the derived group of the centralizer of $\mathfrak{a}$ in H. Its Dynkin diagram is obtained by removing the black vertices. In each case C has an outer automorphism of order 2. (In the case where C is of the type $\mathbf{A}_2 \times \mathbf{A}_2$ we pick the automorphism that, on the level of the Dynkin diagrams, is a composite of the automorphism of the $\mathbf{E}_6$ diagram followed by the automorphism of one of the two $\mathbf{A}_2$ factors.) Again, we pick the automorphism so that it commutes with the Chevalley involution. Let G' be the group of fixed points of the automorphism, and G its centralizer in H. Since G' fixes a line in each short space it follows that G is of the type $\mathbf{G}_2$. Arguing exactly as above we have $K \subset K_H$, where K is a maximal compact subgroup of G, and $A^+ \subset A_H^+$.

3. Minimal representations

3.1. We retain the setting of the previous section. Recall that we have fixed a Cartan decomposition $H = K_H A_H^+ K_H$ where $A_H^+ = \exp(\mathfrak{a}_H^+)$ and $\mathfrak{a}_H^+$ is a cone whose elements are sums $\sum_i t_i \omega_i^\vee$, where $t_i \geq 0$, over all fundamental co-weights. Let $||\cdot||$ be an euclidean norm on $\mathfrak{a}_H$. For any $\lambda \in \mathfrak{a}_H^*$, let a^λ be the character of A_H defined by $a^\lambda = \exp(\lambda(\log a))$, for all $a \in A_H$. Let $\omega_i \in \mathfrak{a}_H^*$ be the fundamental characters, and $\rho_H \in \mathfrak{a}_H^*$ be the half sum of all positive roots of H. Let ω_b be the fundamental weight corresponding to the branching vertex. Let $\mathbf{V}$ be the Harish-Chandra module of the minimal representation of H. The normalized leading exponent of $\mathbf{V}$ is $\rho_H - \omega_b$. For our purposes, this means that the restriction of any matrix coefficient of $\mathbf{V}$ to A_H^+ is bounded by a multiple of $(1 + ||\log a||)^d a^{-\omega_b}$, for some integer d. Let p_H be the smallest positive real number such that

$$p_H \langle \omega_b, \omega_i^\vee \rangle \geq 2 \langle \rho_H, \omega_i^\vee \rangle$$

for all fundamental co-characters $\omega_i^\vee$. Then $a^{-\omega_b} \leq a^{-2\rho_H/p_H}$ for all $a \in A_H^+$ and, by [Kn, Theorem 8.48], the matrix coefficients of $\mathbf{V}$ are contained in $L^{p_H+\epsilon}(H)$, for all $\epsilon > 0$. One easily checks (see [LS06]) that p_H is as in the following table:

H	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
p_H	8	9	29/3

3.2. We now study integrability properties of the matrix coefficients of $\mathbf{V}$ restricted to G. We have a Cartan decomposition $G = KA^+K$, where $A^+ = \exp(\mathfrak{a}^+)$, where $\mathfrak{a}^+$ is a subcone of $\mathfrak{a}_H^+$ whose elements are sums $\sum_i t_i\omega_i^\vee$, where $t_i \geq 0$, over the fundamental co-weights corresponding to the black vertices in the marked Dynkin diagrams in Section 2.2. Let $\rho \in \mathfrak{a}^*$ be the half-sum of all positive roots of G. Let p be the smallest positive real number such that

$$p\langle\omega_b, \omega_i^\vee\rangle \geq 2\langle\rho, \omega_i^\vee\rangle$$

for considering only $\omega_i^\vee$ of G, i.e. those corresponding to the black vertices in the marked Dynkin diagrams. Then $a^{-\omega_b} \leq a^{-2\rho/p}$ for all $a \in A^+$ and the restriction to G of a matrix coefficient of $\mathbf{V}$ is contained in $L^{p+\epsilon}(G)$, for all $\epsilon > 0$. For G arising form the first family of marked Dynkin diagrams, i.e. the centralizer of G is of the type $\mathbf{G}_2$, we have the following values of p:

G	$\mathbf{A}_2$	$\mathbf{C}_3$	$\mathbf{F}_4$
p	1	2	8/3

For G arising from the second family of marked Dynkin diagrams we have the following values of p:

G	$\mathbf{G}_2$	$\mathbf{G}_2$	$\mathbf{G}_2$
p	2	3/2	1

We record what we have found in a form that is useful for later.

Lemma 3.1. *Assume that $G \times G'$ is the dual pair in H with G smaller than G'. Then there exists a positive real number $p < 2$ such that the restriction to G of matrix coefficients of the minimal representation of H are contained in $L^{p+\epsilon}(G)$, for all $\epsilon > 0$.* $\square$

3.3. Let $P = MAN$ the minimal parabolic subgroup of G containing A such that the Lie algebra of the unipotent radical N is spanned by the positive root spaces. Every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ defines a character $a^\lambda = \lambda(\log(a))$ of A. We can extend this character to P, so that it is trivial on MN. The normalized induction from P to G produces a smooth representation π_λ of G, naturally isomorphic to the space of smooth functions on $M\backslash K$. We have a G-invariant pairing $(\cdot,\cdot)$ between $\pi_{-\lambda}$ and π_λ given by integrating functions over K. Let V_λ be the Harish-Chandra module of π_λ. Let $v_\lambda \in V_\lambda$ be the unique K-invariant vector such that its value on $M\backslash K$ is one. Let

$$\varphi_\lambda(g) = (\pi_\lambda(g)v_\lambda, v_{-\lambda}) \tag{1}$$

be the spherical function on G corresponding to π_λ where, abusing the notation, $\pi_\lambda(g)$ denotes the group action. It is well known that, for $\lambda \in i\mathfrak{a}^*$, the matrix coefficients of V_λ, in particular φ_λ, are contained in $L^{2+\epsilon}(G)$, for all $\epsilon > 0$. Moreover if we fix a $q > 2$, then [Kn, Theorem 8.48] implies that there exists a tubular neighborhood of $i\mathfrak{a}^*$ in $\mathfrak{a}_{\mathbb{C}}^*$, defined by the inequality $||\Re(\lambda)|| < \delta$ for some $\delta > 0$ depending on q, such that, for λ contained in the tubular neighborhood, the matrix coefficients of V_λ are contained in $L^q(G)$.

Let $\mathbf{V}^\vee$ be the contragradient of the minimal representation $\mathbf{V}$. Let $(\cdot,\cdot)$ denote the natural pairing between $\mathbf{V}$ and $\mathbf{V}^\vee$. We shall fix, once for all a pair of spherical vectors $v_0 \in \mathbf{V}$ and $\tilde{v}_0 \in \mathbf{V}^\vee$ of spherical vectors such that $(v_0, \tilde{v}_0) = 1$. Let $\Phi(g) =$

$(\pi(g)v_0, \tilde{v}_0)$, $g \in H$, be the spherical function. Here $\pi(g)$ denotes the group action of H in a globalization of $\mathbf{V}$.

Proposition 3.2. *Assume that G is the smaller member of the dual pair. Then there is a tubular neighborhood of $i\mathfrak{a}^*$ such that for every λ in the tubular neighborhood, the integral*

$$\langle v, u\rangle = \int_G (\pi(g)v, \tilde{v}_0)(\pi_\lambda(g)u, v_{-\lambda})dg$$

is absolutely convergent for all v in $\mathbf{V}$ and $u \in V_\lambda$. The pairing $\langle v, u\rangle$ is $\mathfrak{g}$-invariant, i.e. for every $X \in \mathfrak{g}$, $\langle \pi(X)v, u\rangle + \langle v, \pi_\lambda(X)u\rangle = 0$, and $\langle v_0, v_\lambda\rangle \neq 0$ for all λ in an open neighborhood of 0.

Proof. Since G is smaller, Lemma 3.1 states that the restriction of the matrix coefficients of V to G is $L^{p+\epsilon}$ for some $p < 2$ and every $\epsilon > 0$. Hence the integral is convergent, by the Hölder's inequality, in a suitable tubular neighborhood of $i\mathfrak{a}^*$, depending on p.

Let $f(t) = (\pi_\lambda(g)\pi_\lambda(\exp(tX))u, v_{-\lambda})$, where $t \in \mathbb{R}$. Then

$$\langle \pi(X)v, u\rangle = \int_G (\pi(g)v, \tilde{v}_0) \lim_{t\to 0} \frac{f(t) - f(0)}{t} dg. \tag{2}$$

The $\mathfrak{g}$-invariance of the pairing is a formal consequence of G-invariance of the measure dg, provided we can switch the order of integration and differentiation in (2). This will be accomplished by the Lebesgue's Dominated Convergence Theorem by finding an appropriate bound on $\frac{f(t)-f(0)}{t}$, uniform for all small t. By the mean value theorem, it suffices to find a bound for $f'(t) = (\pi_\lambda(gg_t)w, v_{-\lambda})$ where $g_t = \exp(tX)$ and $w = \pi(X)u$. Note that $f'(0)$ is a matrix coefficient of V_λ. Let $\mu = \Re(\lambda)$. By [Kn, Proposition 7.14], there exists a constant C_w such that

$$|(\pi_\lambda(g)w, v_{-\lambda})| \leq C_w \varphi_\mu(g)$$

for all $g \in G$. We claim that there exists a constant $C > 0$ such that

$$\varphi_\mu(gg_t) \leq C\varphi_\mu(g)$$

for all $g \in G$ and $|t| \leq 1$. To that end, recall that V_μ is realized as functions on $M\backslash K$, and v_μ is the constant function equal to one. Let C be such that, for all $|t| \leq 1$, the function $\pi_\mu(g_t)v_\mu$ is bounded by C on $M\backslash K$. Since the pairing between V_μ and $V_{-\mu}$ is defined as integration of functions over K, it follows that

$$\varphi_\mu(gg_t) = (\pi_\mu(gg_t)v_\mu, v_{-\mu}) = (\pi_\mu(g_t)v_\mu, \pi_{-\mu}(g^{-1})v_{-\mu}) \leq C(v_\mu, \pi_{-\mu}(g^{-1})v_{-\mu}) = C\varphi_\mu(g).$$

This proves our claim. Thus we have a uniform estimate of $|f'(t)|$, for $|t| \leq 1$, by $C_w C\varphi_\mu$ and we can switch the order of integration and differentiation by the Lebesgue's Dominated Convergence Theorem.

Finally, to prove non-vanishing of the pairing, note that $\langle v_0, v_\lambda\rangle$ is simply the spherical transform

$$s(\lambda) = \int_G \Phi(g)\varphi_\lambda(g)dg.$$

Using the Harish-Chandra estimates for φ_λ [A, Proposition 3], it follows, from the Lebesgue's Dominated Convergence Theorem that $s(\lambda)$ is a continuous function on a tubular neighborhood of $i\mathfrak{a}^*$. (In fact, it is an analytic function, see [TV], but we shall not need that.) Recall that $\varphi_\lambda(g) > 0$ for λ real. For the same reason $\Phi(g) > 0$ since the parameter of the minimal representation is also real. Thus $s(0) \neq 0$ since $s(0)$ is an integral of a positive, analytic function. □

Proposition 3.2 implies that the matrix coefficient pairing gives a family of $\mathfrak{g}$-intertwining maps

$$m_\lambda \colon \mathbf{V} \to V_\lambda \tag{3}$$

in the tubular neighborhood such that $m_\lambda(v_0)$ is a non-zero multiple of v_λ for λ in a neighborhood of 0.

4. Main Results

4.1. Let $G \times G'$ be the dual pair in H as before. We shall now give a description of the maximal compact subgroups $K \times K' \subset G \times G'$ and the K_H-types of the minimal representation

$$\mathbf{V} = \bigoplus_{n=0}^{\infty} V(n\omega) \tag{4}$$

where $V(n\omega)$ denotes the irreducible representation of K_H of the highest weight $n\omega$. If we assume that G' is of the type $\mathbf{G}_2$, then $K' \cong \mathrm{SU}_2 \times_2 \mathrm{SU}_2$ while the rest of the data is given by the following table:

H	K_H	ω	G	K
$\mathbf{E}_6$	Sp_8	$\omega_4 = (1,1,1,1)$	$\mathrm{SL}_3(\mathbb{R})$	SO_3
$\mathbf{E}_7$	SU_8	$\omega_4 = (1,1,1,1,0,0,0,0)$	$\mathrm{Sp}_6(\mathbb{R})$	U_3
$\mathbf{E}_8$	Spin_{16}	$\omega_8 = \frac{1}{2}(1,1,1,1,1,1,1,1)$	$\mathbf{F}_4$	$\mathrm{SU}_2 \times_2 \mathrm{Sp}_6$.

Here ω_i denotes the i-th fundamental weight of K_H as given in [Bou]. We note that the groups listed in the second column are in fact 2-fold covers of K_H, the maximal compact subgroup of H. For notational convenience we shall often work with these 2-fold covers, instead of K_H proper.

In order to obtain the strong reciprocity statement in the case of the dual pair in $\mathbf{E}_6$, it is necessary to consider larger, disconnected groups. Let $\tilde{H}$ be a semi-direct product of H and its group of outer automorphisms $\mathbb{Z}/2\mathbb{Z}$. The maximal compact subgroup $\tilde{K}_H$ of $\tilde{H}$ is isomorphic to $\mathrm{Sp}_8 \times \mathbb{Z}/2\mathbb{Z}$. We let the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ act on the type $V(n\omega) \subset \mathbf{V}$ by $(-1)^n$. In this way $\mathbf{V}$ becomes a $(\mathfrak{h}, \tilde{K}_H)$-module. The centralizer of G' in $\tilde{H}$ is $\tilde{G}$, a semi-direct product of G and $\mathbb{Z}/2\mathbb{Z}$. The maximal compact subgroup $\tilde{K} \subset \tilde{G}$ is isomorphic to O_3.

4.2. Assume for the rest of the section that G is the smaller member of the dual pair. Using the symmetrizing map, $S(\mathfrak{p})$ can be viewed as a vector subspace of $U(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt theorem, multiplication of elements in $S(\mathfrak{p})$ and $U(\mathfrak{k})$ gives an isomorphism $U(\mathfrak{g}) \cong S(\mathfrak{p}) \otimes U(\mathfrak{k})$, compatible with the adjoint action of K. Let $\mathbb{C}$ be the trivial representation of $U(\mathfrak{k})$, and let

$$E = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbb{C} \cong S(\mathfrak{p}).$$

The center $Z(\mathfrak{g})$ maps into E by $x \mapsto x \otimes 1$. Since G is split, this map is an isomorphism of $Z(\mathfrak{g})$ and $E^K \cong S(\mathfrak{p})^K$. We also have a decomposition $S(\mathfrak{p}) \cong \mathcal{H} \otimes S(\mathfrak{p})^K$, where $\mathcal{H}$ is an K-invariant subspace of, so-called, harmonic polynomials. Note that E is naturally a left $U(\mathfrak{g})$-module by left multiplication. Considering $Z(\mathfrak{g})$ as a subalgebra of $U(\mathfrak{g})$, E is also a right $Z(\mathfrak{g})$-module. Recall that $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ defines a one-dimensional character χ_λ of $Z(\mathfrak{g})$ acting on the line $\mathbb{C}_\lambda$. Let

$$E_\lambda = E \otimes_{Z(\mathfrak{g})} \mathbb{C}_\lambda \cong \mathcal{H}.$$

Let V_λ be the $(\mathfrak{g}, K)$-module of the principal series representation as in Section 3.3. By a Frobenius Reciprocity Theorem, we have a canonical map from E_λ to V_λ, sending $1 \otimes 1$ to v_λ. By [WII, Section 11.3.6], this is an isomorphism for a generic λ.

Lemma 4.1. *Let $X \in S(\mathfrak{p})$ be non-zero. Then $\pi_\lambda(X) \cdot v_\lambda$ is non-zero for generic $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.*

Proof. Write $X = \sum X_i \otimes Y_i$ under the isomorphism $S(\mathfrak{p}) = \mathcal{H} \otimes S(\mathfrak{p})^K$, where the X_i's are linearly independent elements in $\mathcal{H}$ and the Y_i's are non-zero. Let $Z_i \in Z(\mathfrak{g})$ correspond to Y_i under the isomorphism of $S(\mathfrak{p})^K$ and $Z(\mathfrak{g})$. Let $c_i = \chi_\lambda(Z_i)$. Note that $c_i \neq 0$ for a generic λ. Under the isomorphisms $V_\lambda \cong E_\lambda \cong \mathcal{H}$, the element $\pi_\lambda(X) \cdot v_\lambda \in V_\lambda$ corresponds to $\sum_i c_i X_i \in \mathcal{H}$, and this is non-zero for a generic λ. □

Corollary 4.2. *The map $m \colon S(\mathfrak{p}) \to \mathbf{V}^{K'}$ given by $X \mapsto X \cdot v_0$ is an injection.*

Proof. Let X be a non-zero element in $S(\mathfrak{p})$. By (3) there is a family of $\mathfrak{g}$-intertwining maps $m_\lambda \colon \mathbf{V} \to V_\lambda$ such that $m_\lambda(v_0)$ is a non-zero multiple of v_λ for all λ in a neighborhood B of 0. Let $X \in S(\mathfrak{p})$ be non-zero. It follows that $m_\lambda(X \cdot v_0)$ is a non-zero multiple of $\pi_\lambda(X) \cdot v_\lambda$ for all $\lambda \in B$. By Lemma 4.1, $\pi_\lambda(X) \cdot v_\lambda$ is nonzero for some λ in B. This proves that $X \cdot v_0 \neq 0$. □

Proposition 4.3. *The injection $m \colon S(\mathfrak{p}) \to \mathbf{V}^{K'}$ in the last corollary is a bijection. In particular, $U(\mathfrak{g}) \cdot v_0 = \mathbf{V}^{K'}$.*

Proof. Let $S^n(\mathfrak{p})$ be the n-th symmetric power of $\mathfrak{p}$. For every non-negative integer n, the injection m restricts to an injection

$$m_n \colon \oplus_{k=0}^n S^k(\mathfrak{p}) \to \oplus_{k=0}^n V(k\omega)^{K'}$$

In order to prove that m is a bijection, it suffices to show that each m_n is a bijection, and this is a dimension check. The branching rule in Proposition 6.2(i) and Proposition 5.2 implies that $\dim S^n(\mathfrak{p}) = \dim V(n\omega)^{K'}$. This proves the proposition. □

We now derive several consequences of Proposition 4.3. Let S_λ be the irreducible spherical sub-quotient of V_λ. Recall that the isomorphism classes of irreducible K-spherical $(\mathfrak{g}, K)$-modules correspond to the maximal ideals in $Z(\mathfrak{g})$.

Corollary 4.4. *For every $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, S_λ is a quotient of $\mathbf{V}$, i.e. $\Theta(S_\lambda) \neq 0$.*

Proof. Note that $\mathbf{V}^{K'}$ is a summand of $\mathbf{V}$ and $\mathbf{V}^{K'} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbb{C}$, as $\mathfrak{g}$-modules. The corollary follows because S_λ is a quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbb{C}$. $\square$

We recall that G is the smaller member of the dual pair. Let $Z(\mathfrak{g}) = U(\mathfrak{g})^G$ be the center of $U(\mathfrak{g})$. There exists a homomorphism

$$\gamma\colon Z(\mathfrak{g}') \to Z(\mathfrak{g}) \tag{5}$$

such that, for every $z \in Z(\mathfrak{g}')$, $z = \gamma(z)$ on $\mathbf{V}$. The map γ is surjective in this case if we replace $Z(\mathfrak{g})$ by $\tilde{Z}(\mathfrak{g}) = U(\mathfrak{g})^{\tilde{G}}$. After taking $\tilde{K}$-invariants of both sides in Proposition 4.3, we get

$$\tilde{Z}(\mathfrak{g}) \cdot v_0 = \mathbf{V}^{\tilde{K}\times K'} = Z(\mathfrak{g}') \cdot v_0.$$

Let $\tilde{S}_\lambda$ be the irreducible $\tilde{K}$-spherical $(\mathfrak{g}, K)$-module containing S_λ. The isomorphism classes of irreducible $\tilde{K}$-spherical modules correspond to the maximal ideals in $\tilde{Z}(\mathfrak{g})$. Let $S_{\lambda'}$ be the irreducible K'-spherical $(\mathfrak{g}', K')$-module whose infinitesimal character is the pullback, via γ, of the infinitesimal character of $\tilde{S}_\lambda$.

Corollary 4.5. *If $\Theta(S_{\lambda'}) \neq 0$ then it is a finite length $(\mathfrak{g}, \tilde{K})$-module with the unique irreducible quotient isomorphic to $\tilde{S}_\lambda$.*

Proof. Since $\Theta(S_{\lambda'}) \otimes S_{\lambda'}$ is a quotient of $\mathbf{V}$, it follows that $\Theta(S_{\lambda'})$ is a quotient of $\mathbf{V}^{K'}$ and, therefore, a quotient of the cyclic $(\mathfrak{g}, \tilde{K})$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbb{C}$ where $\mathbb{C}$ is the trivial representation of $\tilde{K}$. The module $\Theta(S_{\lambda'})$ is also annihilated by the maximal ideal of $\tilde{Z}(\mathfrak{g})$ corresponding to $\tilde{S}_\lambda$. By [WI, Corollary 3.4.7] $\Theta(S_{\lambda'})$ is admissible and by [WI, Theorem 4.2.1] it is of finite length. $\square$

4.3. Now we want to go in the opposite direction, and show that $U(\mathfrak{g}') \cdot v_0 = \mathbf{V}^{\tilde{K}}$. Since the restriction of the matrix coefficients of $\mathbf{V}$ to G' are not contained in $L^2(G')$, the strategy we used for G cannot be applied. However now we have another way. Let

$$U = U(\mathfrak{g}') \cdot v_0 \subseteq \mathbf{V}^{\tilde{K}}.$$

Note that $U^{K'} = Z(\mathfrak{g}') \cdot v_0$, and this is a direct summand of U, considered a $Z(\mathfrak{g}')$-module. (This is because U is a direct sum of its K'-type subspaces, each of which is a $Z(\mathfrak{g}')$-submodule.) Let χ_λ be a character of $Z(\mathfrak{g})$ and let $\chi_{\lambda'} = \gamma^*(\chi_\lambda)$ be the character of $Z(\mathfrak{g}')$ on a one-dimensional space $\mathbb{C}_{\lambda'}$, obtained by pulling back by γ. Then $U_{\lambda'} = U \otimes_{Z(\mathfrak{g}')} \mathbb{C}_{\lambda'}$ is a quotient of U, generated by the unique K'-fixed line. This proves the following lemma.

Lemma 4.6. *If $\chi_{\lambda'} = \gamma^*(\chi_\lambda)$ then $S_{\lambda'}$ is a quotient of U.* $\square$

We will now state the result corresponding to Proposition 4.3.

Proposition 4.7. *Assume H is of the type $\mathbf{E}_6$ or $\mathbf{E}_7$. Then $U(\mathfrak{g}') \cdot v_0 = \mathbf{V}^{\tilde{K}}$.*

Proof. We need to discuss on a case by case basis.

Case $\mathbf{E}_6$: In this case the map γ is a bijection. Thus $S_{\lambda'}$ is irreducible for a generic λ' and it is a quotient of U. We can apply Lemma 4.1 to G' and, arguing as in the proof

Corollary 4.2, it follows that the map $X \mapsto X \cdot v_0$ is an injection of $S(\mathfrak{p}')$ into $\mathbf{V}^{\tilde{K}}$. The branching rule in Proposition 5.2 implies that this map is also a bijection.

Case $\mathbf{E}_7$: In this case the map γ is a surjection and the kernel is generated by a polynomial which under the isomorphism $S(\mathfrak{p}')^{K'} \cong Z(\mathfrak{g}')$ corresponds to a homogeneous polynomial p of degree 4. If $\chi_{\lambda'} = \gamma^*(\chi_\lambda)$ then by [LS07, Theorem 6] $S_{\lambda'}$ is irreducible for a generic λ and, by Lemma 4.6, it is a quotient of U. An obvious alteration of the proof of Corollary 4.2 implies that the map $X \mapsto X \cdot v_0$ is an injection of $S(\mathfrak{p}')/pS(\mathfrak{p}')$ into $\mathbf{V}^K$ (in this case $K = \tilde{K}$). The branching rule in Proposition 6.2(ii) implies that this is also a bijection. □

Corollary 4.8. *Assume H is of the type $\mathbf{E}_6$ or $\mathbf{E}_7$. Let $S_{\lambda'}$ be an irreducible K'-spherical module whose infinitesimal character is the pullback, via γ in (5), of the infinitesimal character of an irreducible $\tilde{K}$-spherical module $\tilde{S}_\lambda$. Then*

- $\Theta(S_{\lambda'}) \neq 0$ *and*
- $\Theta(\tilde{S}_\lambda)$ *is a finite length $(\mathfrak{g}', K')$-module with the unique irreducible quotient isomorphic to $S_{\lambda'}$.* □

5. Some simple branching rules

The goal of this section is to compute the dimensions of $V(n\omega)^K$ where K is the maximal compact subgroup of the centralizer of G_2, and $V(n\omega)$ are the types of the minimal representation (4).

5.1. We start with three families of branching rules in a more general setting than what we need, but the proofs are not more difficult. All groups are complex algebraic groups in this subsection unless otherwise stated.

Let $R = \mathrm{Sp}_{2m}$ (respectively GL_{2m} and Spin_{4m}). Note that R acts naturally on its standard representation $V = \mathbb{C}^{2m}$ (respectively $\mathbb{C}^{2m}$, $\mathbb{C}^{4m}$). Let P be the maximal parabolic subgroup of R stabilizing an isotropic subspace of V of dimension m (respectively m, $2m$). Let $P = MN$ be its Levi decomposition. We tabulate M and define a fundamental weight ω of R which defines a one-dimensional character of P.

R	ω	M
Sp_{2m}	$\omega_m = (1^m)$	GL_m
GL_{2m}	$\omega_m = (1^m, 0^m)$	$\mathrm{GL}_m \times \mathrm{GL}_m$
Spin_{4m}	$\omega_{2m} = \frac{1}{2}(1^{2m})$	$\widetilde{\mathrm{GL}}_{2m} = \mathrm{GL}_1 \times_m \mathrm{SL}_{2m}$

Here 1^m denotes the m-tuple $(1, \ldots, 1)$ etc. Let $V(n\omega)$ be the the irreducible representation of R with the highest weight $n\omega$. The next proposition describes the restriction of $V(n\omega)$ from R to M. (The highest weights for M are with respect to the Borel subgroup of upper-triangular matrices.)

Proposition 5.1. *Let n be a positive integer. We have*

(i)

$$\mathrm{Res}^{\mathrm{Sp}_{2m}}_{\mathrm{GL}_m} V(n\omega_m) = \bigoplus_{\lambda} V(\lambda)$$

where the sum is taken over all highest weights $\lambda = (\lambda_1, \dots, \lambda_m)$ *of* GL_m *such that*

$$n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq -n$$

and $\lambda_i \equiv n \pmod 2$ *for all* i*;*

(ii)

$$\mathrm{Res}^{\mathrm{GL}_{2m}}_{\mathrm{GL}_m \times \mathrm{GL}_m} V(n\omega_m) = \bigoplus_{\lambda} V(\lambda)$$

where the sum is taken over all highest weights $\lambda = (\lambda_1, \dots, \lambda_{2m})$ *of* $\mathrm{GL}_m \times \mathrm{GL}_m$ *such that*

$$n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$$

and $\lambda_{m+i} = n - \lambda_{m-i+1}$ *for all* $i = 1, \dots, m$*;*

(iii)

$$\mathrm{Res}^{\mathrm{Spin}_{4m}}_{\widetilde{\mathrm{GL}}_{2m}} V(n\omega_{2m}) = \bigoplus_{\lambda} V(\lambda)$$

where the sum is taken over all highest weights $\lambda = (\lambda_1, \dots, \lambda_{2m})$ *of* $\widetilde{\mathrm{GL}}_{2m}$ *such that*

$$\frac{n}{2} \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2m} \geq -\frac{n}{2}$$

$\lambda_{2i-1} = \lambda_{2i}$ *for* $i = 1, \dots, m$ *and* $\lambda_j \equiv \frac{n}{2} \pmod{\mathbb{Z}}$ *for all* $j = 1, \dots, 2m$*.*

Proof. First we observe that a highest weight of M is also a weight of R. It follows that the bounds on λ_i hold.

By the Borel-Weil theorem, we interpret $V(n\omega)$ as the space of holomorphic sections of an R-equivariant line bundle on R/P. We also interpret an irreducible representation $V(\lambda)$ of M with the highest weight λ as the space of sections of an M-equivariant line bundle on M/B where B is a Borel subgroup of M.

Claim. *The Levi subgroup* M *has an open orbit on* $R/P \times M/B$*.*

Proof. Let $\bar{P} = M\bar{N}$ be the opposite parabolic subgroup to P. Then the image of $\bar{N}$ is an open subset of R/P. The Levi subgroup M acts on $\bar{N}$ with an open orbit. By choosing an appropriate point x in the open orbit, we may assume that the stabilizer S of x in M is isomorphic to O_m, $\triangle\mathrm{GL}_m$ and $\mathrm{Sp}(2m)$ respectively in the three cases. It is well known that S has an open orbit in M/B. This proves our claim. □

Pick B in a standard position, i.e. containing the torus T of diagonal matrices in M such that $S \cdot B/B$ is an open orbit in M/B. The claim implies that the restriction of $V(n\omega)$ to M is multiplicity free. Moreover, only $V(\lambda)$ such that $\lambda = n\omega$ on $S \cap T$ can appear in the restriction. We will compute $S \cap T$ in each case, and then we will derive a necessary condition on λ such that $V(\lambda) \subset V(n\omega)$:

- If $R = \mathrm{Sp}(2m)$ then $S \cap T = \{t \in \mathrm{GL}_m : t \text{ diagonal and } t^2 = 1\}$. Hence
$$\lambda_i \equiv n \pmod 2 \text{ for } i = 1, \dots, m.$$
- If $R = \mathrm{GL}_{2m}$ then $S \cap T = \{(t,t) \in \mathrm{GL}_m^2 : t \text{ diagonal}\}$. Hence
$$\lambda_{m+i} = n - \lambda_{m-i+1} \text{ for } i = 1, \dots, m.$$

- If $R = \mathrm{Spin}_{4m}$ then $S \cap T = \{(t_1, t_1^{-1}, t_2, t_2^{-1}, \dots t_m, t_m^{-1}) : t_i \in \mathbb{C}\}$. Hence
$$\lambda_{2i-i} = \lambda_{2i} \text{ for } i = 1, \dots, m.$$

Summing up, we have shown that in all three cases of the proposition, the left hand sides are contained in the respectively right hand sides.

Finally we have to show the containments are equalities. We will prove this by induction on n. If $n = 1$, the proposition holds by checking the weights of $V(n\omega)$. Suppose $n-1$ is true. Then we have an R-equivariant map
$$V((n-1)\omega) \otimes V(\omega) \to V(n\omega)$$
given by multiplying holomorphic sections. If $f \in V((n-1)\omega)$ and $f' \in V(\omega)$ are nonzero highest weight vectors of M of weights λ and λ' respectively, then the product $f \cdot f' \in V(n\omega)$ is a nonzero highest weight vector of weight $\lambda + \lambda'$. It follows by induction that all irreducible representations of M on the right hand sides occur in $V(n\omega)$. This completes the induction and proves the proposition. □

5.2. Let $\tilde{H}$ denote the semi-direct product of H and its group of outer automorphisms in all three cases, in particular, $H = \tilde{H}$ unless the type of H is $\mathbf{E}_6$. Let $\tilde{K}_H$ be the maximal compact subgroup of $\tilde{H}$. The representations discussed in Proposition 5.1, for $m = 4$, are essentially the $\tilde{K}_H$-types of the minimal representations (4). Recall that, if $\tilde{K}_H \cong \mathrm{Sp}_8 \times (\mathbb{Z}/2\mathbb{Z})$, we extend $V(n\omega)$ to a representation of $\tilde{K}_H$ such that the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ acts on by $(-1)^n$. We assume now that G' is of the type $\mathbf{G}_2$. Let $\tilde{G}$ be the centralizer of G' in $\tilde{H}$.

Proposition 5.2. *Let $\tilde{K}$ be the maximal compact subgroup of $\tilde{G}$ and let $\mathfrak{k}' \oplus \mathfrak{p}'$ be the Cartan decomposition of the Lie algebra of G', the split group of type $\mathbf{G}_2$. Then, for all n,*
$$\dim V(n\omega)^{\tilde{K}} = \dim S^n(\mathfrak{p}').$$

Proof. We define an n-string to be a sequence of integers
$$n \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq 0.$$
The number of n-strings is equal to the dimension of the space of homogeneous polynomials of degree n in 8 variables. Indeed, to every n string we can attach a monomial
$$x_1^{n-a_1} x_2^{a_1-a_2} \cdot \dots \cdot x_8^{a_7}$$
and vice versa. Since the dimension of $\mathfrak{p}'$ is 8, the strategy of the proof is to show that $\dim V(n\omega)^K$ is equal to the number of n-strings. We consider the three cases separately.

<u>Case $\mathbf{E}_6$</u>: $\tilde{K}_H \cong \mathrm{Sp}_8 \times \langle \epsilon \rangle$ and $\tilde{K} \cong \mathrm{O}_3$. The embedding of $\tilde{K}$ into $\tilde{K}_H$ is given by $x \mapsto (i(x), \det(x))$ where i is the composite of the embeddings
$$\mathrm{O}_3 \subset \mathrm{GL}_3 \subset \mathrm{GL}_4 \subset \mathrm{Sp}_8.$$
Since ϵ acts on $V(n\omega)$ by $(-1)^n$, our our task is to determine the dimension of the subspace of $V(n\omega)$ on which $i(\mathrm{O}_3)$ acts by the character $\det^n$. Let $W = V(a,b,c)$ be a representation of GL_3 with the highest weight (a,b,c). By the Cartan-Helgason Theorem, $\dim W^{\mathrm{SO}_3} \leq 1$. The representation W contains a nonzero SO_3 invariant vector

v if and only if $a \equiv b \equiv c \pmod 2$. Moreover O_3 acts on v by $\det^{a+b+c}$. We have thus shown that W contains $\det^n_{\mathrm{O}_3}$ with multiplicity at most one, and it is one if and only if $a \equiv b \equiv c \equiv n \pmod 2$. Let $V = V(\lambda_1, \ldots, \lambda_4)$ be the irreducible representation of GL_4 of the highest weight $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. By Proposition 5.1, $V(n\omega)$ is a multiplicity free direct sum of V such that $n \geq \lambda_i \geq -n$ and $\lambda_i \equiv n \pmod 2$. By the Gelfand-Zetlin pattern, a summand V contains W if and only if

$$n \geq \lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3 \geq c \geq \lambda_4 \geq -n$$

and when this happens, it contains W with multiplicity one. Thus each such string of integers, where all are odd or all are even, contributes one dimension to $V(n\omega)^{\tilde{K}}$. Since these strings are clearly in a bijection with the n-strings, we have completed the proof in this case.

Case $\mathbf{E}_7$: $K_H \cong \mathrm{SL}_8$ and $K \cong \mathrm{GL}_3$. The embedding of K into K_H is given by the composite of the embeddings

$$\triangle \mathrm{GL}_3 \subset \mathrm{GL}_3^2 \subset \mathrm{SL}_4^2 \subset \mathrm{SL}_8.$$

An irreducible representation W of GL_3^2 contains a nonzero K-invariant if and only if W is of the form $V(a,b,c) \otimes V(a,b,c)^*$. Let V be the restriction to SL_4^2 of the irreducible representation $V(\lambda_1, \ldots, \lambda_4) \otimes V(\lambda_1, \ldots, \lambda_4)^*$ of GL_4^2. By Proposition 5.1, $V(n\omega)$ is a multiplicity free direct sum of such V with $n \geq \lambda_i \geq 0$. By the Gelfand-Zetlin pattern, V contains W if and only if

$$n \geq \lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3 \geq c \geq \lambda_4 \geq 0.$$

In other words, we have an n-string of integers. Each such n-string contributes one dimension to $V(n\omega)^K$. This completes the proof in this case.

Case $\mathbf{E}_8$: $K_H \cong \mathrm{Spin}_{16}$ and $K \cong \mathrm{SL}_2 \times \mathrm{Sp}_6$. The embedding of K into K_H is given by the composite of the embeddings

$$\mathrm{SL}_2 \times \mathrm{Sp}_6 \subset \widetilde{\mathrm{GL}}_2 \times_2 \widetilde{\mathrm{GL}}_6 \subset \widetilde{\mathrm{GL}}_8 \subset \mathrm{Spin}_{16}$$

where the tildes above the groups denote double covers. By the Cartan-Helgason theorem, an irreducible representation W of $\widetilde{\mathrm{GL}}_2 \times_2 \widetilde{\mathrm{GL}}_6$ contains a nonzero K-invariant if and only if W is of the form $V(d,d) \otimes V(a,a,b,b,c,c)$ where $a,b,c,d \in \frac{1}{2}\mathbb{Z}$, and a,b,c are congruent to one another modulo $\mathbb{Z}$. Let $V = V(\lambda_1, \lambda_1, \ldots, \lambda_4, \lambda_4)$ be the irreducible representation of $\widetilde{\mathrm{GL}}_8$ where λ_i are half-integers, congruent modulo $\mathbb{Z}$. By Proposition 5.1, $V(n\omega)$ is a multiplicity free direct sum of such V with $n/2 \geq \lambda_i \geq -n/2$ and λ_i congruent to $n/2$ modulo $\mathbb{Z}$. By the Gelfand-Zetlin pattern, a summand V contains W if and only if $d = \sum_{i=1}^4 \lambda_i - a - b - c$ and

$$n/2 \geq \lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3 \geq c \geq \lambda_4 \geq -n/2$$

where all half-integers are congruent modulo $\mathbb{Z}$. When this happens, V contains W with multiplicity one. Each such string of half integers contributes one dimension to $V(n\omega)^K$. Since these strings are in a bijection with the n-strings, we have completed the proof in this case, as well. $\square$

6. A quaternionic see-saw

The goal of this section is to compute the dimensions of $V(n\omega)^{K'}$ where K' is the maximal compact subgroup of G_2, and $V(n\omega)$ are the types of the minimal representation in (4). We shall compute these dimensions in the case $H = \mathbf{E}_6$ and $H = \mathbf{E}_7$. We do not know how to do this in the case $H = \mathbf{E}_8$.

6.1. Recall that the maximal compact of $\mathrm{G}_{2,2}$ is $K' \cong \mathrm{SU}_{2,l} \times \mathrm{SU}_{2,s}$. Here the subscripts l and s denote a long root and a short root respectively. The embedding of K' into K_H, for all three H, is given by the sequence of embeddings

$$\mathrm{SU}_{2,l} \times \mathrm{SU}_{2,s} \subset \mathrm{SU}_2^4 \subset \mathrm{Sp}_8 \subset \mathrm{SU}_8 \subset \mathrm{Spin}_{16}$$

where $\mathrm{SU}_{2,s} \subset \mathrm{SU}_2^3$ diagonally, and $\mathrm{SU}_2^4 \subset \mathrm{Sp}_8$ is given by 4 orthogonal long roots. The first and easy step is to compute $\mathrm{SU}_{2,l}$-invariants in $V(n\omega)$. (i) If $K_H \cong \mathrm{Sp}_8$, then $V(n\omega)^{\mathrm{SU}_{2,l}}$ is naturally an Sp_6-module. By the branching rule in [WY], it follows that

$$V(n\omega)^{\mathrm{SU}_{2,l}} \cong V(n,n,0),$$

the irreducible representation of Sp_6 with the highest weight $(n,n,0)$. (ii) If $K_H \cong \mathrm{SU}_8$, then $V(n\omega)^{\mathrm{SU}_{2,l}}$ is naturally an SU_6-module. By the Littlewood-Richardson rule, it follows that

$$V(n\omega)^{\mathrm{SU}_{2,l}} \cong \oplus_{a+b=n} V_{a,b}.$$

Here $V_{a,b}$ is the irreducible representation of SU_6 with the highest weight $a\omega_2 + b\omega_4$ where ω_2 and ω_4 are the second and the fourth fundamental weights of SU_6 respectively.

The key result is the following lemma:

Lemma 6.1. *We have*

$$\dim V_{a,b}^{\mathrm{SU}_{2,s}} = \binom{a+5}{5}\binom{b+5}{5} - \binom{a+3}{5}\binom{b+3}{5}.$$

In the above lemma if $0 \le c \le 1$, then we set $\binom{c+3}{5} = 0$. The proof of the lemma will be given in Section 6.3. It uses quaternionic representations and a see-saw dual pair in the quaternionic group of type $\mathbf{E}_7$. Before that, we shall derive our branching rules.

Proposition 6.2. *Let K' be the maximal compact subgroup of G_2, and let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the centralizer $\mathfrak{g}$ of G_2 in $\mathfrak{h}$.*

(i) If $\mathfrak{h}$ is of the type $\mathbf{E}_6$ and $\mathfrak{g}$ of the type A_2 then, for all n,

$$\dim V(n\omega)^{K'} = \binom{n+4}{4} = \dim S^n(\mathfrak{p}).$$

(ii) If $\mathfrak{h}$ is of the type $\mathbf{E}_7$ and $\mathfrak{g}$ of the type C_3 then, for all n,

$$\dim V(n\omega)^{K'} = \binom{n+11}{11} - \binom{n+7}{11} = \dim S^n(\mathfrak{p}) - \dim S^{n-4}(\mathfrak{p}).$$

Proof. We have $V(n\omega)^{\mathrm{SU}_{2,l}} = V(n,n,0)$. By [FH, (25.39) on p. 427]), the restriction of $V_{n,0}$ to Sp_6 decomposes as

$$V_{n,0} \cong \oplus_{c\leq n} V(c,c,0).$$

Hence $V(n,n,0) \cong V_{n,0}/V_{n-1,0}$, as $\mathrm{SU}_{2,s}$-modules, and by Lemma 6.1,

$$\dim V(n\omega)^{K'} = \dim V_{n,0}^{\mathrm{SU}_{2,s}} - \dim V_{n-1,0}^{\mathrm{SU}_{2,s}} = \binom{n+5}{5} - \binom{n+4}{5} = \binom{n+4}{4}.$$

Next we prove (ii). We have $V(n\omega)^{\mathrm{SU}_{2,l}} \cong \oplus_{a+b=n} V_{a,b}$. Hence by Lemma 6.1,

$$\begin{aligned}\dim V(n\omega)^{K'} &= \sum_{a+b=n} V_{a,b}^{\mathrm{SU}_{2,s}} = \sum_{a+b=n} \binom{a+5}{5}\binom{b+5}{5} - \sum_{a+b=n} \binom{a+3}{5}\binom{b+3}{5} \\ &= \binom{n+11}{11} - \binom{n+7}{11}.\end{aligned}$$

□

6.2. **Quaternionic representations.** We recall some results from [GW] and [Lo00] on quaternionic representations. Let G be a quaternionic simple, simply connected real Lie group as in [GW], i.e. it has a maximal compact subgroup K isomorphic to $\mathrm{SU}_2 \times_2 M$. Let $T \subset \mathrm{SU}_2$ be a maximal Cartan subgroup. Let $\mathfrak{g}$, $\mathfrak{t}$ etc be the complexified Lie algebras of G, T etc, respectively. The adjoint action of T gives a $\mathbb{Z}$-grading of $\mathfrak{g}$. The grading defines a theta-stable maximal parabolic subalgebra $\mathfrak{q} = (\mathfrak{t} \oplus \mathfrak{m}) \oplus \mathfrak{n}$, where $\mathfrak{n}$ is a two step nilpotent Lie subalgebra with one-dimensional center $Z(\mathfrak{n})$. We identify T with U_1 (the group of complex numbers of norm one) so that $z \in \mathrm{U}_1$ acts by multiplication by z on $\mathfrak{n}/Z(\mathfrak{n})$ and by multiplication by z^2 on $Z(\mathfrak{n})$. Let $L = T \times_2 M$. Let W be an irreducible finite dimensional representation of L. The group $T \cong \mathrm{U}_1$ acts on W by multiplication by z^k for some integer k. Hence we shall also use the notation $W = W[k]$ to emphasize the action of T. In particular, $\mathbb{C}[k]$ will denote the representation of L trivial on M. Extend W trivially over $\mathfrak{n}$ so that it is a representation of $\mathfrak{q}$. For $k \geq 2$, the (unnormalized) cohomologically induced representation $\mathcal{R}^1(W) = \Gamma^1_{K/L}(\mathrm{Hom}_{\mathfrak{q},L}(U(\mathfrak{g}), W[k]))$ has $\mathrm{SU}_2 \times_2 M$-types of the form

$$\mathcal{R}^1(W) = \bigoplus_{n=0}^{\infty} S^{k+n-2}(\mathbb{C}^2) \otimes (\mathrm{Sym}^n(\mathfrak{n}_M) \otimes W_M)$$

where $\mathbb{C}^2$ is the standard, 2-dimensional representation of SU_2, W_M is the restriction of W to M and $\mathfrak{n}_M$ is the restriction of $\mathfrak{n}/Z(\mathfrak{n})$ to M. Let $\overline{\mathcal{R}}^1(W)$ be the unique irreducible subquotient $\mathcal{R}^1(W)$ containing the lowest K-type $S^{k-2}(\mathbb{C}^2) \otimes W_M$.

Let $\mathrm{E}_{7,4}$ be the simply connected quaternionic real form of type $\mathbf{E}_7$, and let σ_Z be its minimal representation constructed in [GW]. Let $\mathrm{F}_{4,4}$ be the split group of type $\mathbf{F}_4$. It is a quaternionic group, and its maximal compact subgroup is isomorphic to $\mathrm{SU}_2 \times_2 \mathrm{Sp}_6$. Let σ_X be the representation $\overline{\mathcal{R}}^1(\mathbb{C}[6])$ of $\mathrm{F}_{4,4}$. By [GW, Prop. 8.5] we have an exact sequence of Harish-Chandra modules of $\mathrm{F}_{4,4}$:

$$0 \to \sigma_X \to \mathcal{R}^1(\mathbb{C}[6]) \to \mathcal{R}^1(\mathbb{C}[10]) \to 0. \tag{6}$$

The group $\mathrm{E}_{7,4}$ contains a see-saw of dual pairs [Lo00]

$$\begin{tikzcd} \mathrm{F}_{4,4} \arrow[dr, dash] & \mathrm{SU}_6 \arrow[dl, dash] \\ \mathrm{SU}_{2,1} & \mathrm{SU}_{2,s} \end{tikzcd}$$

We shall restrict the minimal representation σ_Z to these two dual pairs. By [Lo97, Eqn. 6.6],

$$\sigma_Z^{\mathrm{SU}_{2,s}} \cong \sigma_X \tag{7}$$

as representations of $\mathrm{F}_{4,4}$. The restriction of σ_X to $\mathrm{SU}_{2,1} \times \mathrm{SU}_6$ is given in [Lo00]. In order to describe the result, note that $\mathrm{SU}_{2,1}$ is also a quaternionic group with the maximal compact subgroup isomorphic to $\mathrm{SU}_2 \times_2 \mathrm{U}_1$, i.e. $M = \mathrm{U}_1$ in this case. Irreducible representations of $M = \mathrm{U}_1$ are χ^n, where $\chi(z) = z$ for all $z \in \mathrm{U}_1$, and $n \in \mathbb{Z}$. If a, b be are non-negative integers, then $\mathcal{R}^1(\chi^{a-b}[6+a+b])$ is a quaternionic discrete series representation of $\mathrm{SU}_{2,1}$. Recall that $V_{a,b}$ is the irreducible representation of SU_6 of the highest weight $a\omega_2 + b\omega_4$.

Theorem 6.3. [Lo00, Theorem 6.1.2] *We have*

$$\mathrm{Res}^{\mathrm{E}_{7,4}}_{\mathrm{SU}_{2,1}\times\mathrm{SU}_6} \sigma_Z = \bigoplus_{a,b=0}^{\infty} \mathcal{R}^1(\chi^{a-b}[6+a+b]) \otimes V_{a,b}.$$

6.3. **Proof of Lemma 6.1.** It follows from this theorem and the see-saw pair that

$$\dim V_{a,b}^{\mathrm{SU}_{2,s}} = \dim \mathrm{Hom}_{\mathrm{SU}_{2,1}}\left(\sigma_Z^{\mathrm{SU}_{2,s}}, \mathcal{R}^1(\chi^{a-b}[6+a+b])\right),$$

where $\sigma_Z^{\mathrm{SU}_{2,s}} \cong \sigma_X$ by (7).

Let $V_6 = S^2\mathbb{C}^3$ be the 6 dimensional representation of SU_3. By the restriction formula in [Lo00], for $r \geq 6$,

$$\mathrm{Res}^{\mathrm{F}_{4,4}}_{\mathrm{SU}_{2,1}\times\mathrm{SU}_3} \mathcal{R}^1(\mathbb{C}[r]) = \bigoplus_{a,b=0}^{\infty} \mathcal{R}^1(\chi^{a-b}[r+a+b]) \otimes \left(\mathrm{Sym}^a V_6 \otimes \mathrm{Sym}^b(V_6^\vee)\right).$$

In particular, the above restriction is a direct sum of discrete series representation of $\mathrm{SU}_{2,1}$. Combining the above for $r = 6, 10$ and (6) we obtain

$$\begin{aligned} \dim V_{a,b}^{\mathrm{SU}_{2,s}} &= \dim\left(\mathrm{Sym}^a V_6 \otimes \mathrm{Sym}^b(V_6^\vee)\right) - \dim\left(\mathrm{Sym}^{a-2} V_6 \otimes \mathrm{Sym}^{b-2}(V_6^\vee)\right) \\ &= \binom{a+5}{5}\binom{b+5}{5} - \binom{a+3}{5}\binom{b+3}{5}. \end{aligned} \tag{8}$$

This proves Lemma 6.1. $\square$

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