TRANSFERS OF K-TYPES ON LOCAL THETA LIFTS OF CHARACTERS AND UNITARY LOWEST WEIGHT MODULES

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ABSTRACT. In this paper we study representations of the indefinite orthogonal group $\mathrm{O}(n,m)$ which are local theta lifts of one dimensional characters or unitary lowest weight modules of the double covers of the symplectic groups. We apply the transfer of K-types on these representations of $\mathrm{O}(n,m)$, and we study their effects on the dual pair correspondences. These results provide examples that the theta lifting is compatible with the transfer of K-types. Finally we will use these results to study subquotients of some cohomologically induced modules.

1. INTRODUCTION

In this paper we study representations of the indefinite orthogonal group $\mathrm{O}(n,m)$ which are local theta lifts of one dimensional characters or unitary lowest weight modules of the double covers of the symplectic groups. We apply the transfer of K-types on these representations of $\mathrm{O}(n,m)$, and we study their effects on the dual pair correspondences. Finally we will use these results to study subquotients of some cohomologically induced modules. Our methods could also be applied to $\mathrm{U}(n,m)$ and $\mathrm{Sp}(n,m)$.

We introduce the Harish-Chandra module $\theta_p^{n,m}(1)$ of $\mathrm{O}(n,m)$. Let $\widetilde{\mathrm{Sp}}(p(n+m),\mathbb{R})$ be the metaplectic double cover of $\mathrm{Sp}(p(n+m),\mathbb{R})$. It contains a dual pair $\left(\widetilde{\mathrm{Sp}}(p,\mathbb{R}),\widetilde{\mathrm{O}}(n,m)\right)$. Here $\widetilde{\mathrm{Sp}}(p,\mathbb{R})$ and $\widetilde{\mathrm{O}}(n,m)$ are (possibly split) double covers of $\mathrm{Sp}(p,\mathbb{R})$ and $\mathrm{O}(n,m)$ respectively. A Harish-Chandra module of $\widetilde{\mathrm{Sp}}(p,\mathbb{R})$ (resp. $\widetilde{\mathrm{O}}(n,m)$) is called genuine if it does not factor through the linear group $\mathrm{Sp}(p,\mathbb{R})$ (resp. $\mathrm{O}(n,m)$).

We will always assume that $\widetilde{\mathrm{Sp}}(p,\mathbb{R})$ splits over $\mathrm{Sp}(p,\mathbb{R})$, i.e. $\widetilde{\mathrm{Sp}}(p,\mathbb{R})\simeq\mathrm{Sp}(p,\mathbb{R})\times(\mathbb{Z}/2\mathbb{Z})$. This happens if and only if $m+n$ is even. Let ς' denote the genuine one dimensional character of $\widetilde{\mathrm{Sp}}(p,\mathbb{R})$ which is trivial on $\mathrm{Sp}(p,\mathbb{R})$ and nontrivial on $\mathbb{Z}/2\mathbb{Z}$. We fix an oscillator representation of $\widetilde{\mathrm{Sp}}(p(n+m),\mathbb{R})$ and we let U_0 denote the local theta lift of ς' to $\widetilde{\mathrm{O}}(n,m)$ with respect to this oscillator representation [H2]. The module U_0 is an irreducible Harish-Chandra module of $\widetilde{\mathrm{O}}(n,m)$. The group $\widetilde{\mathrm{O}}(n,m)$ splits over $\mathrm{O}(n,m)$ if and only if p is even. Since $\widetilde{\mathrm{O}}(n,m)$ has 8 connected components and it has four genuine one dimensional characters. We will explain a choice of a genuine character ς in the paragraph before (6). We set $\theta_p^{n,m}(1)=\varsigma U_0$ which is an irreducible Harish-Chandra module of $\mathrm{O}(n,m)$. We will call $\theta_p^{n,m}(1)$ the theta lift of the trivial character of $\mathrm{Sp}(p,\mathbb{R})$.

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We say that the dual pair $\left(\widetilde{\mathrm{Sp}}(p,\mathbb{R}), \widetilde{\mathrm{O}}(n,m)\right)$ is in the *stable range* if $2p \leq \min(n,m)$ and $2p < \max(n,m)$. Our definition excludes the case $m = n = 2p$. For simplicity, we will say that (p,n,m) is in stable range. By [Li], the Harish-Chandra module $\theta_p^{n,m}(1)$ is nonzero and unitarizable in the stable range.

Let $K_{n,m} = \mathrm{O}(n) \times \mathrm{O}(m)$ denote a maximal compact subgroup of $\mathrm{O}(n,m)$. The $K_{n,m}$-types of $\theta_p^{n,m}(1)$ are described in (6). It is $K_{n,m}$-multiplicity free and it is both $\mathrm{O}(n) \times 1$-admissible and $1 \times \mathrm{O}(m)$-admissible. Let $\mathfrak{g} = \mathfrak{so}(n+m,\mathbb{C})$, let $K_{n,r,m-r} = \mathrm{O}(n) \times \mathrm{O}(r) \times \mathrm{O}(m-r)$ be the subgroup of $K_{n,m}$ and let $\mathcal{F}_{\mathfrak{g},K_{n,m}}^{\mathfrak{g},K_{n,r,m-r}} \theta_p^{n,m}(1)$ denote the restriction of $\theta_p^{n,m}(1)$ as a $(\mathfrak{g}, K_{n,r,m-r})$-module. We will apply the transfer of K-types due to Enright and Wallach [W2]. More precisely we apply the middle degree pr-th derived functor $\Gamma^{pr} = (\Gamma_{\mathfrak{g},K_{n,r,m-r}}^{\mathfrak{g},K_{n+r,m-r}})^{pr}$ of the Zuckerman functor to $\mathcal{F}_{\mathfrak{g},K_{n,m}}^{\mathfrak{g},K_{n,r,m-r}} \theta_p^{n,m}(1)$. Since $K_{n+r,m-r}$ is not connected, special care is necessary. See Section 2.2 for a discussion. Let ρ_N denote the half sum of positive roots of $\mathfrak{so}(N)$ and let $\mathbf{1}_p = (1,\ldots,1) \in \mathbb{Z}^p$. We can now state the first main theorem of this paper.

Theorem 1.1. *Suppose $n+m$ is even and (p,n,m) is in the stable range. Let $0 < r < m$.*

(i) If $2p > m-r$, then $\Gamma^{pr}(\mathcal{F}_{\mathfrak{g},K_{n,m}}^{\mathfrak{g},K_{n,r,m-r}} \theta_p^{n,m}(1)) = 0$.

(ii) If $2p \leq m-r$, then

$$\Gamma^{pr}(\mathcal{F}_{\mathfrak{g},K_{n,m}}^{\mathfrak{g},K_{n,r,m-r}}(\theta_p^{n,m}(1))) = \theta_p^{n+r,m-r}(1)$$

as $(\mathfrak{so}(n+m,\mathbb{C}), K_{n+r,m-r})$-modules.

(iii) We set $N = n+m$. Then every representation in the collection

$$\{\theta_p^{a,b}(1) : a+b = N, a \geq 2p, b \geq 2p\}$$

has the same annihilator ideal in the universal enveloping algebra of $\mathfrak{so}(n+m,\mathbb{C})$. The annihilator ideal is the maximal primitive ideal with infinitesimal character $\rho_N - p\mathbf{1}_{N/2}$. See Section 2.1 for the notation on weights and infinitesimal characters.

(iv) Let $a = a_1 + a_2$ and $b = b_1 + b_2$. Suppose $\theta_p^{a_1+b_1,a_2+b_2}(1) \to \pi_a \boxtimes \pi_b$ is a nontrivial quotient where π_a and π_b are irreducible modules of $(\mathfrak{so}(a,\mathbb{C}), K_{a_1,a_2})$ and $(\mathfrak{so}(b,\mathbb{C}), K_{b_1,b_2})$ respectively. Then the infinitesimal characters of π_a and π_b must respect the following correspondence:

$$(\lambda_1,\ldots,\lambda_p,\rho_{a-2p}) \longleftrightarrow (\lambda_1,\ldots,\lambda_p,\rho_{b-2p}) \qquad \text{if } b \geq a \geq 2p, \tag{1}$$

$$(\lambda_1,\ldots,\lambda_s) \longleftrightarrow (\lambda_1,\ldots,\lambda_s,\rho_{b-a} - \frac{2p-a}{2}\mathbf{1}_{\frac{b-a}{2}}) \qquad \text{if } a < 2p \tag{2}$$

where $s = [\frac{a}{2}]$.

The module $\theta_p^{n,m}(1)$ has been investigated in a number of papers [HZ], [KØ], [LZ], [Z1].

Irreducible representations of $\mathrm{O}(r')$ are parameterized by certain arrays of nonnegative integers $\mu = (\mu_1,\ldots,\mu_{r'}) \in \mathbb{Z}^{r'}$. See Section 2.1. We will use $\mu_{\mathrm{O}(r')}$ or simply μ to denote the corresponding irreducible finite dimensional representation of $\mathrm{O}(r')$. Then the local theta lift $L(\mu)$ of μ to a (possibly split) double cover $\widetilde{\mathrm{Sp}}(p,\mathbb{R})$ is a unitarizable lowest weight module. A result of [EHW] states that conversely, a unitarizable lowest weight module of the connected component of a double cover of $\mathrm{Sp}(p,\mathbb{R})$ is the restriction of a unique $L(\mu)$. Let U_1 be the local theta lift of $L(\mu)$ to $\widetilde{\mathrm{O}}(n,r)$. If it is nonzero, we will choose a genuine character ς'' of $\widetilde{\mathrm{O}}(n,r)$ so that $\theta_p^{n,r}(L(\mu)) := \varsigma'' U_1$ is a Harish-Chandra module of $\mathrm{O}(n,r)$ and it has a $K_{n,r}$-type decomposition as in (8). We remark that $\theta_p^{n,r}(L(\mu))$ is $\mathrm{O}(n)\times 1$-admissible but it is almost never K-multiplicity free. Let $\mathfrak{g}_1 = \mathfrak{so}(n+r,\mathbb{C})$. We will state our second main theorem.

Theorem 1.2. *Suppose $(p,n,r+r')$ is in the stable range. Let μ and $\theta_p^{n,r}(L(\mu))$ as above. Let $0<t<r$. Let $\Gamma_1^{pt} = (\Gamma_{\mathfrak{g}_1,K_{n,t,r-t}}^{\mathfrak{g}_1,K_{n+t,r-t}})^{pt}$ denote the pt-th derived Zuckerman functor.*

(i) If $2p > r+r'-t$, then $\Gamma_1^{pt}(\mathcal{F}_{\mathfrak{g}_1,K_{n,r}}^{\mathfrak{g}_1,K_{n,t,r-t}}\theta_p^{n,r}(L(\mu))) = 0$.
(ii) If $2p \le r+r'-t$, then

$$\Gamma_1^{pt}(\mathcal{F}_{\mathfrak{g}_1,K_{n,r}}^{\mathfrak{g}_1,K_{n,t,r-t}}\theta_p^{n,r}(L(\mu))) = \theta_p^{n+t,r-t}(L(\mu))$$

as $(\mathfrak{g}_1, K_{n+r,r-t})$-modules.

In (ii), it is possible that $\theta_p^{n+t,r-t}(L(\mu)) = 0$ and when this happens, the above theorem says that the left hand side of (ii) is also zero. If we set $r'=0$ in the above theorem, then we recover parts (i) and (ii) of Theorem 1.1. However the proof of Theorem 1.2 requires Theorem 1.1.

We will prove Theorems 1.1 and 1.2 in Section 3.

In Theorems 1.1 and 1.2, we have assumed that (p,n,m) and $(p,n,r+r')$ lie in the stable range. In Section 3.4, we will address the situation when we are outside of this range.

We will briefly explain our motivation. It is a well known fact that the correspondences of the infinitesimal characters induced by the local theta lifts are independent of the real forms of the dual pair [H1] [Pz]. We expect that the local theta lifts of a representation to different real forms share many more interesting properties which are related by the transfer of K-types. For example, Conjecture 5.1 in [WZ] predicts an explicit relationship between the transfers of K-types and the local theta lifts of characters of $\mathrm{O}(n,m)$ to $\widetilde{\mathrm{Sp}}(p,\mathbb{R})$. In a forth coming paper of the second author, J.-j. Ma will show that theta lifts of one dimensional representations and the transfers of K-types are compatible operations and extend Theorems 1.1 and 1.2 to dual pairs $(\mathrm{U}(n,m),\mathrm{U}(p,q))$ and $(\mathrm{Sp}(n,m),\mathrm{O}^*(2p))$. Also see [Ta].

1.3. **Cohomological inductions.** We will apply the above two theorems to some cohomological induced modules. In order to state our next main result, we recall some basic definitions and notation from [KV] and [W1].

We suppose $2p \le n \le m$, $2p < m$ and $m = r+r'$. Let $\mathfrak{g}_1 = \mathfrak{so}(n,r)\otimes\mathbb{C} = \mathfrak{so}(n+r,\mathbb{C})$. Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{so}(n)\oplus\mathfrak{so}(r)$ in $\mathfrak{so}(n,r)$. Let $\lambda_0' = (p,p-1,\ldots,1,0,0,\ldots,0) \in \sqrt{-1}\mathfrak{t}_0^*$. Let $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{n}'$ be the maximal θ-stable parabolic subalgebra in $\mathfrak{g}_1$ defined by λ_0'. The real form of the Levi subalgebra is $\mathfrak{l}_0' = \mathfrak{u}(1)^p \oplus \mathfrak{so}(n-2p,m)$. Let $L' = \mathrm{U}(1)^p \times \mathrm{O}(n-2p,r)$ be the subgroup in $\mathrm{O}(n,r)$ whose Lie algebra is $\mathfrak{l}_0'$. Let $\mathbb{C}_\lambda$ be a character of $\mathrm{U}(1)^p$. We extend it to a character of L' such that $\mathrm{O}(n-2p,r)$ acts on it by

$\det^p_{\mathrm{O}(n-2p,r)}$. Let $\mathfrak{g}_1 = \mathfrak{k}_{n,r} \oplus \mathfrak{p}_{n,r}$ denote the complexification of the Cartan decomposition of $\mathfrak{so}(n,r)$. Let $M' = L' \cap K_{n,r} = \mathrm{U}(1)^p \times K_{n-2p,r}$ and

$$s'_0 = \dim(\mathfrak{k}_{n,r} \cap \mathfrak{n}') = p(n-p-1).$$

We refer to (5.3a) on page 328 in [KV] and define a $(\mathfrak{g}_1, K_{n,m})$-module

$$A(\lambda) = \mathcal{L}_{s'_0}(\mathbb{C}_\lambda) = (P^{\mathfrak{g}_1,K_{n,r}}_{\overline{\mathfrak{q}}',M'})_{s'_0}(\mathcal{F}^{\overline{\mathfrak{q}}',M'}_{\mathfrak{l}',M'}(\mathbb{C}^\sharp_\lambda)),$$

where $P^{\mathfrak{g}_1,K_{n,r}}_{\overline{\mathfrak{q}}',M'}$ is the induction functor in Section II.1 in [KV] and $(P^{\mathfrak{g}_1,K_{n,r}}_{\overline{\mathfrak{q}}',M'})_{s'_0}$ is its s'_0-th derived functor. Also see Section 4. The module $A(\lambda)$ has infinitesimal character $\lambda + \rho_{n+r}$.

Given a λ as in Theorem 1.4 below, we will show in Lemma 4.1 that the bottom layer $K_{n,r}$-type is the minimal $K_{n,r}$-type of $A(\lambda)$. Let $\overline{A}(\lambda)$ denote the irreducible subquotient of $A(\lambda)$ generated by the minimal $K_{n,r}$-type. We can now state our main result on cohomological inductions.

Theorem 1.4. *Suppose $2p \leq n \leq m$, $2p < m$.*

(i) The irreducible $(\mathfrak{so}(n+m,\mathbb{C}), K_{n,m})$-modules $\theta^{n,m}_p(1)$ and $\overline{A}(\lambda)$ are isomorphic where $\lambda = -\frac{m+n}{2}\mathbf{1}_p$.

(ii) Let $0 \leq r < m$ and let $\mu = (\mu_1, \ldots, \mu_{r'})$ such that $\mu_i = 0$ if $i > p$. Then the irreducible $(\mathfrak{so}(n+r,\mathbb{C}), K_{n,r})$-modules $\theta^{n,r}_p(L(\mu))$ and $\overline{A}(\lambda)$ are isomorphic where $\lambda = (\mu_1, \ldots, \mu_p) + \frac{m-n-2r}{2}\mathbf{1}_p$.

In (i), we set $r = m$. In (ii), both $L(\mu)$ and $\theta^{n,r}_p(L(\mu))$ are zero if $\mu_{p+1} > 0$.

We refer to page 330 in [KV] for the definition of the $(\mathfrak{g}_1, K^0_{n,m})$-module $A_{\mathfrak{q}'}(\lambda) = \mathcal{L}_{s'_0}(\mathbb{C}_\lambda)$. Let $\mathcal{F}^{\mathfrak{g}_1,K^0_{n,r}}_{\mathfrak{g}_1,K_{n,r}}A(\lambda)$ denote the restriction of $A(\lambda)$ as a $(\mathfrak{g}_1, K^0_{n,m})$-module. We will show in Proposition 4.2 that $\mathcal{F}^{\mathfrak{g}_1,K^0_{n,r}}_{\mathfrak{g}_1,K_{n,r}}A(\lambda)$ contains the $(\mathfrak{g}_1, K^0_{n,m})$-module $A_{\mathfrak{q}'}(\lambda)$ as a submodule. If $n > 2p$, then $\mathcal{F}^{\mathfrak{g}_1,K^0_{n,r}}_{\mathfrak{g}_1,K_{n,r}}A(\lambda) = A_{\mathfrak{q}'}(\lambda)$. We remark that $A_{\mathfrak{q}'}(\lambda)$ in the above theorem is not always in the good or weakly good range (see Definition 0.49 in [KV]). It is interesting to find unitarizable subquotients generated by the images of the bottom layer maps.

We will explain the motivation of the above theorem. First we consider the special case when $n = 2p < r$, $\mu = 0$ and $\lambda = \frac{m-n-2r}{2}\mathbf{1}_p$. In [Kn], Knapp constructed a unitarizable quotient A' of $A_{\mathfrak{q}'}(\lambda)$ by extending the method of Gross and Wallach [GW]. The quotient module A' contains the image of the bottom layer map. He asked if A' is irreducible and if A' is related to local theta lifts. The first question was answered by Trapa where he showed that A' is irreducible [T]. He also computed its associated cycle. In [PT], Trapa and Paul show that A' is a submodule of $\theta^{2p,r}_p(L(0))$. Theorem 1.4 could be considered a generalization of these results.

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2. The Zuckerman functors

The objectives of this section are to set up some notation and define the Zuckerman functors in Theorems 1.1 and 1.2.

2.1. **Weights.** Let $\mathbf{1}_n := (1, 1, \ldots, 1)$ and $\mathbf{0}_n := (0, 0, \ldots, 0)$ in $\mathbb{R}^n$. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^m$, then we denote $(\lambda_1, \ldots, \lambda_n, \xi_1, \ldots, \xi_m) \in \mathbb{R}^{n+m}$ by (λ, ξ).

Let $r = [\frac{n}{2}]$ and let $\rho_n \in \mathbb{R}^r$ denote the half sum of positive roots of $\mathfrak{so}(n, \mathbb{C})$. Let $\mu \in \mathbb{R}^r$ be a highest weight of $\mathfrak{so}(n, \mathbb{C})$. We will use $\mu_{\mathfrak{so}(n)}$ or simply μ to denote the corresponding irreducible finite dimensional representation of $\mathfrak{so}(n)$.

We will denote the trivial and the determinant representation of $\mathrm{O}(n)$ by $\mathbb{C}_{\mathrm{O}(n)}$ and d_n respectively. Irreducible representations of $\mathrm{O}(n)$ are parameterized by arrays of the form

$$\mu = (a_1, \ldots, a_k, \mathbf{0}_{n-k}) \text{ or } (a_1, \ldots, a_k, \mathbf{1}_{n-2k}, \mathbf{0}_k) \tag{3}$$

in $\mathbb{Z}^n$ where a_i are positive integers, $a_i \geq a_{i+1}$ and $k \leq [\frac{n}{2}]$. See [GoW]. We will call these arrays *weights* of $\mathrm{O}(n)$. Let $\Lambda(\mathrm{O}(n))$ denote the set of such weights. Given such a weight μ, we will use $\mu_{\mathrm{O}(n)}$ or simply μ denote the corresponding irreducible finite dimensional representation of $\mathrm{O}(n)$. The two representations corresponding to the two highest weights in (3) differ by the character d_n. In general, we would ignore the string of zeros at the end of a weight $(a_1, \ldots, a_k, \mathbf{0}_{n-k})$ and write it as $(a_1, \ldots, a_k)$ instead.

Finally we recall a branching rule: The dimension of $\mathrm{Hom}_{\mathrm{O}(n)}(\mu_{\mathrm{O}(n)}, \mu'_{\mathrm{O}(n+1)})$ is at most one and it is one if and only if $\mu'_i \geq \mu_i \geq \mu'_{i+1}$ for all $i = 1, \ldots, n$.

Throughout this paper, we denote $K_n = \mathrm{O}(n)$, $K_{n,m} = \mathrm{O}(n) \times \mathrm{O}(m)$, $K_{n,r,r'} = \mathrm{O}(n) \times \mathrm{O}(r) \times \mathrm{O}(r')$ and $K^0_{n,m} = \mathrm{SO}(n) \times \mathrm{SO}(m)$. Let $\mathfrak{g}_1 = \mathfrak{so}(n + r, \mathbb{C})$. Given an irreducible $(\mathfrak{g}_1, K_{n,r})$-module, we will follow Harish-Chandra parametrization and use a weight μ of $\mathfrak{g}_1$ to denote its infinitesimal character. We note that the infinitesimal character is a character of $U(\mathfrak{g}_1)^{\mathrm{O}(n+r,\mathbb{C})}$. Two weights μ and μ' give the same infinitesimal character if and only if $\mu = w\mu'$ for some w in the Weyl group of $\mathrm{O}(n + r, \mathbb{C})$.

2.2. **Zuckerman functors.** Let W be a $(\mathfrak{g}_1, K_{n,r})$-module. We will follow Section II.8.5 in [BW] and [W1] where it is established that i-th Zuckerman functor

$$\Gamma^i(W) = (\Gamma^{\mathfrak{g}_1, K_{n+r}}_{\mathfrak{g}_1, K_{n,r}})^i(W) = \bigoplus_F \mathrm{H}^i(\mathfrak{g}_1, K_{n,r}; W \otimes F^*) \otimes F. \tag{4}$$

Here the sum is taken over all irreducible finite dimensional representations F of $\mathrm{O}(n+r)$. Hence $\Gamma^i(W)$ is nonzero if and only if the Lie algebra cohomology

$$\mathrm{H}^i(\mathfrak{g}_1, K_{n,r}; W \otimes F^*)$$

is nonzero for some F. Suppose the above Lie algebra cohomology is nonzero, then by Wigner's lemma, W and F have the same infinitesimal character. In particular W has regular infinitesimal character. Since $K_{n,r}$ contains the center of $\mathrm{O}(n, r)$, W and F have the same central character, again by Wigner's lemma. Let $\mathfrak{g}_1 = \mathfrak{k}_{n,r} \oplus \mathfrak{p}_{n,r}$ be the complexified Cartan decomposition. By Proposition 9.4.3 in [W1], if W is unitarizable, then

$$\mathrm{H}^i(\mathfrak{g}_1, K_{n,r}; W \otimes F^*) = \mathrm{Hom}_{K_{n,r}}(\wedge^i \mathfrak{p}_{n,r}, W \otimes F^*) = \mathrm{Hom}_{K_{n,r}}(F \otimes \wedge^i \mathfrak{p}_{n,r}, W). \tag{5}$$

We remark that [W1] requires that the maximal compact subgroup is connected. However the proof there works for $K_{n,r}$ too without any modification.

Let $\mathfrak{g} = \mathfrak{so}(n+r+r', \mathbb{C})$. Suppose W is a $(\mathfrak{g}, K_{n,r,r'})$-module. Let $\mathcal{F}W := \mathcal{F}^{\mathfrak{g}_1, K_{n,r}}_{\mathfrak{g}, K_{n,r,r'}} W$ denote the restriction of W as a $(\mathfrak{g}_1, K_{n,r})$-module. By the naturality property of the Zuckerman functor in Chapter 6 of [W1], the $(\mathfrak{g}, K_{n,r,r'})$-module structure on W gives a natural $(\mathfrak{g}, K_{n+r,r'})$-module structure on $\Gamma^i(\mathcal{F}W)$. This $(\mathfrak{g}, K_{n+r,r'})$-module $\Gamma^i(\mathcal{F}W)$ is isomorphic to $(\Gamma^{\mathfrak{g}, K_{n+r,r'}}_{\mathfrak{g}, K_{n,r,r'}})^i W$ by Section II.8.5(1) in [BW]. In other words, $(\Gamma^{\mathfrak{g}, K_{n+r,r'}}_{\mathfrak{g}, K_{n,r,r'}})^i (W)$ is computed by $\Gamma^i(\mathcal{F}W)$ in the category of $(\mathfrak{g}_1, K_{n,r})$-modules.

3. Proofs of Theorems 1.1 and 1.2

First we recall some facts about the local theta lift U_0 of the character ς' of $\widetilde{\mathrm{Sp}}(p, \mathbb{R})$ in [Lo]. We will assume that $n+m$ is even and (p, n, m) is in the stable range. There are four choices of the genuine character ς of $\widetilde{\mathrm{O}}(n, m)$ as mentioned in the introduction. We will fix a choice of ς so that $\theta^{n,m}_p(1) = \varsigma U_0$ has $K_{n,m}$-types

$$\theta^{n,m}_p(1) = \bigoplus_{l=(l_1,\ldots,l_p)} \left(\mathrm{d}^p_n \, (l + \frac{m-n}{2}\mathbf{1}_p, \mathbf{0}_{n-p})_{\mathrm{O}(n)} \right) \boxtimes (l, \mathbf{0}_{m-p})_{\mathrm{O}(m)} \tag{6}$$

where $\mathrm{d}_n = \det_{\mathrm{O}(n)}$ and the sum is taken over $l = (l_1, \ldots, l_p)$ such that l_i are non-negative integers and $l_1 \geq l_2 \geq \ldots \geq l_p \geq \min(\frac{n-m}{2}, 0)$. The module $\theta^{n,m}_p(1)$ is unitarizable. The minimal $K_{n,m}$-type $\tau_{\min}$ is $\mathrm{d}^p_n(\frac{m-n}{2}\mathbf{1}_p, \mathbf{0}_{n-p})_{\mathrm{O}(n)} \boxtimes \mathbb{C}_{\mathrm{O}(m)}$ if $m \geq n$ and $\mathrm{d}^p_n \boxtimes (\frac{n-m}{2}\mathbf{1}_p, \mathbf{0}_{m-p})_{\mathrm{O}(m)}$ if $m < n$.

Let $\mathfrak{g} = \mathfrak{so}(n+m, \mathbb{C})$ and $K^0_{n,m} = \mathrm{SO}(n) \times \mathrm{SO}(m)$. If $\min(n, m) = 2p$ then $\theta^{n,m}_p(1)$ splits into a direct sum of two irreducible $(\mathfrak{g}, K^0_{n,m})$-modules. If $n, m > 2p$, then it is an irreducible $(\mathfrak{g}, K^0_{n,m})$-module.

Let $m = r + r'$ and let $\mathfrak{g}_1 = \mathfrak{so}(n+r, \mathbb{C})$. Since $\theta^{n,m}_p(1)$ is unitarizable and $\mathrm{O}(n) \times 1$-admissible, the restriction of $\theta^{n,m}_p(1)$ as a $(\mathfrak{g}_1, K_{n,r}) \times \mathrm{O}(r')$-module decomposes discretely as a direct sum

$$\theta^{n,m}_p(1) = \bigoplus_{\mu \in \Lambda(\mathrm{O}(r'))} \Omega(\mu) \boxtimes \mu_{\mathrm{O}(r')}. \tag{7}$$

It follows that $\Omega(\mu)$ is unitarizable and it has $K_{n,r}$-types

$$\Omega(\mu) = \bigoplus_{l=(l_1,\ldots,l_p)} \left(\mathrm{d}^p_n \, (l + \frac{m-n}{2}\mathbf{1}_p, \mathbf{0}_{n-p})_{\mathrm{O}(n)} \boxtimes \bigoplus_{\kappa} m(l, \mu, \kappa) \kappa_{\mathrm{O}(r)} \right) \tag{8}$$

where the sum is as in (6) and $m(l, \mu, \kappa)$ is the multiplicity of $\kappa_{\mathrm{O}(r)} \boxtimes \mu_{\mathrm{O}(r')}$ in $(l, \mathbf{0}_{m-p})_{\mathrm{O}(m)}$.

Suppose $k \leq p$ and $k \leq r'$. We lift $\mu_{\mathrm{O}(r')} = (\mu_1, \ldots, \mu_k, \mathbf{0}_{r'-k})_{\mathrm{O}(r')}$ to a unitarizable lowest weight module $L(\mu)$ of a double cover of $\mathrm{Sp}(p, \mathbb{R})$. The lowest $\widetilde{\mathrm{U}}(p)$-type has highest weight (see [KaV])

$$(\mu_1 \ldots, \mu_k, \mathbf{0}_{p-k}) + \frac{r'}{2}\mathbf{1}_p.$$

Let U_1 be the local theta lift of $L(\mu)$ to $\widetilde{\mathrm{O}}(n,r)$. In [Lo] we proved that U_1 is an irreducible and unitarizable Harish-Chandra module of $\widetilde{\mathrm{O}}(n,r)$. There exists a genuine character ς'' of $\widetilde{\mathrm{O}}(n,r)$ such that $\Omega(\mu) = \varsigma'' U_1$. Furthermore $\Omega(\mu) \neq 0$ if and only if $U_1 \neq 0$. From now on, we will denote $\Omega(\mu)$ by $\theta_p^{n,r}(L(\mu))$.

Let $\delta = \frac{n-m+2r}{2}$. Let $s = [r'/2]$ and let $\lambda = (\lambda_1, \dots, \lambda_s)$ denote the infinitesimal character of $\mu_{\mathrm{O}(r')}$.

Lemma 3.1. *Suppose (p,n,m) is in stable range and $m = r + r'$.*

(i) If $r' \geq 2p$, $\theta_p^{n,r}(L(\mu))$ has infinitesimal character $(\mu - \delta \mathbf{1}_p, \mathbf{0}) + \rho_{n+r}$. The infinitesimal character is regular if and only if $\mu_p \geq \delta$.

(ii) If $r' < 2p$, then $\theta_p^{n,r}(L(\mu))$ has infinitesimal character $(\lambda_1, \dots, \lambda_s, \rho_{2\delta} - \frac{2p-r'}{2}\mathbf{1}_\delta)$. The infinitesimal character is singular.

Proof. This follows from the correspondences of infinitesimal characters $\mathfrak{so}(r', \mathbb{C}) \leftrightarrow \mathfrak{sp}(p, \mathbb{C})$ and $\mathfrak{sp}(p,\mathbb{C}) \leftrightarrow \mathfrak{so}(n+r, \mathbb{C})$ established by the oscillator representations [H1] [Pz]. We will leave the details to the reader. □

We apply $(\Gamma^{\mathfrak{g}, K_{n+r,r'}}_{\mathfrak{g}, K_{n,r,r'}})^{pr}$ defined in Section 2.2 to (7) and we get

$$(9) \qquad (\Gamma^{\mathfrak{g}, K_{n+r,r'}}_{\mathfrak{g}, K_{n,r,r'}})^{pr} (\mathcal{F}^{\mathfrak{g}, K_{n,r,r'}}_{\mathfrak{g}, K_{n,m}} \theta_p^{n,m}(1)) = \bigoplus_{\mu \in \Lambda(\mathrm{O}(r'))} (\Gamma^{\mathfrak{g}_1, K_{n+r}}_{\mathfrak{g}_1, K_{n,r}})^{pr} (\theta_p^{n,r}(L(\mu))) \boxtimes \mu_{\mathrm{O}(r')}.$$

As explained in Section 2.2, the equality in the above equation holds by the naturality the Zuckerman functor in Chapter 6 in [W1].

By Lemma 3.1(ii), if $r' < 2p$ then $\theta_p^{n,r}(L(\mu))$ has singular infinitesimal characters and $(\Gamma^{\mathfrak{g}_1, K_{n+r}}_{\mathfrak{g}_1, K_{n,r}})^{pr}(\theta_p^{n,r}(L(\mu))) = 0$. This proves Theorem 1.1(i).

From now on, we will assume that $n, r' \geq 2p$ and $\mu_p \geq \delta$. Let $\mathfrak{t}$ be a Cartan subalgebra of the complexified Lie algebra $\mathfrak{k}_{n,r}$ of $K_{n,r}$. We set

$$(10) \qquad \xi = \xi(\mu) := (\mu - \delta \mathbf{1}_p, \mathbf{0}_{n+r-p})$$

and $\bar{\xi} = \bar{\xi}(\mu) := (\mu - \delta \mathbf{1}_p, \mathbf{0}) \in \mathfrak{t}^*$. Then both $\theta_p^{n,r}(L(\mu))$ and $\xi(\mu)_{\mathrm{O}(n+r)}$ have regular infinitesimal character $\bar{\xi}(\mu) + \rho_{n+r}$.

We refer to the $K_{n,r}$-types of $\theta_p^{n,r}(L(\mu))$ in (8). By the branching rule described in Section 2.1, we see that the minimal $K_{n,r}$-type of $\theta_p^{n,r}(L(\mu))$ is

$$(11) \qquad \tau_{\min}(\mu) := \mathrm{d}_n^p(\mu + \frac{m-n}{2}\mathbf{1}_p, \mathbf{0}_{n-p})_{\mathrm{O}(n)} \boxtimes \mathbb{C}_{\mathrm{O}(r)}.$$

Proposition 3.2 below computes $(\Gamma^{\mathfrak{g}_1, K_{n+r}}_{\mathfrak{g}_1, K_{n,r}})^{pr}(\theta_p^{n,r}(L(\mu)))$. For connected groups, this is a special case of Section 5 in [VZ] or Theorem 9.5.3 in [W1]. We will give a proof for completeness.

Proposition 3.2. *If $\theta_p^{n,r}(L(\mu))$ has regular infinitesimal character $\bar{\xi}(\mu) + \rho_{n+r}$, then*

$$(\Gamma^{\mathfrak{g}_1, K_{n+r}}_{\mathfrak{g}_1, K_{n,r}})^{pr}(\theta_p^{n,r}(L(\mu))) = \mathrm{d}_{n+r}^p \xi(\mu)_{\mathrm{O}(n+r)}.$$

Proof. Let $\mathfrak{so}(n+r,\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. We set $i = pr$ and $W = \theta_p^{n,r}(L(\mu))$ in (4) and (5). By Section 2.2, (5) is nonzero only if F is an irreducible finite dimensional representation of $\mathrm{O}(n+r)$ with the same infinitesimal character as $\theta_p^{n,r}(L(\mu))$. Hence we may assume F is either $\xi(\mu)_{\mathrm{O}(n+r)}$ or $\mathrm{d}_{n+r}\xi(\mu)_{\mathrm{O}(n+r)}$. Now it is suffices to show that

$$\mathrm{Hom}_{K_{n,r}}(F \otimes \wedge^{pr}\mathfrak{p}, \theta_p^{n,r}(L(\mu))) \tag{12}$$

is one dimensional if $F = \mathrm{d}_{n+r}^p\xi(\mu)_{\mathrm{O}(n+r)}$, and zero if otherwise.

We note that F restricts irreducibly to $\mathrm{SO}(n+r)$ so it is a quotient of the generalized Verma module $U(\mathfrak{g}_1) \otimes_{\mathfrak{q}'} \mathbb{C}_{\overline{\xi}} = U(\overline{\mathfrak{n}'}) \otimes_{\mathbb{C}} \mathbb{C}_{\overline{\xi}}$. Hence a $\mathfrak{t}$-weight of F is of the form $\mu - \delta\mathbf{1}_p - \sum\alpha$ where $\sum\alpha$ is a sum some positive roots in $\mathfrak{n}'$.

Let ϕ be a nonzero homomorphism of (12). Let $\tau = \mathrm{d}_n^p(l + \frac{m-n}{2}\mathbf{1}_p)_{\mathrm{O}(n)} \otimes \kappa_{\mathrm{O}(r)}$ be a $K_{n,r}$-type of $\theta_p^{n,r}(L(\mu))$ in (8). Suppose τ is in the image of ϕ. Then the image contains the $\mathfrak{t}$-weight $l + \frac{m-n}{2}\mathbf{1}_p + \bar{\kappa}$ where $\bar{\kappa}$ denotes a highest $\mathfrak{t}$-weight of $\kappa_{\mathrm{O}(r)}$. There is a $\mathfrak{t}$-weight $\mu - \delta\mathbf{1}_p - \sum\alpha$ of F and a $\mathfrak{t}$-weight $\sum\beta + \sum\gamma - \sum\gamma'$ of $\wedge^{pr}\mathfrak{p}$ such that

$$\mu - \delta\mathbf{1}_p - \sum\alpha + \sum\beta + \sum\gamma - \sum\gamma' = l + \frac{m-n}{2}\mathbf{1}_p + \bar{\kappa}. \tag{13}$$

Here $\sum\beta$ is a sum of distinct roots in $\mathfrak{p} \cap \mathfrak{l}'$ and, $\sum\gamma$ and $\sum\gamma'$ are sums of distinct roots in $\mathfrak{p} \cap \mathfrak{n}'$. Taking inner products with $\eta = (\mathbf{1}_p, \mathbf{0})$ on both sides and rearrange the terms give

$$\sum_{i=1}^{p}(l_i - \mu_i) + \langle \eta, \sum\alpha \rangle = -\Big(pr - \langle \eta, \sum\gamma \rangle\Big) - \langle \eta, \sum\gamma' \rangle \le 0.$$

By the description of the $K_{n,r}$-types in (11), the left hand side is non-negative. Hence the left hand side is zero. This gives $l = \mu$, $\sum\alpha = 0$, $\sum\gamma' = 0$, $\sum\gamma = r\mathbf{1}_p$. Since $l = \mu$, τ is the minimal $K_{n,r}$-type $\tau_{\min}(\mu)$ and $\kappa = 0$. Putting these back into (13) gives $\sum\beta = 0$.

Let v_F be a highest weight vector of F and let v be a nonzero vector in $\wedge^{pr}(\mathfrak{p} \cap \mathfrak{n}') = \wedge^{\mathrm{top}}(\mathfrak{p} \cap \mathfrak{n}')$. Then $v_F \otimes v$ is a highest weight vector with respect to $\mathfrak{so}(n) \oplus \mathfrak{so}(r)$ in $F \otimes \wedge^{pr}\mathfrak{p}$. Let F' denote the irreducible $K_{n,r}$-submodule generated by $v_F \otimes v$. Now (12) becomes

$$\mathrm{Hom}_{K_{n,r}}(F \otimes \wedge^{pr}\mathfrak{p}, \theta_p^{n,r}(L(\mu))) = \mathrm{Hom}_{K_{n,r}}(F', \tau_{\min}(\mu)).$$

Indeed the $\mathfrak{t}$-weights argument above showed that the right hand side contains the left hand side. The left hand side clearly contains the right hand side. Finally the right hand side has dimension 0 or 1. It is 1 if and only if $F = \mathrm{d}_{n+r}^p\xi(\mu)_{\mathrm{O}(n+r)}$. This proves the lemma. □

Before completing the proof of Theorem 1.1, we need a characterization of $\theta_p^{n,m}(1)$.

Lemma 3.3. *Suppose (p,n,m) is in stable range. Let W_0 be a Harish-Chandra module of $\mathrm{O}(n,m)$ which has the same infinitesimal character and $K_{n,m}$-types as $\theta_p^{n,m}(1)$. If $n, m > 2p$, then we further assume that W_0 respects the correspondence of infinitesimal characters in Lemma 3.1(i) for the pairs $\mathfrak{so}(n+1) \times \mathfrak{so}(m-1)$ and $\mathfrak{so}(n-1) \times \mathfrak{so}(m+1)$. Then W_0 is isomorphic to $\theta_p^{n,m}(1)$.*

Proof. If $m = n$, then $\theta_p^{n,n}(1)$ is a representation with scalar K-type by (6). Hence it is uniquely determined by its infinitesimal character and K-types. For example see [Z2]. This proves that $\theta_p^{n,n}(1) = W_0$.

We now suppose that $m > n \geq 2p$. We consider the restriction of W_0 as an $(\mathfrak{so}(n+1,\mathbb{C}), K_{n,1}) \times \mathrm{O}(m-1)$-module:

$$W_0 = \bigoplus_{\mu \in \Lambda(\mathrm{O}(m-1))} W_0(\mu) \boxtimes \mu_{\mathrm{O}(m-1)}.$$

We claim $W_0(\mu)$ and $\theta_p^{n,1}(L(\mu))$ are isomorphic $(\mathfrak{so}(n+1,\mathbb{C}), K_{n,1})$-modules. Indeed by our hypothesis, $W_0(\mu)$ has the same $K_{n,1}$-types as $\theta_p^{n,1}(L(\mu))$. If $n = 2p$, then $\theta_p^{n,1}(L(\mu))$ is an irreducible discrete series representation. It is a well known result of Schmid that an irreducible discrete series representation is uniquely determined by its $K_{n,1}$-types [Sc]. Hence $W_0(\mu) \simeq \theta_p^{n,1}(L(\mu))$. If $n > 2p$, then $\theta_p^{n,1}(L(\mu))$ is an irreducible $(\mathfrak{so}(n+1,\mathbb{C}), \mathrm{SO}(n))$-module. If we set $r = 1$, then $2p < n+1 \leq m-1$ and Lemma 3.1(i) applies. Hence $W_0(\mu)$ has the same infinitesimal character as $\theta_p^{n,1}(L(\mu))$. It is well known that irreducible $(\mathfrak{so}(n+1,\mathbb{C}), \mathrm{SO}(n))$-modules are characterized by its infinitesimal characters and $\mathrm{SO}(n)$-types. This implies that $\theta_p^{n,1}(L(\mu))$ and W_0 are isomorphic $(\mathfrak{so}(n+1,\mathbb{C}), \mathrm{SO}(n))$-modules. They extend to the same $K_{n,1}$-module because they have the same $K_{n,1}$-types. This proves our claim.

Let $\theta = \theta_p^{n,m}(1)$. Let $\tau^l = \mathrm{d}_n^p(l + \frac{m-n}{2}\mathbf{1}_p, \mathbf{0}_{n-p})_{\mathrm{O}(n)} \boxtimes (l, \mathbf{0}_{m-p})_{\mathrm{O}(m)}$ denote the K-type of θ in (6). Let τ_0^l denote the isomorphic $K_{n,m}$-type in W_0. We would like to construct a $K_{n,m}$-module isomorphism $\phi_l : \tau^l \to \tau_0^l$ such that $\phi := \oplus_l \phi_l : \theta \to W_0$ becomes an $(\mathfrak{so}(n+m,\mathbb{C}), K_{n,m})$-isomorphism. This would prove the lemma. In order to achieve this, we will follow the method in Proposition 8.2.1(i) in [Lo]. There we construct an $(\mathfrak{so}(n+m,\mathbb{C}), K_{n,m})$-isomorphism from θ to its Hermitian dual θ^h. The construction is valid if we replace θ^h with W_0. A main ingredient of the proof is the claim above that $W_0(\mu)$ and $\theta_p^{n,1}(L(\mu))$ are isomorphic $(\mathfrak{so}(n+1,\mathbb{C}), K_{n,1})$-modules. We refer the reader to Section 8.5 in [Lo] for details.

Finally suppose $m < n$. Then we interchange the role of m and n and proceed as before. This proves the lemma. □

Remark. If $m \geq n = 2p$, then the above lemma could be deduced from [T] where Trapa computes the characteristic cycles and annihilator ideal of θ. It is very likely that his method could be extended to give a proof the lemma without the assumption on the correspondences of infinitesimal characters.

Proof of Theorem 1.1. We have proved (i) in the paragraph after (9). Next we prove (ii). The condition $r' \geq 2p$ implies that $\theta_p^{n,r}(L(\mu))$ could have regular infinitesimal character. Let $\Gamma_0 = \Gamma_{\mathfrak{g},K_{n,r,r'}}^{\mathfrak{g},K_{n+r,r'}}$ be the Zuckerman functor. We set $W_0 = \Gamma_0^{pr}(\mathcal{F}_{\mathfrak{g},K_{n,m}}^{\mathfrak{g},K_{n,r,r'}} \theta_p^{n,m}(1))$ to be the left hand side of (9). By Proposition 3.2, $(\Gamma_{\mathfrak{g}_1,K_{n,r}}^{\mathfrak{g}_1,K_{n+r}})^{pr}(\theta_p^{n,r}(L(\mu))) = \mathrm{d}_{n+r}^p \xi(\mu)_{\mathrm{O}(n+r)}$. Putting this back into (9) and comparing it with (6), we conclude that W_0 has the same $K_{n+r,r'}$-types as $\theta_p^{n+r,r'}(1)$. By the naturality of the Zuckerman functor, the annihilator ideal of a $U(\mathfrak{so}(n+m,\mathbb{C}))$-module V will also annihilate $\Gamma_0^i(V)$. Let $I_p^{n,m}$ denote the annihilator ideal of $\theta_p^{n,m}(1)$. Then $I_p^{n,m}$ annihilates W_0. In particular the correspondence of infinitesimal characters in Lemma 3.1(i) holds for W_0. By Lemma 3.3, $W_0 = \theta_p^{n+r,r'}(1)$. This proves (ii).

Next we prove (iii). In the proof of (ii), we have shown that $I_p^{n,m}$ annihilates $W_0 = \theta_p^{n+r,r'}(1)$, i.e. $I_p^{n,m} \subseteq I_p^{n+r,r'}$. Similarly if we interchange the role of n and m, then we get $(\Gamma^{\mathfrak{g},K_{n,m}}_{\mathfrak{g},K_{n,r,r'}})^{pr}(\mathcal{F}^{\mathfrak{g},K_{n,r,r'}}_{\mathfrak{g},K_{n+r,r'}}\theta_p^{n+r,r'}) = \theta_p^{n,m}(1)$ which implies $I_p^{n,m} \supseteq I_p^{n+r,r'}$. This proves the first assertion of (iii). The last assertion follows immediately from [T] which states that the annihilator ideal of $\theta_p^{2p,m+n-2p}(1)$ is the maximal primitive ideal with infinitesimal character $\rho_{n+m} - p\mathbf{1}_{\frac{n+m}{2}}$.

We will now prove (iv). The correspondence of infinitesimal characters is a result of the annihilator ideal so it is independent of the real form. Hence we may assume that $a_1 = 0$, $a_2 = r$, $(b_1, b_2) = (n, r')$, and $m = a_2 + b_2 = r + r'$. In this case we have verified the correspondence in Lemma 3.1. This proves (iv). This also completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We set $m = r + r'$. Applying $(\Gamma^{\mathfrak{g},K_{n+t,m-t}}_{\mathfrak{g},K_{n,t,m-t}})^{pt}$ to (7), we get, as $(\mathfrak{g}_1, K_{n,r}) \times \mathrm{O}(r')$-module,

$$(\Gamma^{\mathfrak{g},K_{n+t,m-t}}_{\mathfrak{g},K_{n,t,m-t}})^{pt}(\mathcal{F}^{\mathfrak{g},K_{n+t,m-t}}_{\mathfrak{g},K_{n,t,m-t}}\theta_p^{n,m}(1)) = \bigoplus_{\mu\in\Lambda(\mathrm{O}(r'))} (\Gamma^{\mathfrak{g}_1,K_{n+t,r-t}}_{\mathfrak{g}_1,K_{n,t,r-t}})^{pt}(\mathcal{F}^{\mathfrak{g}_1,K_{n,t,r-t}}_{\mathfrak{g}_1,K_{n,r}}\theta_p^{n,r}(L(\mu))) \boxtimes \mu_{\mathrm{O}(r')}.$$

By Theorem 1.1, the left hand side is $\theta_p^{n+t,m-t}(1)$. Theorem 1.2 follows by comparing the above equation with (7) for $\theta_p^{n+t,m-t}(1)$. □

3.4. **Outside of stable range.** Let $m = r + r'$. We will discuss the situation when (p, n, m) is outside of the stable range. As in the stable range case, we will multiply the local theta lifts by suitable genuine characters to get Harish-Chandra modules $\theta_p^{n,m}(1)$ and $\theta_p^{n,r}(L(\mu))$ of the linear groups $\mathrm{O}(n,m)$ and $\mathrm{O}(n,r)$ respectively. We will also refer $\theta_p^{n,m}(1)$ and $\theta_p^{n,r}(L(\mu))$ as the local theta lifts.

First we look at $\theta_p^{n,m}(1)$. By symmetry, we assume that $n \le m$. Outside the stable range, $\theta_p^{n,m}(1)$ is nonzero if and only if

(I) $p \le n \le 2p - 1$ and $m = n + 2$ or
(II) $n = m \le 2p$.

First we consider (I). Let d_n denote the one dimensional character of $\mathrm{O}(n, n+2)$ which is $\det_{\mathrm{O}(n)}$ on $\mathrm{O}(n)$ and trivial on $\mathrm{O}(n+2)$. Then by Lemma 3.5.1 in [Lo], $\mathrm{d}_n\theta_p^{n,n+2}(1)$ has the same $K_{n,n+2}$-types as those of $\theta_{n-p}^{n,n+2}(1)$ in the stable range. By Proposition 6.3.1 in [Lo], the correspondence of the infinitesimal characters between $\mathfrak{so}(n+1, \mathbb{C})$ and $\mathfrak{so}(m-1, \mathbb{C}) = \mathfrak{so}(n+1, \mathbb{C})$ is given by the identity map. Hence Lemma 3.3 gives $\mathrm{d}_n\theta_p^{n,n+2}(1) = \theta_{n-p}^{n,n+2}(1)$, and we are back to the stable range. Equation (7) is valid for $\theta_p^{n,m}(1)$ so we also have $\theta_p^{n,r}(L(\mu)) = \theta_{n-p}^{n,r}(L(\mu))$ if $m = n + 2 = r + r'$.

For (II), $\theta_p^{n,n}(1)$ is the dual representation of $\theta_{n-1-p}^{n,n}(1)$ in the stable range [Z1]. Since all the $K_{n,r}$-types are self dual and -1 belongs to the Weyl group, Lemma 3.1(i) holds for $\theta_p^{n,n}(1)$. By Lemma 3.3, $\theta_p^{n,n}(1) = \theta_{n-1-p}^{n,n}(1)$, and we are back to the stable range.

For all the cases considered so far, we could reduce the computation to the stable range. We will leave it to the readers to formulate the corresponding Theorems 1.1 and 1.2.

Unfortunately in Case (II), (7) is no longer valid. It is only true if we replace $\theta_p^{n,m}(1)$ and $\theta_p^{n,r}(L(\mu))$ by the maximal Howe quotients $\Theta_p^{n,m}(1)$ and $\Theta_p^{n,r}(L(\mu))$ respectively. We refer to [Lo] for the definition of maximal Howe quotients. Then $\theta_p^{n,m}(1)$ and $\theta_p^{n,r}(L(\mu))$ are the unique irreducible quotients of $\Theta_p^{n,m}(1)$ and $\Theta_p^{n,r}(L(\mu))$ respectively. This complicates the investigation of the transfers of $K_{n,r}$-types for $\theta_p^{n,r}(L(\mu))$. One could manage by a careful analysis of the submodules and the K-types of $\Theta_p^{n,m}(1)$ and $\Theta_p^{n,r}(L(\mu))$ as given in [Lo]. However it is tedious so we will omit this case.

4. Cohomologically induced modules

The objective of this section is to setup some notation for computing $A(\lambda)$ and $\overline{A}(\lambda)$ in Theorem 1.4. We will follow the notation in [KV].

From now on, we assume that $2p \leq n \leq m$, $2p < m$ and $m = r+r'$. We use a subscript 0 to denote a real Lie algebra. Those without are complex Lie algebras. Let $\mathfrak{g}_1 = \mathfrak{so}(n+r, \mathbb{C})$. In Section 1.3, we have chosen a compact Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{so}(n, \mathbb{R}) \oplus \mathfrak{so}(r, \mathbb{R})$ in $\mathfrak{so}(n, r)$. Let $\mathfrak{t} = \mathfrak{t}_0 \otimes \mathbb{C}$ and $t = \dim \mathfrak{t} = [\frac{n}{2}] + [\frac{r}{2}]$. We choose a positive root system $\Phi^+(\mathfrak{t})$ with respect to $\mathfrak{t}$ such that the positive roots $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq [\frac{n}{2}]$ belongs to $\mathfrak{so}(n)$, and $\varepsilon_i \pm \varepsilon_j$ for $[\frac{n}{2}] + 1 \leq i < j \leq [\frac{n}{2}] + [\frac{r}{2}]$ belongs to $\mathfrak{so}(r)$.

Let $\lambda_0 = (\mathbf{1}_p, \mathbf{0}_{t-p})$ and $\lambda_0' = (p, p-1, \ldots, 1, \mathbf{0}_{t-p}) \in \sqrt{-1}\mathfrak{t}_0^*$. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{n}'$ be the θ-stable parabolic subalgebras in $\mathfrak{g}_1$ defined by λ_0 and λ_0' respectively. We have $\mathfrak{l}_0 = \mathfrak{u}(p) \oplus \mathfrak{so}(n-2p, r)$ and $\mathfrak{l}_0' = \mathfrak{u}(1)^p \oplus \mathfrak{so}(n-2p, r)$. Let

$$L = \mathrm{U}(p) \times \mathrm{O}(n-2p, r) \text{ and } L' = \mathrm{U}(1)^p \times \mathrm{O}(n-2p, r)$$

be the subgroups in $\mathrm{O}(n, r)$ whose Lie algebras are $\mathfrak{l}_0$ and $\mathfrak{l}_0'$ respectively. Let $M = \mathrm{U}(p) \times K_{n-2p,r}$ and $M' = \mathrm{U}(1)^p \times K_{n-2p,r}$ be maximal compact subgroups of L and L' respectively. We note that $\mathfrak{q}'$, L' and M' are the same as those in Section 1.3.

Under the adjoint action of L, the radical $\mathfrak{n}$ of $\mathfrak{q}$ decomposes as

$$\mathfrak{n} = \wedge^2(\mathbb{C}^p) \oplus (\mathbb{C}^p \otimes \mathbb{C}^{n-2p+r})$$

where $\mathbb{C}^p$ is the standard representation of $\mathrm{U}(p)$ and $\mathbb{C}^{n-2p+r}$ is the standard representation of $\mathrm{O}(n-2p, r)$. Hence

$$\wedge^{\mathrm{top}}\mathfrak{n} \cong \det{}_{\mathrm{U}(p)}^{n+r-p-1} \otimes \det{}_{\mathrm{O}(n-2p,r)}^{p}. \tag{14}$$

Let $\mu = (\mu_1, \ldots, \mu_{m-r})$ be a highest weight of $\mathrm{O}(m-r)$ such that $\mu_i = 0$ for $i > p$. We set $Z = Z(\lambda)$ to be the irreducible finite dimensional $\mathrm{U}(p)$-module of highest weight

$$\lambda = \lambda_\mu = (\mu_1, \ldots, \mu_p) + \frac{m-n-2r}{2}\mathbf{1}_p. \tag{15}$$

We let $\mathrm{O}(n-2p, r)$ act on it by $\det^p_{\mathrm{O}(n-2p,r)}$. We continue to denote its highest $\mathfrak{t}$-weight by λ. Let $\overline{\mathfrak{q}}$ denote the parabolic subalgebra that is opposite to $\mathfrak{q}$. We put $Z^\sharp = Z \otimes \wedge^{\mathrm{top}}\mathfrak{n}$.

Let $M = L \cap K_{n,r} \cong \mathrm{U}(p) \times \mathrm{O}(n-2p) \times \mathrm{O}(r)$. Let $\mathcal{F}_{\mathfrak{l},M}^{\bar{\mathfrak{q}},M} Z^\sharp$ denote the extension of $Z^\sharp$ to a $\bar{\mathfrak{q}}$-module where $\bar{\mathfrak{n}}$ acts trivially. We define the $(\mathfrak{g}_1, M)$-module

$$\mathrm{ind}_{\bar{\mathfrak{q}},M}^{\mathfrak{g}_1,M}(\mathcal{F}_{\mathfrak{l},M}^{\bar{\mathfrak{q}},M} Z^\sharp) = U(\mathfrak{g}_1) \otimes_{\bar{\mathfrak{q}}} Z^\sharp.$$

We will write $\mathrm{ind} Z^\sharp$ if there is no fear of confusion.

We set $s_0 = \dim(\mathfrak{n} \cap \mathfrak{k}) = p(n-2p) + \frac{p(p-1)}{2}$. Note that s_0 is independent of r. Let $s_1 = \frac{p(p-1)}{2}$ be the dimension of a maximal nilpotent subalgebra of $\mathfrak{gl}(p)$. Let $K = K_{n,r}$ and let $(\Pi_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K})_j$ be the j-th derived functor of the Bernstein functor. Let $\mathbb{C}_\lambda$ be a character of L' as in Section 1.3 and let $\mathbb{C}_\lambda^\sharp = \mathbb{C}_\lambda \otimes \wedge^{\mathrm{top}} \mathfrak{n}'$. We recall $A(\lambda)$ in Section 1.3. We claim that

$$A(\lambda) = (\Pi_{\mathfrak{g}_1,M'}^{\mathfrak{g}_1,K})_{s_0+s_1}(\mathrm{ind}_{\mathfrak{q}',M'}^{\mathfrak{g}_1,M'} \mathbb{C}_\lambda^\sharp) = (\Pi_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K})_{s_0}(\mathrm{ind} Z^\sharp). \tag{16}$$

The claim follows from the Borel-Weil-Bott-Kostant theorem and a standard spectral sequence argument. For example see Corollary 11.86(a) on page 683 in [KV].

Instead of working with Bernstein functor modules, we prefer to use Zuckerman functor modules. By the Hard Duality Theorem 3.5 in [KV], we have

$$A(\lambda) = (\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K})^{s_0} \mathrm{ind} Z^\sharp. \tag{17}$$

The module $A(\lambda)$ has infinitesimal character $\lambda + \rho(\mathfrak{g}_1)$.

Lemma 4.1. *Let $\lambda = \lambda_\mu = \mu + \frac{m-n-2r}{2}\mathbf{1}_p$, Z and $A(\lambda)$ as above.*

(i) The module $A(\lambda)$ is $\mathrm{O}(n)$-admissible. An $\mathrm{O}(n)$-type of $A(\lambda)$ has highest weight

$$\mathrm{d}_n^p\left(\mu_1 + \kappa_1 + \frac{m-n}{2}, \ldots, \mu_p + \kappa_p + \frac{m-n}{2}, \mathbf{0}_{n-p}\right)$$

where κ_i are non-negative integers.

(ii) The module $A(\lambda)$ contains the $K_{n,r}$-type $\tau_{\min} = \mathrm{d}_n^p(\mu + \frac{m-n}{2}\mathbf{1}_p)_{\mathrm{O}(n)} \otimes \mathbb{C}_{\mathrm{O}(m)}$ with multiplicity one. It is the minimal $K_{n,r}$-type. It is also the image of the bottom layer map.

In (ii), we refer to page 642 in [KV] for the definition of minimal K-types for disconnected K.

We postpone the proof of the above lemma to Section 5.4. Alternatively one could also verify this lemma directly using the Blattner's formula (see Theorem 5.64 in [KV]). The fact that $A(\lambda)$ is admissible with respect to $\mathrm{SO}(n)$ also follows from a very general criterion in [Ko].

By (ii), $\overline{A}(\lambda)$ is the irreducible $(\mathfrak{g}_1, K_{n,r})$-subquotient of $A(\lambda)$ generated by the minimal $K_{n,r}$-type $\tau_{\min}$ in (ii).

Let $K = K_{n,r}$ and $K^1 = MK_{n,r}^0$. If $n > 2p$ then $K^1 = K$. If $n = 2p$ then $K^1 = \mathrm{SO}(n) \times \mathrm{O}(r)$ is a subgroup of index two in K. Let $\mathcal{C}(\mathfrak{g}_1, K)$ denote the category of $(\mathfrak{g}_1, K)$-modules etc... For $V \in \mathcal{C}(\mathfrak{g}_1, K^1)$, we define

$$\mathrm{induced}_{\mathfrak{g}_1,K^1}^{\mathfrak{g}_1,K} V = \{f : K \to V : f(k_1 k) = k_1 f(k) \text{ for all } k_1 \in K^1, k \in K\}.$$

Applying Proposition 2.77 in [KV] to (17) gives

$$A(\lambda) = \mathrm{induced}_{\mathfrak{g}_1,K^1}^{\mathfrak{g}_1,K}(\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K^1})^{s_0}(\mathrm{ind}Z^\sharp) \tag{18}$$

where $(\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K^1})^{s_0}$ is the s_0-th derived functor of the Zuckerman functor. Let $\mathcal{C}(\mathfrak{g}_1, K)$ denote the category of $(\mathfrak{g}_1, K)$-modules etc... The above equation shows that we could compute $A(\lambda)$ first in $\mathcal{C}(\mathfrak{g}_1, K^1)$ and then apply the exact functor $\mathrm{induced}_{\mathfrak{g}_1,K^1}^{\mathfrak{g}_1,K}$.

Given a module V in $\mathcal{C}(\mathfrak{g}_1, K)$, we let $\mathcal{F}_{\mathfrak{g}_1,K}^{\mathfrak{g}_1,K^0} V \in \mathcal{C}(\mathfrak{g}_1, K^0)$ denote the restriction of V as a $(\mathfrak{g}_1, K^0)$-module. We recall the $(\mathfrak{g}_1, K^0)$-module $A_{\mathfrak{q}'}(\lambda) = (P_{\bar{\mathfrak{q}}',(M')^0}^{\mathfrak{g},K^0})_{s_0'}\mathbb{C}_\lambda^\sharp$ on page 330 in [KV].

Proposition 4.2. *We have* $A_{\mathfrak{q}'}(\lambda) = \mathcal{F}_{\mathfrak{g}_1,K^1}^{\mathfrak{g}_1,K^0}(\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K^1})^{s_0}(\mathrm{ind}Z^\sharp) \subseteq \mathcal{F}_{\mathfrak{g}_1,K}^{\mathfrak{g}_1,K^0} A(\lambda)$.

If $n > 2p$, then $K = K^1$ and $A_{\mathfrak{q}'}(\lambda) = \mathcal{F}_{\mathfrak{g}_1,K}^{\mathfrak{g}_1,K^0} A(\lambda)$.

Proof. The inclusion on the right follows from (18). By the spectral sequence and the Hard Duality Theorem (cf. (16) and (17)), we have

$$A_{\mathfrak{q}'}(\lambda) = (\Pi_{\mathfrak{g}_1,M^0}^{\mathfrak{g}_1,K^0})_{s_0}(\mathrm{ind}_{\bar{\mathfrak{q}},M^0}^{\mathfrak{g}_1,M^0}(\mathcal{F}_{\mathfrak{l},M^0}^{\bar{\mathfrak{q}},M^0} Z^\sharp)) = (\Gamma_{\mathfrak{g}_1,M^0}^{\mathfrak{g}_1,K^0})^{s_0}(\mathrm{ind}_{\bar{\mathfrak{q}},M^0}^{\mathfrak{g}_1,M^0}(\mathcal{F}_{\mathfrak{l},M^0}^{\bar{\mathfrak{q}},M^0} Z^\sharp)). \tag{19}$$

Note that $M \cap K^0$ is the identity connected component M^0 of M. By Section 6.2 of [Vo] or Section I.8.9 in [BW],

$$\mathcal{F}_{\mathfrak{g}_1,K^1}^{\mathfrak{g}_1,K^0} \circ (\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K^1})^{s_0} = (\Gamma_{\mathfrak{g}_1,M^0}^{\mathfrak{g}_1,K^0})^{s_0} \circ \mathcal{F}_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,M^0}. \tag{20}$$

Applying this functor to $\mathrm{ind}Z^\sharp$ and comparing it with (19) gives the first equality of the proposition. □

4.3. Although the group M is different from $K_{n,r,m-r}$, Section 2.2 applies to $\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K}V$. More precisely, let $V \in \mathcal{C}(\mathfrak{g}_1, M)$. We define $(\Gamma_M^K)^i V = H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ where $\mathcal{H}(K)$ denotes the bi-K-finite continuous functions on K. Then $(\Gamma_M^K)^i V$ has a natural $(\mathfrak{g}_1, K)$-module structure and $(\Gamma_M^K)^i V = (\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K})^i V$ in $\mathcal{C}(\mathfrak{g}_1, K)$. Since $K/M \simeq \mathrm{O}(n)/(\mathrm{U}(p) \times \mathrm{O}(n-2p))$, one could naturally identify $(\Gamma_M^K)^i \simeq (\Gamma_{\mathrm{U}(p)\times\mathrm{O}(n-2p)}^{\mathrm{O}(n)})^i$. This allows us to compute $(\Gamma_{\mathfrak{g}_1,M}^{\mathfrak{g}_1,K})^i V$ within the category $\mathcal{C}(\mathfrak{so}(n, \mathbb{C}), \mathrm{U}(p) \times \mathrm{O}(n-2p))$.

5. Proof of Theorem 1.4

This section contains the proofs of Lemma 4.1 and Theorem 1.4. The initial argument leading to Lemma 5.5 follows Section 4 in [LS] so we will omit details and refer the reader to that paper. Unfortunately we cannot totally dispense with it because we will need them later in the proof of Theorem 1.4.

We set $\mathfrak{g} = \mathfrak{so}(n+m, \mathbb{C})$ and $\mathfrak{g}_1 = \mathfrak{so}(n+r, \mathbb{C})$. We recall the theta stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ of $\mathfrak{g}$ where $\mathfrak{l} = \mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{so}(n+m-2p, \mathbb{C})$. We define $\mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{so}(n+r, \mathbb{C}) = \mathfrak{l}_1 \oplus \mathfrak{n}_1$. We have a decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ such that $\mathfrak{n}_2 = \mathbb{C}^p \otimes \mathbb{C}^{m-r}$ is a tensor product of standard representations of $\mathrm{U}(p)$ and $\mathrm{O}(m-r)$, while the group

$\mathrm{O}(n-2p,r)$ acts trivially on it. We extend $\mathfrak{n}_2$ to a representation of $\mathrm{U}(p)\times\mathrm{U}(m-r)$. It is well known that (see [GoW] and [H3])

$$\mathrm{Sym}^N\mathfrak{n}_2=\sum_\mu \mu_{\mathrm{U}(p)}\otimes\mu_{\mathrm{U}(m-r)}$$

where the sum is taken over all partitions μ of N such that the length of μ is not longer than $\min(p,m-r)$. We restrict the summand $\mu_{\mathrm{U}(m-r)}$ to $\mathrm{O}(m-r)$ and we denote it by $\mu_{\mathrm{U}(m-r)}|_{\mathrm{O}(m-r)}$.

With reference to (14) and (15), let $\lambda=-\frac{n+m}{2}\mathbf{1}_p$, $Z=\det_{\mathrm{U}(p)}^{-\frac{n+m}{2}}\otimes\det^p_{\mathrm{O}(n-2p,m)}$ and

$$Z^\sharp=\det{}_{\mathrm{U}(p)}^{\frac{n+m}{2}-p-1}\otimes\mathbb{C}_{\mathrm{O}(n-2p,m)}.$$

Let symm : $\mathrm{Sym}(\mathfrak{g})\to U(\mathfrak{g})$ denote the symmetrization map (see §0.4.2 in [W1]). Let $M=\mathrm{U}(p)\times K_{n-2p,m}$. By the Poincare-Birkhoff-Witt theorem,

$$\mathrm{ind}Z^\sharp=\mathrm{ind}_{\bar{\mathfrak{q}},M}^{\mathfrak{g},M}Z^\sharp=U(\mathfrak{g})\otimes_{\bar{\mathfrak{q}}}Z^\sharp=U(\mathfrak{n}_1)\otimes\mathrm{symm}(\mathrm{Sym}(\mathfrak{n}_2))\otimes Z^\sharp \tag{21}$$

as $\mathrm{U}(p)\times K_{n-2p,r}\times\mathrm{O}(m-r)$-module. We define F_N to be the $\mathfrak{g}_1$-submodule of $\mathrm{ind}Z^\sharp$ generated by $1\otimes\mathrm{symm}(S_N(\mathfrak{n}_2))\otimes Z^\sharp$. Hence $\{F_N : N=0,1,2,\ldots\}$ forms an exhaustive increasing filtration of $(\mathfrak{g}_1,\mathrm{U}(p)\times K_{n-2p,r})\times\mathrm{O}(m-r)$-submodules of $\mathrm{ind}Z^\sharp$. We set

$$V_r(\mu)=\mathrm{ind}_{(\bar{\mathfrak{q}}_1,\mathrm{U}(p)\times K_{n-2p,r})}^{(\mathfrak{g}_1,\mathrm{U}(p)\times K_{n-2p,r})}\mathcal{F}_{\mathfrak{l}_1,\mathrm{U}(p)\times K_{n-2p,r}}^{\bar{\mathfrak{q}}_1,\mathrm{U}(p)\times K_{n-2q,r}}\left((\mu+(\frac{m+n}{2}-2p)\mathbf{1}_p)_{\mathrm{U}(p)}\otimes\mathbb{C}_{\mathrm{O}(n-2p,r)}\right).$$

Let $\mathcal{B}$ (resp. $\mathcal{B}(N)$) denote the partitions (resp. partitions of N) of length not more than $\min(p,m-r)$. We will now state a special case of a known fact which is used in proof of the Blattner's formula in [KV].

Lemma 5.1. *For every positive integer N, we have an isomorphism of $(\mathfrak{g}_1,\mathrm{U}(p)\times K_{n-2p,r})\times\mathrm{O}(m-r)$-modules*

$$F_N/F_{N-1}=\bigoplus_{\mu\in\mathcal{B}(N)}V_r(\mu)\otimes(\mu_{\mathrm{U}(m-r)}|_{\mathrm{O}(m-r)}).\quad\square$$

Case 1. We first consider the case $r=0$, $\mathfrak{g}_1=\mathfrak{so}_n(\mathbb{C})$ and $V_r(\mu)=V_0(\mu)$. By an irreducibility criterion in Theorem 9.12 in [Hu], for $\mu\in\mathcal{B}$, $V_0(\mu)$ is an irreducible generalized Verma module of $\mathfrak{so}(n)$. The infinitesimal character of $V_0(\mu)$ is the same as the infinitesimal of $(\mu+(\frac{m-n}{2})\mathbf{1}_p)_{\mathrm{O}(n)}$. Hence these infinitesimal characters are pairwise different for different partitions μ. It follows that the filtration F_N splits and (21) becomes

$$\mathrm{ind}Z^\sharp=\bigoplus_{\mu\in\mathcal{B}}V_0(\mu)\otimes(\mu_{\mathrm{U}(m)}|_{\mathrm{O}(m)}) \tag{22}$$

where the sum is taken over all partitions μ of length no more than $\min(p,m)$. Let $\Gamma=\Gamma_M^{K_{n,m}}=\Gamma_{\mathrm{U}(p)\times\mathrm{O}(n-2p)}^{\mathrm{O}(n)}$ as in Section 4.3. Applying its derived functor Γ^j to (22) gives

$$\Gamma^j(\mathrm{ind}Z^\sharp)=\bigoplus_{\mu\in\mathcal{B}}\Gamma^j(V_0(\mu))\otimes(\mu_{\mathrm{U}(m)}|_{\mathrm{O}(m)}).$$

where we recall $\lambda = -\frac{n+m}{2}\mathbf{1}_p$. Since $s_0 = \dim(\mathfrak{n}_1)$ in this case, by the Borel-Weil-Bott-Kostant theorem, $\Gamma^j(V_0(\mu)) = 0$ if $j \neq s_0$ and $\Gamma^{s_0}(V_0(\mu)) = \mathrm{d}_n^p(\mu + \frac{m-n}{2}\mathbf{1}_p)_{\mathrm{O}(n)}$. When $j = s_0$, the above equation becomes

$$A(\lambda) = \bigoplus_{\mu\in\mathcal{B}} \mathrm{d}_n^p(\mu + \frac{m-n}{2}\mathbf{1}_p)_{\mathrm{O}(n)} \otimes (\mu_{\mathrm{U}(m)}|_{\mathrm{O}(m)}) \tag{23}$$

and it has minimal $K_{n,m}$-type $\tau_{\min} = \mathrm{d}_n^p(\frac{m-n}{2}\mathbf{1}_p)_{\mathrm{O}(n)} \otimes \mathbb{C}_{\mathrm{O}(m)}$ which occurs with multiplicity one.

Lemma 5.2. *Suppose W_1, W_2, W_3 are $(\mathfrak{so}(n), \mathrm{U}(p) \times \mathrm{O}(n-2p))$-subquotients of* $\mathrm{ind}Z^\sharp$ *and suppose they satisfy an exact sequence*

$$0 \to W_1 \to W_2 \to W_3 \to 0.$$

(i) Each W_k is a direct summand of (22) *and $W_k = \bigoplus_{\mu'} V_0(\mu')$ is a direct sum taken over certain partitions μ' of length no more than* $\min(p, m)$.
(ii) If $j \neq s_0$, then $\Gamma^j W_k = 0$. Moreover we have an exact sequence

$$0 \to \Gamma^{s_0} W_1 \to \Gamma^{s_0} W_2 \to \Gamma^{s_0} W_3 \to 0.$$

(iii) Each $\Gamma^{s_0} W_j$ is $\mathrm{O}(n)$*-admissible.*

Proof. Part (i) follows from (22). Part (ii) follows from (i) and the Borel-Weil-Bott-Kostant theorem alluded above. By (ii) each $\Gamma^{s_0} W_j$ is an $\mathrm{O}(n)$-subquotient of $A(\lambda)$ in (23) which is $\mathrm{O}(n)$-admissible. This proves (iii). $\square$

Case 2. Now we return to the general r for $\mathfrak{g}_1 = \mathfrak{so}(n+r, \mathbb{C})$. Let $V_r(\mu)$ be a subquotient in Lemma 5.1. By Lemma 5.2(i), $V_r(\mu) = \bigoplus_{\mu'} V_0(\mu')$.

Lemma 5.3. *If $V_0(\mu') \subseteq V_r(\mu)$, then $\mu' = \mu + \kappa$ where κ is a p-tuple of non-negative integers. Furthermore $V_r(\mu)$ contains $V_0(\mu)$ with multiplicity one and $\mathrm{O}(r)$ acts trivially on it.*

The proof consists of setting up a filtration of $(\mathfrak{so}(n, \mathbb{C}), \mathrm{U}(p) \times K_{n-2p})$-submodules of $V_r(\mu)$ in the same fashion as Lemma 5.1 and study the graded module. We refer to Lemma 4.5 in [LS] for a detailed proof of a similar result.

5.4. *Proof of Lemma 4.1.* By Lemma 5.3 above, $V_r(\mu) = \bigoplus_\kappa V_0(\mu+\kappa)$ where the sum κ is taken with multiplicities over a set of p-tuples of non-negative integers. Let $\lambda = -\frac{n+m}{2}\mathbf{1}_p$ and $\lambda_\mu = \mu + (\frac{m-n-2r}{2})\mathbf{1}_p$. Applying $\Gamma^{s_0} = (\Gamma^{K_{n,r}}_{\mathrm{U}(p)\times K_{n-2p,r}})^{s_0} = (\Gamma^{\mathrm{O}(n)}_{\mathrm{U}(p)\times \mathrm{O}(n-2p)})^{s_0}$ to $V_r(\mu)$ gives

$$A(\lambda_\mu) = \Gamma^{s_0}(V_r(\mu)) = \bigoplus_\kappa \Gamma^{s_0} V_0(\mu+\kappa) = \bigoplus_\kappa \mathrm{d}_n^p(\mu + \kappa + \frac{m-n}{2}\mathbf{1}_p)_{\mathrm{O}(n)}. \tag{24}$$

It is $\mathrm{O}(n)$-admissible by Lemma 5.2(iii). Furthermore by the second assertion of Lemma 5.3, $A(\lambda_\mu)$ above contains $\mathrm{d}_n^p(\mu + \frac{m-n}{2}\mathbf{1}_p)_{\mathrm{O}(n)} \otimes \mathbb{C}_{\mathrm{O}(r)}$ with multiplicity one. Clearly this is the minimal $K_{n,r}$-type of $A(\lambda_\mu)$. This proves Lemma 4.1. $\square$

Applying Lemma 5.2 to the filtration F_N gives the next lemma.

Lemma 5.5. *In the category of* $(\mathfrak{g}_1, K_{n,r}) \times \mathrm{O}(m-r)$*-modules,* $\Gamma^{s_0}(F_N)$ *is an exhaustive increasing filtration of* $A(\lambda)$ *and*

$$\Gamma^{s_0}(F_N)/\Gamma^{s_0}(F_{N-1}) = \Gamma^{s_0}(F_N/F_{N-1}) = \bigoplus_{\mu \in \mathcal{B}(N)} A(\lambda_\mu) \otimes (\mu_{\mathrm{U}(m-r)}|_{\mathrm{O}(m-r)}). \quad \square$$

Proof of Theorem 1.4(i). Let $c = \frac{n+m}{2}$, $\lambda = -c\mathbf{1}_p$ and $\mathfrak{g} = \mathfrak{so}(c, \mathbb{C})$. We will denote the $(\mathfrak{g}, K_{n,m})$-module $\overline{A}(\lambda)$ by $\overline{A}_{n,m}(\lambda)$. Let $M_{n,m} = \mathrm{U}(p) \times K_{n-2p,m}$ and let $\Gamma^i_{n,m}$ denote the derived Zuckerman functor $(\Gamma^{K_{n,m}}_{M_{n,m}})^i$ as in Section 4.3.

First we consider the special case when $n = 2p < m = 2c - 2p$ and $s_0 = p(p-1)/2$. Let $\mathrm{ind}Z^\sharp$ denote $\mathrm{ind}^{\mathfrak{g},\mathrm{U}(p)\times\mathrm{O}(2c-2p)}_{\bar{\mathfrak{q}},\mathrm{U}(p)\times\mathrm{O}(2c-2p)} Z^\sharp$ where $\mathfrak{q} = (\mathfrak{u}(p) \oplus \mathfrak{so}(2c-2p)) \oplus \mathfrak{n}$. In [Kn], Knapp constructs an exact sequence of $\mathfrak{g}$-modules

$$0 \to Q' \to \mathrm{ind}Z^\sharp \to Q \to 0. \tag{25}$$

We check that the above extends to an exact sequence of $(\mathfrak{g}, \mathrm{U}(p) \times \mathrm{O}(2c-2p))$-modules. By Lemma 5.2(i) $\mathrm{ind}Z^\sharp$, Q and Q' are direct sums of $V_0(\mu)$'s. Furthermore Q contains the bottom layer $V_0(\mu = 0)$. Lemma 5.2(ii) gives

$$0 \to \Gamma^{s_0}_{2p,2c-2p} Q' \to \Gamma^{s_0}_{2p,2c-2p} \mathrm{ind}Z^\sharp \to \Gamma^{s_0}_{2p,2c-2p} Q \to 0. \tag{26}$$

Since Q contains $V_0(0)$, $\Gamma^{s_0}_{2p,2c-2p} Q$ contains the minimal $K_{2p,2c-2p}$-type of $A_{2p,2c-2p}(\lambda) = \Gamma^{s_0}_{2p,2c-2p}\mathrm{ind}Z^\sharp$. Hence $\overline{A}_{2p,2c-2p}(\lambda)$ is a subquotient of $\Gamma^{s_0}_{2p,2c-2p} Q$.

Let $\Gamma_0 = \Gamma^{K^0_{2p,2c-2p}}_{M^0_{2p,2c-2p}}$, $\Gamma_1 = \Gamma^{K^1_{2p,2c-2p}}_{M_{2p,2c-2p}}$ and $\pi_0 = \Gamma^{s_0}_1 Q$. By (20), $\mathcal{F}^{\mathfrak{g},K^0_{2p,2c-2p}}_{\mathfrak{g},K^1_{2p,2c-2p}} \pi_0 = \Gamma^{s_0}_0 Q$. Knapp shows that π_0 is a unitarizable $(\mathfrak{g}, K^1_{2p,2c-2p})$-quotient of $\Gamma^{s_0}_1 \mathrm{ind}Z^\sharp$ and π_0 contains the minimal $K^1_{2p,2c-2p}$-type of $\Gamma^{s_0}_1\mathrm{ind}Z^\sharp$. Trapa proves that π_0 is an irreducible module [T]. We remark that Knapp and Trapa uses Γ_0 instead of Γ_1 but their proofs could be easily adapted to Γ_1 with the help of (20).

Let $\tilde{\pi}_0 = \mathrm{induced}^{\mathfrak{g},K_{2p,2c-2p}}_{\mathfrak{g},K^1_{2p,2c-2p}} \pi_0$. By (18), $\Gamma^{s_0}_{2p,2c-2p} Q = \tilde{\pi}_0$ and $\overline{A}_{2p,2c-2p}(\lambda)$ is its $(\mathfrak{g}, K_{2n,2c-2p})$-subquotient. Using the description of $K^0_{2p,2c-2p}$-types of π_0 in [Kn], one checks that $\tilde{\pi}_0$ has the same $K_{2p,2c-2p}$-types as $\theta^{2p,2c-2p}_p(1)$. Hence $\tilde{\pi}_0 = \theta^{2p,2c-2p}_p(1)$ by Lemma 3.3. In particular $\tilde{\pi}_0$ is irreducible and therefore it is equal to $\overline{A}_{2p,2c-2p}(\lambda)$. This proves that $\overline{A}_{2p,2c-2p}(\lambda) = \theta^{2p,2c-2p}_p(1)$. Alternatively, this also follows from [PT] where Paul and Trapa prove that π_0 is a $(\mathfrak{g}, K^0_{2p,2c-2p})$-submodule of $\theta^{2p,2c-2p}_p(1)$.

Now we turn to the general case, i.e. $2p < n$ and $s_0 = p(n-2p) + \frac{p(p-1)}{2}$. We consider (25) as an exact sequence of $(\mathfrak{g}, \mathrm{U}(p) \times K_{n-2p,m})$-modules. Repeating the argument with n instead of $2p$ in (26) gives

$$0 \to \Gamma^{s_0}_{n,m} Q' \to \Gamma^{s_0}_{n,m}\mathrm{ind}Z^\sharp \to \Gamma^{s_0}_{n,m} Q \to 0.$$

By Section 4.3, $\Gamma_{n,m}$ is essentially the functor $\Gamma_1 := \Gamma^{\mathrm{O}(n)}_{\mathrm{U}(p)\times\mathrm{O}(n-2p)}$. Let $\Gamma_2 = \Gamma^{\mathrm{O}(n)}_{\mathrm{O}(2p)\times\mathrm{O}(n-2p)}$ and $\Gamma_3 = \Gamma^{\mathrm{O}(2p)\times\mathrm{O}(n-2p)}_{\mathrm{U}(p)\times\mathrm{O}(n-2p)}$. By Proposition 6.2.17 in [Vo], we have a spectral sequence $\Gamma^a_2\Gamma^b_3 \Rightarrow \Gamma^{a+b}_1$. By Lemma 5.2(ii), $\Gamma^b_3 Q = 0$ if $b \neq p(p-1)/2$. Hence the spectral sequence gives $\Gamma^{s_0}_{n,m} Q = \Gamma^{s_0}_1 Q = \Gamma^{p(n-2p)}_2 \Gamma^{p(p-1)/2}_3 Q = \Gamma^{p(n-2p)}_2 \theta^{2p,2c-2p}_p(1) = \theta^{n,m}_p(1)$. The last equality follows from Theorem 1.1(ii). Hence $\theta^{n,m}_p(1)$ is an irreducible quotient of

$A_{n,m}(\lambda) = \Gamma^{s_0}_{n,m}\mathrm{ind}Z^\sharp$ containing the minimal $K_{n,m}$-type, i.e. $\overline{A}_{n,m}(\lambda) = \theta^{n,m}_p(1)$. This proves Theorem 1.4(i). □

Proof of Theorem 1.4(ii). Since $\theta^{n,m}_p(1)$ is an irreducible subquotient of $A(\lambda)$, it follows that $\theta^{n,r}_p(L(\mu)) \otimes \mu_{\mathrm{O}(m-r)}$ is an irreducible subquotient of $A(\lambda)$, considered as $(\mathfrak{so}(n+r,\mathbb{C}), K_{n,r}) \times \mathrm{O}(m-r)$-module. This implies that $\theta^{n,r}_p(L(\mu))$ is an irreducible subquotient of $A(\lambda_{\mu'}) = \Gamma^{s_0}(V_r(\mu'))$ as in (24) for some μ' such that $(\mu')_{\mathrm{U}(m-r)}$ contains $\mu_{\mathrm{O}(m-r)}$. We recall that $\theta^{n,m}_p(1)$ contains the $K_{n,r}$-type $W = \mathrm{d}^p_n(\mu + \frac{m-n}{2}\mathbf{1}_m)_{\mathrm{O}(n)} \otimes \mathbb{C}_{\mathrm{O}(r)} = \Gamma^{s_0}(V_0(\mu))$. Hence $A(\lambda_{\mu'})$ contains W.

We claim that $\mu = \mu'$. Indeed we embed the torus $\mathfrak{t}$ of $\mathrm{O}(m-r)$ in the torus of $\mathrm{U}(m-r)$ so that the restriction of positive $\mathrm{U}(m-r)$ roots to $\mathrm{O}(m-r)$ remains positive. By Lemma 5.3 and (24), $\mu = \mu' + \kappa = \mu' + (\kappa_1, \ldots, \kappa_p) \in \mathfrak{t}^*$ where $\kappa_i \geq 0$. On the other hand any weight of $(\mu')_{\mathrm{U}(m-r)}$ is of the form $\mu' - (\sum \alpha)$ where $\sum \alpha$ is a sum of positive roots of $\mathrm{U}(m-r)$ with multiplicities. Hence $\mu' + \kappa = \mu = \mu' - (\sum \alpha)$ on $\mathfrak{t}$. This forces $\kappa_1 = \ldots = \kappa_p = 0$ and $\mu = \mu'$. This proves our claim.

From our claim $\theta^{n,r}_p(L(\mu))$ is the irreducible subquotient of $A(\lambda_\mu)$ generated by $K_{n,r}$-type W. Since W is the unique minimal $K_{n,r}$-type of $A(\lambda_\mu)$, $\theta^{n,r}_p(L(\mu)) = \overline{A}(\lambda_\mu)$. This proves Theorem 1.4(ii). □

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