ON MINIMAL REPRESENTATIONS OF CHEVALLEY GROUPS OF TYPE D_n, E_n AND G_2

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Abstract. Let G be a simply connected Chevalley group of type D_n, E_n or G_2. In this paper, we show that the minimal representation of G is unique for types D_n and E_n and it does not exist for the type G_2.

1. Introduction

Let F be a p-adic field with p odd. Let Φ be a simply laced root system (or of type G_2) and $\mathfrak{g}$ the corresponding split semi-simple Lie algebra over the field F. Then there is a decomposition

$$\mathfrak{g} = (\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha) \oplus \mathfrak{t}$$

where $\mathfrak{g}_\alpha$ are one-dimensional root spaces and $\mathfrak{t}$ a maximal split Cartan subalgebra. Let G be the corresponding simply connected Chevalley group. Let $B = TU$ be a Borel subgroup corresponding to a choice of positive roots Φ^+. Here T is a maximal split torus which is described as follows. For every root α there is a homomorphism $\varphi_\alpha : \mathrm{SL}_2 \to G$ (the image will be denoted by $\mathrm{SL}_2(\alpha)$). Then T is generated by elements

$$\alpha^\vee(t) = \varphi_\alpha(\mathrm{diag}(t, t^{-1})),$$

for $t \in F^\times$. The map $\alpha^\vee : F^\times \to T$ is the co-root corresponding to α. Let Δ denote the set of simple roots. Recall that parabolic subgroups containing B are in one-to-one correspondence with subsets of Δ. For every subset $\Theta \subseteq \Delta$, there is a parabolic subgroup $P_\Theta = L_\Theta U_\Theta$ such that L_Θ is generated by T and $\mathrm{SL}_2(\alpha)$ for all α in Θ. In particular, $G = P_\Delta$ and $B = P_\emptyset$.

Any admissible representation V of G defines a character distribution χ in a neighborhood of 0 in $\mathfrak{g}$. Moreover, by a theorem of Howe and Harish-Chandra [HC], there exists a compact open subset Ω_V of 0 such that for every function f which is compactly supported in Ω_V,

$$\chi(f) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}} \int \hat{f}\mu_{\mathcal{O}}. \tag{1}$$

Here $\mathcal{N}$ is the set of nilpotent G-orbits in $\mathfrak{g}$, $\mu_{\mathcal{O}}$ is a suitably normalized Haar measure on $\mathcal{O}$, and $\hat{f}$ is the Fourier transform of f with respect to the Killing form and a non-trivial

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character $\psi : F \to \mathbb{C}^\times$. Let

$$\mathcal{N}_V = \{\mathcal{O} \in \mathcal{N} \mid c_{\mathcal{O}} \neq 0\}.$$

The wavefront set $\mathrm{WF}(V)$ of V is defined as the subset of $\mathcal{N}_V$ consisting of all maximal elements in $\mathcal{N}_V$ with respect to the partial order $\leq$ defined in the following way:

$$\mathcal{O}_1 \leq \mathcal{O}_2$$

if and only if $\mathcal{O}_1 \subseteq \bar{\mathcal{O}}_2$ where $\bar{\mathcal{O}}$ denotes the topological closure of $\mathcal{O}$. The minimal orbit $\mathcal{O}_{\min}$ is the smallest non-trivial nilpotent orbit in $\mathfrak{g}$. Its Bala-Carter [Ca] notation is A_1. If α is a long root and X a non-zero element in $\mathfrak{g}_\alpha$, then

$$\mathcal{O}_{\min} = \mathrm{Ad}_G(X).$$

Definition. Suppose π is an irreducible admissible smooth representation of G such that the wavefront set of π is the minimal orbit, then we call π a *minimal* representation of G.

The main result of this paper is to determine minimal representations for groups of type D_n and E_n. See Theorem 1.1. In particular, we need to fix some notation for these two types roots systems. The set of simple roots is denoted by

$$\Delta = \{\beta_1, \beta_2, \ldots, \beta_n\}.$$

We pick an indexing of simple roots so that $\beta_1, \beta_2, \beta_3$ and β_4 form the unique subdiagram of type D_4, and

- The root β_2 corresponds to the branching point of the Dynkin diagram.
- The root β_1 is connected to β_2 only and to no other simple roots of G in the Dynkin diagram.

The last two simple roots β_3 and β_4 are picked, in no particular order, to complete the D_4 subdiagram. In terms of Bourbaki [Bo] notation, for type D_n groups, we have $\beta_1 = \alpha_n$ and $\beta_2 = \alpha_{n-2}$, and for type E_n groups, we have $\beta_1 = \alpha_2$ and $\beta_2 = \alpha_4$.

We define a character $\nu : F^\times \to \mathbb{C}^\times$ by $\nu(x) = |x|$. Given a character χ of T and a simple root β_i, we define a character $\chi_i : F^\times \to \mathbb{C}^\times$ by

$$\chi_i(t) = \chi(\beta_i^\vee(t)).$$

The main result of this paper is:

Theorem 1.1. *Let V be a minimal representation of G. Then V is the unique irreducible submodule of $\mathrm{Ind}_B^G \chi$ (normalized induction) where χ is a character such that $\chi_i = \nu^{-1}$ for all $i \neq 2$ and χ_2 is the trivial character.*

Conversely, the unique irreducible submodule of $\mathrm{Ind}_B^G(\chi)$ (where χ is as in Theorem 1.1) is a minimal representation with

$$c_{\mathcal{O}_{\min}} = 1.$$

This is Theorem 2.1 in [Sa]. Our next result deals with the exceptional group of type G_2. In a sense this is the most interesting case. Indeed, a simple argument shows that a minimal representation of a split group of type D_n or E_n must be a representation of a linear group. If the type is B_3, C_n or F_4 then a minimal representation must be a representation of a two-fold cover of a linear group (oscillator representation for C_n). A split group of type B_n for $n > 3$ has no minimal representation. However, if the type is G_2 then the situation is not so clear-cut. A minimal representation is either a representation of a linear group or a representation of a three-fold cover of the linear group. (See also a work of Torasso [To] for an explanation in terms of so-called admissible data.) Thus, for some time, it has remained somewhat a mystery whether a Chevalley group of type G_2 has a minimal representation. In [Ga] Gan showed that there is no minimal representation among spherical representations of G_2. The following result now completely answers this question:

Theorem 1.2. *Let G be a Chevalley group of type* G_2*. Then G has no minimal representation.*

As we have mentioned in the beginning of this introduction our results are subject to the condition $p \neq 2$. This restriction comes from the work of Moeglin and Waldspurger [MW]. Since [MW] makes use of the exponential map from $\mathfrak{g}$ to G the restriction $p \neq 2$ appears to be unavoidable.

Methods of this paper are, of course, applicable to non-split groups. However, we have restricted ourselves to split groups for the following reasons. First, a classification of all non-split groups is quite complicated and, second, parameters of minimal representations may differ considerably from group to group (see [GS] for exceptional groups).

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2. Principal series representations

In this section we review some well known facts about principal series representations (see [Ro]) and prove that the induced representation $\mathrm{Ind}_B^G(\chi)$ where χ is as in the statement of Theorem 1.1 has a unique irreducible submodule.

In this paper, Jacquet functors are normalized Jacquet functors as defined in Section 1.8(2)(b) in [BZ]. It is the left adjoint to the normalized induction functor.

Let E be an admissible representation of G. Now $J_U(E)$, the normalized module of U-coinvariants (Jacquet module) with respect to the maximal unipotent subgroup U, is finite dimensional. As a T-module, it can be decomposed as

$$J_U(E) = \oplus_\chi J_U(E)_\chi^\infty$$

where $J_U(E)^\infty_\chi$ consists of all v in $J_U(E)$ such that $(\pi(t)-\chi(t))^n v = 0$ for a sufficiently large n. The characters χ are called *exponents* of E. The Frobenius reciprocity implies that E is a submodule of an induced representation $\mathrm{Ind}_B^G(\chi)$ if and only if χ is an exponent of E. Moreover, a character χ' is an exponent of $\mathrm{Ind}_B^G(\chi)$ if and only if $\chi' = \chi^w$ for some w is in the Weyl group W of Φ. The multiplicity of an exponent χ is

$$\dim J_U(\mathrm{Ind}_B^G(\chi))^\infty_\chi = |W_\chi|$$

where $W_\chi \subseteq W$ is the stabilizer of χ in the Weyl group W.

Proposition 2.1. *Let E be a submodule of $\mathrm{Ind}_B^G(\chi)$ and β_i a simple root. Let s_i be the reflection defined by β_i. Recall that $\chi_i = \chi \circ \beta_i^\vee$.*

1. *If $\chi \neq \chi^{s_i}$ and $\chi_i \neq \nu^{\pm 1}$ then χ^{s_i} is also an exponent of E.*
2. *If $\chi_i = 1$ then $\dim J_U(E)^\infty_\chi \geq 2$.*

Proof. We shall prove both statements at once. The proof is a simple combination of representation theory for SL_2 and induction in stages. To that end, let $P_i = L_i U_i$ be the parabolic subgroup such that $[L_i, L_i] = \mathrm{SL}_2(\beta_i)$. By representation theory of SL_2, the conditions on χ in each of the two statements imply that $\mathrm{Ind}_B^{P_i}(\chi)$ is irreducible. Since

$$\mathrm{Ind}_B^G(\chi) = \mathrm{Ind}_{P_i}^G(\mathrm{Ind}_B^{P_i}(\chi)),$$

the Frobenius reciprocity implies that $\mathrm{Ind}_B^{P_i}(\chi)$ is a quotient of $J_{U_i}(E)$. It follows that $J_U(\mathrm{Ind}_B^{P_i}(\chi))$ is a quotient of $J_U(E)$. The proposition follows at once since the exponents of $\mathrm{Ind}_B^{P_i}(\chi)$ are χ and χ^{s_i} if $\chi \neq \chi^{s_i}$ and χ with multiplicity 2 if $\chi = \chi^{s_i}$. □

Corollary 2.2. *Let χ be a character of T such that $\chi_i = \nu^{-1}$ for all $i \neq 2$ and $\chi_2 = 1$. Then $\mathrm{Ind}_B^G(\chi)$ has a unique irreducible submodule.*

Proof. Let $V' \oplus V''$ be a submodule of $\mathrm{Ind}_B^G(\chi)$ such that $V' \neq 0$ and $V'' \neq 0$. Since $\chi_2 = 1$, the proposition implies that $\dim J_U(V')^\infty_\chi \geq 2$ and $\dim J_U(V'')^\infty_\chi \geq 2$. By exactness of the Jacquet functor, $\dim J_U(\mathrm{Ind}_B^G(\chi))^\infty_\chi \geq 4$. On the other hand, it can be easily seen that W_χ, the stabilizer of χ in W, consist of only two elements: $W_\chi = \{1, s_2\}$. It follows that $\dim J_U(\mathrm{Ind}_B^G(\chi))^\infty_\chi = 2$. This is a contradiction. □

Our strategy of the proof of Theorem 1.1 is to show that any minimal representation has an exponent χ such that $\chi_i = \nu$ for all $i \neq 2$ and $\chi_2 = 1$.

3. Whittaker models

We state a result of [MW] which relates wavefront sets and generalized Whittaker models of G. Let Y be an element in a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$. Let H be a semisimple element in $\mathfrak{g}$ such that $[H, Y] = -2Y$ and all eigenvalues of H are integral. Existence of one such H is guaranteed by the Jacobson-Morozov theorem, but there are many other choices, especially for Y in a small orbit. This observation is critical to us. Write $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ where $\mathfrak{g}_i$ is the i-eigenspace of H. Let

$$\mathfrak{n}' = (Z_{\mathfrak{g}}(Y) \cap \mathfrak{g}_1) + \sum_{i \geq 2} \mathfrak{g}_i$$

and $N' = \exp(\mathfrak{n}')$. Let $\langle \cdot, \cdot \rangle$ denote the Killing form on $\mathfrak{g}$. The pair (Y, H) defines a character $\psi(Y, H)$ of N' by

$$\psi(Y, H)(\exp X) = \psi(\langle X, Y \rangle)$$

where X is in $\mathfrak{n}'$. If V is a representation of G, we set $J_{\psi(Y,H)}(V)$ to be the twisted Jacquet module with respect to the character $\psi(Y, H)$. Let

$$Wh_\psi(V) = \{\mathcal{O} \in \mathcal{N} \mid J_{\psi(Y,H)}(V) \neq 0 \text{ for some } H\}.$$

The following result is due to Moeglin and Waldspurger [MW].

Theorem 3.1. *Assume that $p \neq 2$. The wavefront set of V coincides with the set of maximal (with respect to the partial order $\leq$) nilpotent orbits in $Wh_\psi(V)$.*

Moeglin and Waldspurger also give a more precise description of $J_{\psi(Y,H)}(V)$ for Y in the wavefront set of V. There are two cases. The first case is when $\mathfrak{g}_1 = 0$. Then

$$\dim J_{\psi(Y,H)}(V) = c_{\mathcal{O}} \tag{2}$$

where $c_{\mathcal{O}}$ is given in (1). The second case, when $\mathfrak{g}_1 \neq 0$, is more complicated. Let $\mathfrak{n} = \oplus_{i>0} \mathfrak{g}_i$ and $N = \exp(\mathfrak{n})$. Let $\mathfrak{n}''$ be the kernel of the functional $X \mapsto \langle X, Y \rangle$ where X is in $\mathfrak{n}'$. Let $N'' = \exp(\mathfrak{n}'')$. Then N/N'' is a Heisenberg group with the center N'/N''. As such, it has a unique irreducible smooth representation W_Y with the central character $\psi(Y, H)$. Since N/N'' acts on $J_{\psi(Y,H)}(V)$ with the central character $\psi(Y, H)$, as an N/N''-module, $J_{\psi(Y,H)}(V)$ is a multiple of W_Y and have

$$\dim \mathrm{Hom}_N(W_Y, J_{\psi(Y,H)}(V)) = c_{\mathcal{O}}. \tag{3}$$

Finally we remark that for the given Y above, (2) or (3) will continue to hold for a different choice of H such that $[H, Y] = -2Y$.

We now describe some of our choices for H and Y. Let Y_i be a non-zero element of $\mathfrak{g}_{-\beta_i}$. Let H_Δ be in $\mathfrak{t}$ such that $[H_\Delta, Y_i] = -2Y_i$ for all i. For any subset Θ of Δ, define

$$Y_\Theta = \sum_{\beta_i \in \Theta} Y_i.$$

According to the recipe given above the pair (H_Δ, Y_Θ) defines a character $\psi(Y_\Theta, H_\Delta)$ of U. Moreover, let $P_\Theta = L_\Theta U_\Theta$ be the parabolic subgroup corresponding to Θ. We remind the reader that L_Θ is generated by T and $\mathrm{SL}_2(\alpha)$ for all simple roots α in Θ. Note that

- $\psi(Y_\Theta, H_\Delta)$ is trivial on U_Θ.
- $\psi(Y_\Theta, H_\Delta)$ restricted to on $U \cap L_\Theta$ is a Whittaker functional for the group L_Θ.

For any representation V we have a natural isomorphism of vector spaces

$$J_{\psi(Y_\Theta,H_\Delta)}(V)=J_{\psi(Y_\Theta,H_\Delta)}(J_{U_\Theta}V) \tag{4}$$

where J_{U_Θ} is the space of U_Θ-coinvariants of V (Jacquet module). Thus, the above formula shows that if $J_{\psi(Y_\Theta,H_\Delta)}(V)\neq 0$ then $J_{U_\Theta}(V)\neq 0$ and it is generic.

The rest of this section is devoted to a proof of Theorem 1.1. The proof consists of a series of lemmas. Let $P_1=L_1U_1$ be the parabolic subgroup corresponding to β_1. Let Y_1 be a non-zero element in $\mathfrak{g}_{-\beta_1}$. Then the pair (Y_1,H_Δ) defines the character $\psi(Y_1,H_\Delta)$ of U which is trivial on U_1. If V is a minimal representation then $J_{\psi(Y_1,H_\Delta)}(V)\neq 0$ so the formula (4) shows that $J_{U_1}(V)\neq 0$ and it has generic (with respect to L_1) subquotients. The center of L_1 clearly contains elements $\alpha^\vee(t)$ for any root α perpendicular to β_1. These include β_i for all $i\neq 1,2$ and the root $\beta=\beta_1+2\beta_2+\beta_3+\beta_4$. By Schur's lemma elements of the center of L_1 have to act by a scalar on every irreducible subquotient of $J_{U_1}(V)$. The first result of this section is the following:

Lemma 3.2. *Let V be a minimal representation. Then $\beta_i^\vee(t)$ for $i\neq 1,2$ acts trivially on any irreducible generic (with respect to L_1) subquotient of $J_{U_1}(V)$.*

Proof. The scalar by which $\beta_i^\vee(t)$ acts on an irreducible generic subquotient can be detected by a Whittaker functional (for L_1). Since every irreducible generic subquotient of $J_{U_1}(V)$ corresponds to a one-dimensional subquotient of $J_{\psi(Y_1,H_\Delta)}$, via a Whittaker functional, it suffices to show that $\beta_i^\vee(t)$ acts trivially on $J_{\psi(Y_1,H_\Delta)}(V)$. Let $P_\Sigma=L_\Sigma U_\Sigma$ be a parabolic subgroup corresponding to $\Sigma=\Delta\setminus\{\beta_1,\beta_2\}$. Let H be in $\mathfrak{t}$ such that $[H,Y_i]=0$ for all $i\neq 1,2$ and $[H,Y_i]=-2Y_i$ for $i=1,2$. The pair (Y_1,H) defines a character $\psi(Y_1,H)$ of U_Σ. Since the restriction of $\psi(Y_1,H_\Delta)$ to U_Σ is equal to $\psi(Y_1,H)$, we have a natural surjection

$$J_{\psi(Y_1,H)}(V)\to J_{\psi(Y_1,H_\Delta)}(V).$$

If $i\neq 1,2$ then $\langle\beta_i,\beta_1\rangle=0$ and the group $\mathrm{SL}_2(\beta_i)$ centralizes Y_1 and H. It follows that the action of G on V descends to an action of $\mathrm{SL}_2(\beta_i)$ on $J_{\psi(Y_1,H)}$. Since $J_{\psi(Y_1,H)}(V)$ is finite dimensional (by minimality of V) the action is trivial as $\mathrm{SL}_2(F)$ has no non-trivial finite dimensional representations. This proves the lemma. □

Lemma 3.3. *Let V be a minimal representation. Then $\beta^\vee(t)$ acts by ν^2 on any irreducible generic subquotient of $J_{U_1}(V)$.*

We shall assume this lemma for a moment. Its proof is given towards the end of Section 4.

Lemma 3.4. *Let τ be an irreducible subquotient of $J_{U_1}(V)$. Then τ is not supercuspidal.*

Proof. Suppose τ is supercuspidal, in which case τ can be considered a quotient of $J_{U_1}(V)$. In particular,

$$V\subseteq \mathrm{Ind}_{P_1}^G(\tau).$$

Let $P_\Theta = L_\Theta U_\Theta$ be the parabolic subgroup corresponding to $\Theta = \{\beta_1, \beta_2\}$. Then we can write

$$\mathrm{Ind}_{P_1}^G(\tau) = \mathrm{Ind}_{P_\Theta}^G(\mathrm{Ind}_{P_1}^{P_\Theta}(\tau)).$$

Next, by [BZ], $\mathrm{Ind}_{P_1}^{P_\Theta}(\tau)$ is an irreducible generic representation of L_Θ, a reductive group of type A_2. Recall that $Y_\Theta = Y_1 + Y_2$ where $Y_i \in \mathfrak{g}_{-\beta_i}$ and, by (4),

$$J_{\psi(Y_\Theta, H_\Delta)}(V) = J_{\psi(Y_\Theta, H_\Delta)}(J_{U_\Theta}(V)) \neq 0$$

since the generic L_Θ-module $\mathrm{Ind}_{P_1}^{P_\Theta}(\tau)$ is a quotient of $J_{U_\Theta}(V)$. This is a contradiction since Y_Θ belongs to an orbit with Bala-Carter notation A_2. The lemma follows. □

In the following corollary we summarize what we have shown thus far:

Corollary 3.5. *Let V be a minimal representation. Then the character χ such that $\chi_i = \nu^{-1}$ for all $i \neq 1, 2$ and $\chi_1 \chi_2^2 = \nu^{-1}$ is an exponent of V.*

Proof. Indeed, we have shown that $J_U(V) \neq 0$ and there is a non-trivial T-invariant subquotient of $J_U(V)$ where $\beta_i^\vee(t)$ acts trivially for all $i \neq 1, 2$ and $\beta^\vee(t)$ acts as ν^2. Since $\beta = \beta_1 + 2\beta_2 + \beta_3 + \beta_4$, and $\beta_3^\vee(t)$ and $\beta_4^\vee(t)$ act trivially, it follows that $\beta_1^\vee(t)\beta_2^\vee(t^2)$ acts as ν^2 as well. Since the modular function satisfies $\delta_B^{1/2}(\beta_i^\vee(t)) = \nu(t)$ it follows that χ is indeed an exponent of V. □

As the corollary shows, we have reduced the problem of finding an exponent of the minimal representation V to figuring out what χ_2 is. The following lemma reduces to three possibilities.

Lemma 3.6. *Assume that V is a minimal representation and χ an exponent such that $\chi_i = \nu^{-1}$ for all $i \neq 1$ and 2. Then χ_2 is ν, 1 or ν^{-1}.*

Proof. Let $\Theta = \{\beta_3, \beta_4\}$ and $P_\Theta = L_\Theta U_\Theta$ the corresponding parabolic subgroup of G. Let $\chi' = \chi^{s_2}$. If $\chi_2 \neq \nu^{\pm 1}$ then χ' is also an exponent of V by Proposition 2.1. If we further assume that $\chi_2 \neq 1$ then

$$\chi'_3 = \chi_3/\chi_2 \neq \nu^{-1} \text{ and } \chi'_4 = \chi_4/\chi_2 \neq \nu^{-1}.$$

Using the induction in stages

$$\mathrm{Ind}_B^G(\chi') = \mathrm{Ind}_{P_\Theta}^G(\mathrm{Ind}_B^{P_\Theta}(\chi'))$$

so $J_{U_\Theta}(V)$ maps to $\mathrm{Ind}_B^{P_\Theta}(\chi')$ by Frobenius reciprocity. Since $\chi'_i \neq \nu^{-1}$ for $i = 3, 4$, any submodule V' of $\mathrm{Ind}_B^{P_\Theta}(\chi')$ is L_Θ-generic. In particular, we can pick $Y_\Theta = Y_3 + Y_4$ with $Y_i \in \mathfrak{g}_{-\beta_i}$ for $i = 3, 4$ such that $J_{\psi(Y_\Theta, H_\Delta)}(V') \neq 0$. Using (4), it follows that

$$J_{\psi(Y_\Theta, H_\Delta)}(V) = J_{\psi(Y_\Theta, H_\Delta)}(J_{U_\Theta}(V)) \neq 0.$$

This contradicts the fact that V is minimal because Y belongs to an orbit with Bala-Carter notation $2A_1$. It follows that χ_2 must be ν^{-1}, 1 or ν. □

The three cases $\chi_2 = 1, \nu$ and ν^{-1} will be referred to as cases a), b) and c) and need separate considerations.
Case a): $\chi_2 = 1$. Since $\chi_1\chi_2^2 = \nu^{-1}$, it follows that $\chi_1 = \nu^{-1}$ and χ is the exponent we have been looking for.
Case b): $\chi_2 = \nu$. Since $\chi_1\chi_2^2 = \nu^{-1}$, it follows that $\chi_1 = \nu^{-3}$. This exponent is eliminated by the following lemma.

Lemma 3.7. *A character χ of T such that $\chi_1 = \nu^{-3}$ and $\chi_2 = \nu$ cannot be an exponent of a minimal representation.*

Proof. Let $\Theta = \{\beta_1, \beta_2\}$ and $P_\Theta = L_\Theta U_\Theta$ be the corresponding parabolic subgroup of G. Now, if V is a submodule of $\mathrm{Ind}_B^G(\chi)$ then, using the induction in stages and Frobenius reciprocity, there is a non-trivial map (of L_Θ-modules) from J_{U_Θ} to $\mathrm{Ind}_B^{P_\Theta}(\chi)$. We need to understand this L_Θ-module. To this end, realize the root subsystem spanned by roots β_1 and β_2 in the space of triples (x, y, z) such that $x + y + z = 0$ and the simple roots are $\beta_1 = (1, -1, 0)$ and $\beta_2 = (0, 1, -1)$. Let $\mathrm{SL}_3 = [L_\Theta, L_\Theta]$. Then the restriction of the unramified character χ of T to $T \cap \mathrm{SL}_3$ can be identified with a triple $\chi = (x, y, z)$ so that

$$\chi(\beta_i(t)) = |t|^{\langle \chi, \beta_i \rangle}.$$

Under this identification the character χ such that $\chi_1 = \nu^{-3}$ and $\chi_2 = \nu$ is represented by $\chi = (-\frac{5}{3}, \frac{4}{3}, \frac{1}{3})$. Notice that this character is regular for the A_2-root system. This implies that $\mathrm{Ind}_B^{P_\Theta}(\chi)$ has a unique irreducible submodule. Furthermore, since $\chi_2 = \nu$, the induction in stages through the parabolic subgroup corresponding to β_2 implies that $\mathrm{Ind}_B^{P_\Theta}(\chi)$ has a generic submodule V_g and a degenerate quotient V_d. Both are irreducible by Rodier [Ro]. One could also deduce this from Theorems 2.2 and 3.5 in [Ze]. It follows that the image of $J_{U_\Theta}(V)$ in $\mathrm{Ind}_B^{P_\Theta}(\chi)$ must contain V_g. Using (4), it follows that

$$J_{\psi(Y_\Theta, H_\Delta)}(V) = J_{\psi(Y_\Theta, H_\Delta)}(J_{U_\Theta}(V)) \neq 0.$$

This contradicts the fact that V is minimal because Y belongs to an orbit with Bala-Carter notation A_2. □

Case c): $\chi_2 = \nu^{-1}$. Since $\chi_1\chi_2^2 = \nu^{-1}$, it follows that $\chi_1 = \nu$. Theorem 1.1 follows from the following lemma:

Lemma 3.8. *Let V be a minimal representation. Assume that V has an exponent χ' such that $\chi'_i = \nu^{-1}$ for all $i \neq 1$ and $\chi'_1 = \nu$. Then the character χ such that $\chi_i = \nu^{-1}$ for all $i \neq 2$ and $\chi_2 = 1$ is also an exponent of V.*

Proof. Notice that $\chi' = \chi^{s_1}$. By the assumption, V is a submodule of $\mathrm{Ind}_B^G(\chi')$. Again, we use the parabolic $P_\Theta = L_\Theta U_\Theta$ such that $\Theta = \{\beta_1, \beta_2\}$. The induction in stages and the Frobenius reciprocity imply that $J_{U_\Theta}(V)$ has a non-trivial quotient contained in $\mathrm{Ind}_B^{P_\Theta}(\chi')$. It remains to understand this L_Θ-module. The restriction of χ' to $T \cap \mathrm{SL}_3$ corresponds to $\chi' = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$. Since $\chi'_2 = \nu^{-1}$ induction in stages through the parabolic subgroup corresponding to β_2 implies that $\mathrm{Ind}_B^{P_\Theta}(\chi)$ has a degenerate submodule V_d and a generic quotient V_g.

We claim that V_d and V_g are irreducible. To see this it suffices to show that they are irreducible as $\mathrm{SL}_3 = [L_\Theta, L_\Theta]$-modules. To that end the restriction of χ' and χ to $T \cap \mathrm{SL}_3$ corresponds to $\chi' = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$ and $\chi = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ respectively. By Example 11.2 in [Ze], $\mathrm{Ind}_B^{P_\Theta}(\chi')$ is a direct sum of two irreducible submodules. Hence V_d and V_g are irreducible and proves our claim.

We will give an alternative proof of the claim in which we will also compute the exponents. Since V_d is fully induced, it is a straightforward exercise to show that the exponents of V_d are χ' and $\chi = (\chi')^{s_1}$ twice. Further decomposing of V_d would imply that it contains, as a subquotient, a module with only one exponent. But there are only two such representations for SL_3: Steinberg and the trivial representation. Their exponents are $(-1, 0, 1)$ and $(1, 0, -1)$, respectively. This is clearly a contradiction which shows that V_d is irreducible. This argument shows that V_g is also irreducible.

In view of minimality of V, the non-trivial quotient of $J_{U_\Theta}(V)$ in $\mathrm{Ind}_B^{P_\Theta}(\chi')$ must be equal to V_d. This shows that χ is an exponent of V. The lemma and main theorem are proved at last. □

4. Heisenberg groups

This section is devoted to the proof of Lemma 3.3. In words, we want to calculate the action of $\beta(t)$ on $J_{\psi(Y,H_\Delta)}(V)$ where Y is a non-zero element in $\mathfrak{g}_{\beta_1}$ and $\beta = \beta_1 + 2\beta_2 + \beta_3 + \beta_4$. This will be accomplished by comparing $J_{\psi(Y,H_\Delta)}(V)$ with $J_{\psi(Y,H)}(V)$ where H belongs to a Jacobson-Morozov triple $\{X, H, Y\}$ generating $\mathfrak{sl}_2(\beta_1)$. Recall how $\psi(Y, H)$ is defined. First, the element H defines a gradation of $\mathfrak{g} = \oplus_i \mathfrak{g}_i$. In order to describe $\mathfrak{g}_i$, let

$$S_i = \{\alpha \in \Phi | \langle \alpha, \beta_1 \rangle = i\}.$$

Then, for every $i \neq 0$, the space $\mathfrak{g}_i$ is a direct sum of $\mathfrak{g}_\alpha$ for all α in S_i. Since $\langle \alpha, \beta_1 \rangle \leq 2$ and is equal to 2 only if $\alpha = \beta_1$, it follows that $\mathfrak{g}_2$ is one dimensional, spanned by X, and $\psi(Y, H)$ is a character of the one-dimensional subgroup $N' = \exp(\mathfrak{g}_2)$. There is more to this story, however. Let $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. This is a two step (Heisenberg) nilpotent algebra with the center $\mathfrak{g}_2$. The normalizer of $\mathfrak{n}$ in $\mathfrak{g}$ is a parabolic subalgebra $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{n}$ where

$$\mathfrak{m} = \mathfrak{g}_0 = \mathfrak{t} \oplus (\oplus_{\alpha \in S_0} \mathfrak{g}_\alpha).$$

Let $Q = MN$ denote the parabolic subgroup in G with the Lie algebra $\mathfrak{q}$. Of course, N is a Heisenberg group with the center N'. As such, N has a unique irreducible representation (π_Y, W_Y) with central character $\psi(Y, H)$. Next, note that $J_{\psi(Y,H)}(V)$ is the maximal quotient of V such that N' acts via $\psi(Y, H)$ on it. In particular, as an N-module, $J_{\psi(Y,H)}(V)$ is a multiple of W_Y and we have an isomorphism of N-modules (see [We])

$$W_Y \otimes \mathrm{Hom}_N(W_Y, J_{\psi(Y,H)}(V)) \cong J_{\psi(Y,H)}(V) \tag{5}$$

defined by $v \otimes A \mapsto A(v)$. In fact, this is also an isomorphism of $[M,M]$-modules under the following actions. First of all, by the usual construction of the Weil representation, there is an action, also denoted by π_Y, of a double cover of $[M,M]$ on W_Y. In fact, as it was shown in [KS], this action descends down to $[M,M]$. Second, the action of $[M,M]$ on V descends down to $J_{\psi(Y,H)}(V)$ and is denoted by π. Putting things together, an element m in $[M,M]$ acts on an element A in $\mathrm{Hom}_N(W_Y, J_{\psi(Y,H)}(V))$ by $\pi^{-1}(m) \circ A \circ \pi_Y(m)$.

If V is minimal then $\mathrm{Hom}_N(W_Y, J_{\psi(Y,H)}(V))$ has finite dimension $c_{\mathcal{O}_{\min}}$ by (3) and the action of the perfect group $[M,M]$ must be trivial. It follows that the action of $[M,M]$ on $J_{\psi(Y,H)}(V)$ can be reconstructed from the action π_Y of $[M,M]$ on W_Y. In order to exploit this idea we need a *polarization* of N/N' to write down W_Y. Define

$$S_1^+ = \{\alpha \in \Phi^+ : \langle \alpha, \beta_1 \rangle = 1\} \text{ and } S_1^- = \{\alpha \in \Phi^- : \langle \alpha, \beta_1 \rangle = 1\}.$$

Let $\mathfrak{n}^+$ and $\mathfrak{n}^-$ be the direct sums of $\mathfrak{g}_\alpha$ with α in S_1^+ and S_1^-, respectively. Then $\mathfrak{g}_1 = \mathfrak{n}^+ \oplus \mathfrak{n}^-$ such that $[\mathfrak{n}^+, \mathfrak{n}^+] = 0$ and $[\mathfrak{n}^-, \mathfrak{n}^-] = 0$. In particular, W_Y has a realization as $S(\mathfrak{n}^-)$, the space of locally constant and compactly supported functions on $\mathfrak{n}^-$. Explicit formulas for the action π_Y of $\mathrm{SL}_2(\beta) \subset [M,M]$ on $S(\mathfrak{n}^-)$ were worked out in [KS]. Roughly speaking, as it is shown in the proof of Proposition 2 in [KS], there is a polarization $\mathfrak{n}_\beta^-$ of $\mathfrak{g}_1$ which is $\mathrm{SL}_2(\beta)$-invariant. Then the action $\mathrm{SL}_2(\beta)$ on $S(\mathfrak{n}_\beta^-)$ is by translations. This is correct, without any sign ambiguities, since $\mathrm{SL}_2(\beta)$ is a perfect group by Proposition 1 in [KS]. The action of $\beta^\vee(t)$ on $S(\mathfrak{n}^-)$ is obtained from the action of $\beta^\vee(t)$ on $S(\mathfrak{n}_\beta^-)$ via a partial Fourier transform. We need the following very special case: For every f in $S(\mathfrak{n}^-)$

$$(\pi_Y(\beta^\vee(t))f)(0) = |t|^{-\frac{\langle \beta, \lambda \rangle}{2}} f(0)$$

where $\lambda = \sum_{\alpha \in S_1^-} \alpha$. This formula gives the action of $\beta^\vee(t)$ on the delta functional $\delta(f) = f(0)$. On a case by case basis one easily verifies that $\langle \beta, \lambda \rangle = -4$ in each case. It follows that the action of $\beta^\vee(t)$ on δ is given by

$$\delta(\pi_Y(\beta^\vee(t))f) = |t|^2 \delta(f). \tag{6}$$

Proof of Lemma 3.3. Recall that $\psi(Y, H_\Delta)$ is a character of U and its restriction to Z is equal to $\psi(Y,H)$. It follows that $J_{\psi(Y,H_\Delta)}(V)$ is a quotient of $J_{\psi(Y,H)}(V)$. Using the isomorphism (5) and $W_Y \cong S(\mathfrak{n}^-)$ we have a surjection

$$S(\mathfrak{n}^-) \otimes \mathrm{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y,H)}(V)) \to J_{\psi(Y,H_\Delta)}(V).$$

Recall that $N^+ = \exp(\mathfrak{n}^+)$ is contained in U and note that the character $\psi(Y, H_\Delta)$ is trivial on N^+. The maximal quotient of $S(\mathfrak{n}^-)$ such that N^+ acts trivially on it is one-dimensional and spanned by the delta function δ. Thus the above surjection descends to a surjection

$$\mathbb{C} \cdot \delta \otimes \mathrm{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y,H)}(V)) \to J_{\psi(Y,H_\Delta)}(V).$$

(This map is in fact an isomorphism since $\mathrm{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y,H)}(V))$ and $J_{\psi(Y,H_\Delta)}(V)$ have the same dimension, equal to the coefficient $c_{\mathcal{O}_{\min}}$ in the character expansion (1) of V - we do not need this, however.) The action of $\beta^\vee(t)$ on δ is by $|t|^2$ by (6) and is trivial on $\mathrm{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y,H)}(V))$. The proof of Lemma 3.3 is now complete.

5. G_2

Let G be a Chevalley group of type G_2 over the local field F. In this section we show that G has no minimal representation. The proof is similar in nature to the proof of uniqueness of the minimal representation for simply laced groups. We use a variety of degenerate Whittaker models to narrow down parameters of a possible minimal representation.

Let $\Delta = \{\beta_1, \beta_2\}$ be a set of simple roots for G_2 such that β_1 is long and β_2 is short. Let $P_1 = L_1U_1$ be the parabolic subgroup corresponding to β_1. Let Y_1 be a non-zero element in $\mathfrak{g}_{-\beta_1}$. The minimal orbit is generated by Y_1. The pair (Y_1, H_Δ) defines the character $\psi(Y_1, H_\Delta)$ of U which is trivial on U_1. If V is a minimal representation then $J_{\psi(Y_1,H_\Delta)}(V) \neq 0$ so the formula (4) shows that $J_{U_1}(V) \neq 0$ and it has generic subquotients.

Let $\beta = \beta_1 + 2\beta_2$. Note that β is perpendicular to β_1. It follows that $\beta^\vee(t)$ is in the center of L_1. In fact, if we identify $L_1 \cong \mathrm{GL}_2$ as in [Mu], then $\beta^\vee(t)$ is a scalar matrix:

$$\beta^\vee(t) = \mathrm{diag}(t, t).$$

By Schur's lemma elements of the center of L_1 must act by a scalar on every irreducible subquotient of $J_{U_1}(V)$.

Lemma 5.1. *Let V be a minimal representation. Then, up to a complex number of norm one, $\beta^\vee(t)$ acts by ν^2 on any irreducible generic subquotient of $J_{U_1}(V)$.*

Proof. The proof of this is completely analogous to the proof of Lemma 3.3 and involves a comparison of $J_{\psi(Y_1,H_\Delta)}(V)$ and $J_{\psi(Y_1,H_1)}(V)$ where Y_1 and H_1 belong to an $\mathfrak{sl}_2$-triple (X_1, H_1, Y_1) spanning $\mathfrak{sl}_2(\beta_1)$. In the simply laced case, however, working out the action of $\beta^\vee(t)$ on the Heisenberg representation W_Y is based on the fact that there is a polarization invariant for $\mathrm{SL}_2(\beta)$. There is no such polarization here, so this is why we have a weaker result here. □

Lemma 5.2. *Let τ be an irreducible subquotient of $J_{U_1}(V)$. Then τ is not supercuspidal.*

Proof. Assume that τ is supercuspidal. Then τ can be considered a quotient of $J_{U_1}(V)$. In particular,

$$V \subseteq \mathrm{Ind}_{P_1}^G(\tau)$$

where induction is not normalized. Using the identification $L_1 \cong \mathrm{GL}_2$, the previous Lemma implies that $\tau = \sigma \otimes |\det|$ where σ has a unitary central character. The square root of the modular character of L_1 acting on U_1 is

$$\delta_1^{1/2}(g) = |\det(g)|^{5/2}.$$

Let $I_1(s,\sigma)$ denote the (normalized) induced representation, where we induce the representation $\sigma \otimes |\det|^s$ on L_1. We have

$$V \subseteq I_1(-3/2, \sigma).$$

Now, if V is minimal, then it is clearly a proper submodule of the induced principal series. Thus the principal series $I_1(s,\sigma)$ reduces for $s = -3/2$. On the other hand, Shahidi [Sh] has shown that if σ is a supercuspidal representation with unitary central character then $I_1(s,\sigma)$ could reduce only for a half integral point between -1 and 1. Since $-3/2$ is outside this range, we have a contradiction. The lemma is proved. □

The previous lemma shows that any minimal representation V is induced from a Borel subgroup. In particular, $J_{U_2}(V) \neq 0$, where $P_2 = L_2U_2$ is the parabolic subgroup corresponding to β_2. We identify L_2 with GL_2 as in [Mu]. Let Y_2 be a non-zero element in $\mathfrak{g}_{-\beta_2}$. The minimal orbit does not contain Y_2. The pair (Y_2, H_Δ) defines a character $\psi(Y_2, H_\Delta)$ of U which is trivial on U_2. Note that the minimal orbit does not contain Y_2. Thus, if V is a minimal representation, $J_{\psi(Y_2,H_\Delta)}(V) = 0$. The formula (4) shows that irreducible subquotients of $J_{U_2}(V)$ are one-dimensional characters of L_2. It follows that

$$V \subseteq I_2(s, \chi \circ \det)$$

where I_2 is a degenerate (normalized) principal series (denoted by I_α in [Mu].) and χ a unitary character. The representation $I_2(s, \chi \circ \det)$ is irreducible unless

$$\chi = 1, s = \pm 3/2, \text{ or } \chi^2 = 1, s = \pm 1/2 \text{ or } \chi^3 = 1, s = \pm 1/2.$$

In order to describe irreducible subquotients of $I_2(s, \chi \circ \det)$ we need some notation. Let $\pi(\mu_1, \mu_2)$ be the tempered principal series representation of GL_2 where μ_1 and μ_2 are two unitary characters. Let $\delta(\chi)$ be the Steinberg representation of GL_2 twisted by the character $\chi \circ \det$.

The following description of non-trivial subquoteints of $I_2(s, \chi \circ \det)$ is taken from Section 4 in [Mu]. (Note that $I_2(-s, \chi \circ \det)$ has the same irreducible subquotients as $I_2(s, \chi^{-1} \circ \det)$, so it suffices to consider s positive.)

Proposition 5.3. *In $R(G_2)$, the Grothendieck group of admissible representations of* G_2*, we have:*

(1) Let χ be of order 2. Then

$$I_2(1/2, \chi \circ \det) = J_1(1, \pi(1, \chi)) + J_1(1/2, \delta(\chi)).$$

(2) Let χ be of order 3. Then

$$I_2(1/2, \chi \circ \det) = J_1(1, \pi(\chi, \chi^{-1})) + J_2(1/2, \delta(\chi^{-1})).$$

(3) Let $\chi = 1$. Then

$$I_2(1/2, 1_{\mathrm{GL}_2}) = \pi(1) + J_1(1, \pi(1, 1)) + J_1(1/2, \delta(1)).$$

(4) Let $\chi = 1$. Then

$$I_2(3/2, 1_{\mathrm{GL}_2}) = 1_{\mathrm{G}_2} + J_1(5/2, \delta(1)).$$

where $J_i(s,\sigma)$ is the unique (Langlands) quotient of the principal series representation $I_i(s,\sigma)$ and $\pi(1)$ a discrete series representation which is a submodule of $I_1(1/2, \delta(1))$.

Now it is easy to see that none of the subquotients described above is a minimal representation. As seen in the proof of Lemma 5.2 a minimal representation V can be a submodule of $I_1(s,\sigma)$ where σ has a unitary central character only if $s = -3/2$. Thus, if we look the case $\chi^3 = 1$, for example, then $J_1(1, \pi(\chi, \chi^{-1}))$ cannot be minimal since $J_1(1, \pi(\chi, \chi^{-1}))$ is a submodule of $I_1(-1, \pi(\chi^{-1}, \chi))$. The same argument applies to all Langlands quotients of type J_1 appearing in the above Proposition and to $\pi(1)$. Finally, $J_2(1/2, \delta(\chi^{-1}))$ cannot be minimal since it is a submodule of $I_2(-1/2, \delta(\chi))$ and, by Frobenius reciprocity, the space of U_2-coinvariants of $J_2(1/2, \delta(\chi^{-1}))$ is L_2-generic. This shows that G has no minimal representations.

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