Concentration of solutions of non-linear elliptic equations involving critical Sobolev exponent

Man Chun Leung
`matlmc@math.nus.edu.sg`

Abstract

In $\mathbb{R}^n$ $(n \geq 3)$, an interesting property of the semi-linear equation

$$\Delta\, u_o + c_n\, K_o u_o^{\frac{n+2}{n-2}} = 0$$

is that, when K_o is a positive constant, solutions can concentrate at any point. When K_o is not a constant, we show that concentration of solutions requires strong conditions on K_o. Through the stereographic projection, the discussion can be extended to S^n, and is related to bubbling, or the blow-up phenomenon.

KEY WORDS: Concentration, blow-up, scalar curvature equation.

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§ 1. Introduction

In this article we consider positive smooth solutions u of the equation

$$\Delta_{g_1} u - c_n\, n(n-1)u + c_n\, K u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ S^n. \tag{1.1}$$

Here Δ_{g_1} is the Laplacian for the standard metric g_1 on the unit sphere S^n $(n \geq 3)$, and $c_n = (n-2)/\,[4(n-1)]$. Equation (1.1) describes the scalar curvature K of the conformal metric $u^{\frac{4}{n-2}}\, g_1$ [4]. Under the stereographic projection $\mathcal{P} : S^n \to \mathbb{R}^n$ (cf. [14] and § 7), with

$$K_o(y) := K(\mathcal{P}^{-1}(y))\,, \qquad u_o(y) = u\,(\mathcal{P}^{-1}(y)) \left(\frac{2}{1 + \|y\|^2} \right)^{\frac{n-2}{2}} \quad \text{for } \ y \in \mathbb{R}^n, \tag{1.2}$$

equation (1.1) can be expressed as

$$\Delta\, u_o + c_n\, K_o u_o^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ \mathbb{R}^n. \tag{1.3}$$

The geometric accent of the equations is reflected analytically in the critical Sobolev exponent. Together with conformal invariance, they may cause bubbles to appear [25]. Active studies are conducted on existence of solutions and fine asymptotic properties, employing powerful ideas in partial differential equations and global geometry (see, for instances, recent publications [1], [3], [7], [8], [9], [10], [17], [22], and the references within). However, key questions like the Nirenberg problem and the Kazdan-Warner problem remain unresolved.

An exquisite result of Gidas, Ni and Nirenberg ([12], [13]; cf. [5], [23]) shows that when K_o is a positive constant, say (after rescaling), $K_o = 4\,n(n-1)$, any positive smooth solution of equation (1.3) is of the form

$$u_{\lambda,\,p}(y) := \left(\frac{\lambda}{\lambda^2 + \|y - p\|^2} \right)^{\frac{n-2}{2}} \quad \text{for } \ y \in \mathbb{R}^n. \tag{1.4}$$

Here p is a fixed point in $\mathbb{R}^n$, and λ a positive number. Thus the rigidity and flexibility of the equation are captured. As an interesting consequence, solutions can concentrate near p when $\lambda \to 0^+$. Indeed, direct calculation reveals that

$$\int_{\mathbb{R}^n} u_{\lambda,p}^{\frac{2n}{n-2}}(y)\,dy \quad \text{is} \quad \text{independent} \quad \text{on} \quad \lambda \ \text{and} \ p,$$

$$\int_{\mathbb{R}^n\setminus B_p(\rho)} u_{\lambda,p}^{\frac{2n}{n-2}}(y)\,dy \,\le\, C\left(\frac{\lambda}{\rho}\right)^n \longrightarrow 0 \qquad \text{as} \qquad \lambda \to 0^+.$$

In the above, C a positive constant, $B_p(\rho)$ the open ball in $\mathbb{R}^n$ with center at p and radius $\rho > 0$. Observe that $u_{\lambda,p}(p) \to \infty$ as $\lambda \to 0^+$. Here we use the term concentration to denote the general phenomenon when the solution is large in a neighborhood of a point, and small outside. The precise meaning is made evident in each theorem.

When K_o is *not* a constant, we show that positive smooth solutions of equation (1.3) in the form of (1.2) can only concentrate on particular places. The first observation is that concentration *cannot* take place at a point p with $K_o(p) \le 0$ (see propositions 2.2 and 2.4 for the precise statements). Here we make use of the fact that conformal deformations on $(S^n,\ g_1)$ tend to increase the $L^{\frac{n}{2}}$-norm of the scalar curvatures [15].

The second obstruction for concentration is $\|\bigtriangledown K_o(p)\| \ne 0$, a consequence of the famed Kazdan-Warner *balance* formula:

$$\int_{S^n} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} = 0\,. \tag{1.5}$$

Here X is an arbitrary conformal Killing vector field on $(S^n,\ g_1)$ (cf. [10] [14]).

Formula (1.5), when projected onto $\mathbb{R}^n$ via $\mathcal{P}$, and when X is generated by rescaling, gives rise to the Pohozaev identity

$$\int_{\mathbb{R}^n} (y\cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy = 0\,. \tag{1.6}$$

From (1.6), we derive the third obstruction, namely, *high concentration cannot take place at a point p with $\Delta K_o(p) \ne 0$* (the precise statement is found in theorem 4.1). Earlier, Chang-Gursky-Yang [6] and Schoen-Zhang [27] consider similar situation. ((1.6) is satisfied by solutions u_o and K_o related to u and K by (1.2), which guarantees the convergence of the integral and non-existence of boundary terms.

It is not hard to relax this requirement by imposing suitable decay condition on u_o. cf. [6] .)

Observe that equation (1.3) is invariant under translations. Using this, we also discover that $\bigtriangledown(\Delta\, K_o)(p) \neq 0$ is an obstruction (theorem 5.1). Further exploration of the Pohozaev identity shows that

$$3\left(\sum_{i=1}^{n} \frac{\partial^4 K_o}{\partial y_i^4}(p)\right) + \Delta^2\, K_o(p) \neq 0 \tag{1.7}$$

places an additional restriction on strong concentration, see theorem 6.1. Here, $\Delta^2\, K_o = \Delta\,(\Delta\, K_o)\,$.

A natural link with the kind of concentrations discussed in this article is found in blow-up or bubbling. Let $\{u_i\} \subset C_+^\infty(S^n)$ be a sequence of solutions of equation (1.1). A point x_b is called a blow-up point of $\{u_i\}$ if there exists a sequence $\{x_i\} \subset S^n$ such that $\lim\limits_{i\to\infty} u_i(x_i) = \infty$ and $\lim\limits_{i\to\infty} x_i = x_b\,$. Point singularity of this type is studied in detail by R. Schoen, Y.Y. Li (cf. [22]), Chen and Lin (cf. [7] [8] [24]), and others. Under suitable conditions [10] [20], u_i can be approximated near x_b by a standard solution as in (1.4).

In order to obtain *a priori* bounds and existence results, methods are developed to eliminate the possibility of blow-up (see the elegant works of Aubin [2], Chang-Gursky-Yang [6], Chen-Li [10], Y.-Y. Li [20] [21], Chen-Lin (op. cit.), Schoen [26], Schoen-Escobar [11], and Schoen-Zhang [27]). Conditions allow, uniform upper bound also implies uniform lower bound, thanks to the Harnack inequality. This becomes crucial as certain blow-up tends to pull down the solutions to zero outside a small neighborhood of the blow-up point x_b (see [8]). The conditions discussed here help to avoid this specific type of concentrations.

Conventions. Throughout this article, $n \geq 3$ is an integer; the functions $u_o \in C_+^\infty(\mathbb{R}^n)$ and $K_o \in C^\infty(\mathbb{R}^n)$ descend from the corresponding functions on S^n via (1.2). We observe the practice of summing over repeated indices, and use $C,$ possibly with sub-indices, to denote various positive constants, which may be rendered differently according to the contents.

§ 2. Zeroth order condition

Let

$$\begin{aligned} V &:= \int_{S^n} u^{\frac{2n}{n-2}}\, dV_{g_1} = \int_{\mathbb{R}^n} u_o^{\frac{2n}{n-2}}\, dy\,, \\ T &:= \int_{S^n} K\, u^{\frac{2n}{n-2}}\, dV_1 = \int_{\mathbb{R}^n} K_o\, u_o^{\frac{2n}{n-2}}\, dy\,. \end{aligned}$$

On account of (1.1), we have

$$T = c_n^{-1} \int_{S^n} \| \nabla_1\ u\|^2\, dV_{g_1} + n(n-1) \int_{S^n} |u|^2\, dV_{g_1} > 0\,. \tag{2.1}$$

Proposition 2.2. *Let K_o be as in (1.2). Assume that $K_o(p) < 0$ for a point $p \in \mathbb{R}^n$. There exist positive constants ρ_o and ε_o such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequality*

$$\int_{\mathbb{R}^n \setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \ \le\ \varepsilon \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \tag{2.3}$$

cannot hold for $\rho \le \rho_o$ and $\varepsilon \le \varepsilon_o$.

Proof. Take ρ_o to be small enough so that $\sup\limits_{B_p(\rho_o)} K_o < 0$. Set

$$\sigma := -\left(\sup_{B_p(\rho_o)} K_o\right) > 0\,, \qquad \text{and} \qquad \varepsilon_o = \left(\sup_{\mathbb{R}^n} K_o\right)^{-1} \sigma\,.$$

Suppose that (2.3) holds for $\rho \le \rho_o$ and $\varepsilon \le \varepsilon_o$. We have

$$\begin{aligned} T &= \int_{B_p(\rho)} K_o\, u_o^{\frac{2n}{n-2}}\, dy + \int_{\mathbb{R}^n \setminus B_p(\rho)} K_o\, u_o^{\frac{2n}{n-2}}\, dy \\ &\le -\sigma \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy + \left(\sup_{\mathbb{R}^n} K_o\right) \int_{\mathbb{R}^n \setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \\ &\le -\sigma \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy + \left(\sup_{\mathbb{R}^n} K_o\right) \varepsilon \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le 0\,, \end{aligned}$$

which contradicts (2.1). □

From above, it is not immediately clear that concentration cannot take place at a point p with $K_o(p) = 0$. This can be shown with the help of a result in [15].

Proposition 2.4. *Let K_o be as in (1.2). Assume that $K_o(p) = 0$ for a point $p \in I\!R^n$. Given any positive number C, there exist positive constants ρ_1 and ε_1 such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequalities*

$$\int_{I\!R^n \setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \;\le\; \varepsilon \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \qquad \text{and} \;\; V \le C \tag{2.5}$$

do not hold for $\rho \le \rho_1$ and $\varepsilon \le \varepsilon_1$.

Proof. By applying lemma 4.5 in [15] on S^n with the standard metric, and then transferring to $\mathbb{R}^n$ by the stereographic projection as in (1.2), we obtain

$$\int_{\mathbb{R}^n} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy \;\ge\; [\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n\,. \tag{2.6}$$

Here ω_n is the volume of the standard n-sphere. Take ρ_1 to be small enough so that

$$|K_o(y)|^{\frac{n}{2}} \le \frac{[\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n}{2\,C}$$

for $y \in B_p(\rho)$. Let

$$\varepsilon_1 = \frac{[\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n}{2\,C} \left(\sup_{\mathbb{R}^n} K_o \right)^{-\frac{n}{2}}\,.$$

Here C is the positive constant in (2.5). Suppose that (2.5) holds for $\rho \le \rho_1$ and $\varepsilon \le \varepsilon_1$. We have

$$\begin{aligned}
& \int_{\mathbb{R}^n} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy \\
= \;& \int_{B_p(\rho)} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy + \int_{\mathbb{R}^n \setminus B_p(\rho)} |K_o|^{\frac{n}{2}}\, u_o^{\frac{2n}{n-2}}\, dy \\
\le \;& \frac{[\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n}{2C} \int_{B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy + \left(\sup_{\mathbb{R}^n} K_o \right)^{\frac{n}{2}} \int_{\mathbb{R}^n \setminus B_p(\rho)} u_o^{\frac{2n}{n-2}}\, dy \\
< \;& \frac{[\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n}{2} + \varepsilon\, V \left(\sup_{\mathbb{R}^n} K_o \right)^{\frac{n}{2}} \\
= \;& [\,n(n-1)\,]^{\frac{n}{2}}\, \omega_n\,.
\end{aligned}$$

The strict inequality above provides a contradiction with (2.6). $\square$

Remark 2.7. In proposition 2.4, whether the bound on V can be removed is not known. Interestingly, there are examples which show that V can become very large due to strong concentration at a point, even though K_o is very close to a positive constant (see [18] and [29]; cf. also [16]). However, under mild conditions on K_o (see [17]), it can be shown that if x_b is a blow-up point as defined in the introduction, then $K(x_b) > 0$.

§ 3. First order property

The stereographic projection $\mathcal{P}$ enables us to bring the discussion from $\mathbb{R}^n$ to S^n, or vice versa. For first order obstruction, it is more convenient to consider S^n. Denote by $\mathcal{B}_q\,(r)$ the open (metric) ball in the standard sphere S^n, where q is the center and $r \in (0,\ \pi)$ the radius of the ball.

Proposition 3.1. *Let $K \in C^\infty(S^n)$. Assume that ($K(q) > 0$ and) $\bigtriangledown_1 K(q) \neq 0$ for a point $q \in S^n$. There exist positive constants ρ_2 and ε_2 such that for any positive smooth solution u of equation (1.1), the concentration inequality*

$$\int_{S^n \setminus \mathcal{B}_q\,(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \le \varepsilon \int_{\mathcal{B}_q\,(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \tag{3.2}$$

cannot hold for $r \le \rho_2$ and $\varepsilon \le \varepsilon_2$.

Proof. Let $\| \bigtriangledown_1\ K(q)\| = \delta^2 > 0$. In the Kazdan-Warner formula (1.5), we can choose the coordinate system so that the conformal Killing vector field X has the property that $\|X(q)\| = 1$ and $X(q)$ is in the direction of $\bigtriangledown_1 K\,(q)$. This is possible because of the innate symmetry of S^n. Furthermore, we may take $\|X\| \le 1$ in S^n. Fix a positive constant ρ_2 such that

$$\langle X\,,\ \bigtriangledown_1 K(x)\rangle_{g_1} \ge \frac{\delta^2}{2} \qquad \text{for} \quad x \in \mathcal{B}_q\,(\rho_2)\,. \tag{3.3}$$

Let D be a positive constant such that

$$\| \bigtriangledown_1\ K\| \le D \quad \text{in} \ \ S^n\,.$$

It follows that $X(K) = \langle X\,,\ \bigtriangledown_1 K\rangle_{g_1} \le \|X\| \cdot \|\bigtriangledown_1\ K\| \le D$. Take

$$\varepsilon_2 = \frac{\delta^2}{3D}\,.$$

For $r \le \rho_2$, we have

$$\begin{aligned}
&\int_{S^n} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} = 0 \\
\Longrightarrow\ &\int_{\mathcal{B}_q\,(r)} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} = -\int_{S^n\setminus\mathcal{B}_q\,(r)} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} \\
\Longrightarrow\ &\frac{\delta^2}{2}\int_{\mathcal{B}_q\,(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \le D\int_{S^n\setminus\mathcal{B}_q\,(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \\
\Longrightarrow\ &\int_{S^n\setminus\mathcal{B}_q\,(r)} u^{\frac{2n}{n-2}}\, dV_{g_1} \ge \frac{\delta^2}{2D}\int_{\mathcal{B}_q\,(r)} u^{\frac{2n}{n-2}}\, dV_{g_1}\,.
\end{aligned}$$

The imbalance renders (3.2) invalid for $\varepsilon \le \varepsilon_2$ and $r \le \rho_2$. $\square$

Related to the above, we refer to [28] and [30] for first order conditions on concentration for certain singularly perturbed elliptic equations in $\mathbb{R}^n$.

§ 4. Second order property

Because equation (1.3) is invariant under translations, for the moment, we focus the discussion on the origin.

Theorem 4.1. *Let K_o be as in (1.2). Assume that $\bigtriangledown K_o(0) = 0$ and $\Delta K_o(0) \ne 0$. Given any positive numbers C and ρ, there exist positive numbers c_1 and c_2, such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration*

$$\left\| \left(\frac{u_o}{u_{\lambda,\,0}}\right)^{\frac{2n}{n-2}} - 1 \right\|_{C^o\,(B_o(\rho))} \le \delta\,, \qquad \int_{\mathbb{R}^n\setminus B_o(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le C\,\lambda^3 \tag{4.2}$$

cannot take place for $\lambda \le c_1$ and $\delta \le c_2$. Here $u_{\lambda,\,0}$ is the standard spherical solution defined in (1.4).

Proof. Consider the case $\Delta K_o(0) = \Lambda > 0$ first. As

$$\begin{aligned}
(4.3) \qquad & \int_{\mathbb{R}^n} (y \cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy &=& \ 0 \qquad \text{(the Pohozaev identity)} \\
\Longrightarrow \ & \int_{B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy &=& -\int_{\mathbb{R}^n \setminus B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy\,,
\end{aligned}$$

we intend to show that, under concentration as expressed in (4.2), the left hand side of the above is $O(\lambda^2)$, and the other side $O(\lambda^3)$. Thus (4.3) cannot be balanced for small λ. To this end we apply Taylor's expansion and the fact that $\bigtriangledown K_o(0) = 0$, obtaining

$$K_o(y) = K_o(0) + \frac{1}{2} \sum_{i\,,\,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) + R(y)\,.$$

Here R is a smooth function with vanishing first and second order derivatives at the origin. It follows that

$$(4.4) \qquad y \cdot \bigtriangledown K_o(y) = \sum_{i\,,\,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) + y \cdot \bigtriangledown R(y)\,.$$

By the remainder theorem for Taylor's expansions, we have

$$(4.5) \qquad |y \cdot \bigtriangledown R(y)| \le |y| \cdot \| \bigtriangledown \ R(y)\| \le C_1 \|y\|^3 \quad \text{for} \ \ \|y\| \le \rho\,.$$

It follows from (4.4) that

$$(4.6) \qquad |y \cdot \bigtriangledown K_o(y)| \le C_2 |y|^2 \quad \text{for} \ \ |y| \le \rho\,.$$

Assuming that (4.2) holds. We have

$$\begin{aligned}
(4.7) \qquad & \int_{B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy \\
= \ & \int_{B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy + \int_{B_o(\rho)} (y \cdot \bigtriangledown K_o) \left[u_o^{\frac{2n}{n-2}} - u_{\lambda,0}^{\frac{2n}{n-2}} \right] dy \\
\ge \ & \int_{B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy - \int_{B_o(\rho)} |y \cdot \bigtriangledown K_o| \left| u_o^{\frac{2n}{n-2}} - u_{\lambda,0}^{\frac{2n}{n-2}} \right| dy \\
\ge \ & \sum_{i\,,\,j} \left[\frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) \right] \int_{B_o(\rho)} y_i\, y_j\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy - C_2\, \delta \int_{B_o(\rho)} \|y\|^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \\
& \qquad - C_1 \int_{B_o(\rho)} \|y\|^3\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \qquad \text{(using (4.2), (4.4) − (4.6))}\,.
\end{aligned}$$

As $u_{\lambda,\,0}$ depends only on $r = \|y\|\,,$ by symmetry, one obtains

$$\begin{aligned} \int_{B_o\,(\rho)} y_i\, y_j\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy &= 0 \quad \text{for} \;\; i \neq j\,, \\ \int_{B_o\,(\rho)} y_i^2\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy &= \int_{B_o\,(\rho)} y_j^2\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy\,, \\ \Longrightarrow \;\; \int_{B_o\,(\rho)} y_i^2\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy &= \frac{1}{n} \int_{B_o\,(\rho)} r^2\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy\,. \end{aligned}$$

We compute

$$\begin{aligned} (4.8) \qquad & \int_{B_o\,(\rho)} r^2\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy \\ &= \omega_{n-1} \int_0^\rho \left(\frac{\lambda}{\lambda^2 + r^2} \right)^n r^{n+1}\, dr \\ &= \omega_{n-1} \int_0^{\arctan\,(\rho/\lambda)} \left(\frac{\lambda}{\lambda^2\, \sec^2\, \phi} \right)^n \lambda^{n+2}\, \tan^{n+1} \phi\, \sec^2 \phi\, d\phi \qquad (r = \lambda \tan\, \phi) \\ &= \lambda^2\, \omega_{n-1} \int_0^{\arctan\,(\rho/\lambda)} \cos^{2(n-1)} \phi \left(\frac{\sin^{n+1} \phi}{\cos^{n+1} \phi} \right) d\phi \\ &= \lambda^2\, \omega_{n-1} \int_0^{\arctan\,(\rho/\lambda)} \sin^{n+1} \phi\, \cos^{n-3} \phi\, d\phi \qquad (n \geq 3) \\ &= \lambda^2\, I_{\rho/\lambda}\,. \end{aligned}$$

Here

$$(4.9) \qquad I_{\rho/\lambda} := \omega_{n-1} \int_0^{\arctan\,(\rho/\lambda)} \sin^{n+1} \phi\, \cos^{n-3} \phi\, d\phi\,,$$

and ω_{n-1} is the volume of the standard sphere S^{n-1}. As it can be seen in (4.9), $I_{\rho/\lambda}$ is bounded from above, and its value is larger for smaller λ, assuming that ρ is fixed. Similarly,

$$(4.10) \qquad \int_{B_o(\rho)} \|y\|^3\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy = \lambda^3\, \omega_{n-1} \int_0^{\arctan\,(\rho/\lambda)} \sin^{n+2} \phi\, \cos^{n-4} \phi\, d\phi\,.$$

When $n \geq 4\,,$

$$\int_{B_o(\rho)} \|y\|^3\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy \leq C_4\, \lambda^3.$$

When $n = 3\,,$

$$\begin{aligned}
&\int_0^{\arctan(\rho/\lambda)} \sin^{n+2}\phi\, \sec\phi\, d\phi \\
\le\ & \int_0^{\arctan(\rho/\lambda)} \sec\phi\, d\phi \\
=\ & \ln|\sec y + \tan y|_{y=\arctan(\rho/\lambda)} = \ln\left|\sqrt{1+\tan^2 y} + \tan y\right|_{y=\arctan(\rho/\lambda)} \\
=\ & \ln\left(\sqrt{1+\frac{\rho^2}{\lambda^2}} + \frac{\rho}{\lambda}\right) \le \ln\left(\frac{3\,\rho}{\lambda}\right) \le \sqrt{\frac{3\rho}{\lambda}} \qquad (\text{provided } \ \rho/\lambda \ge 1)\,.
\end{aligned}$$

Thus

$$\int_{B_o(\rho)} \|y\|^3\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \le C_5\, \lambda^{\frac{5}{2}} \quad \text{for} \ \ \lambda \le \rho \quad (n \ge 3)\,,$$

where C_5 is a positive constant that depends on ρ and n only. It follows from the symmetry and (4.7)–(4.10) that

$$\int_{B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy \ge \left[\frac{\Delta K_o(0)}{n} - C\,\delta\right] I_{\rho/\lambda}\, \lambda^2 - C_6\, \lambda^{\frac{5}{2}}\,. \tag{4.11}$$

We choose

$$c_2 = \frac{\Lambda}{2n\, C}$$

so that when $\delta \le \Lambda_2$, we have

$$\left[\frac{\Delta K_o(0)}{n} - C\,\delta\right] \ge \frac{\Lambda}{n} - \frac{\Lambda}{2n} = \frac{\Lambda}{2n}\,.$$

By the decay property of K_o as in (1.2), there exists a positive constant C_7 such that $|y \cdot \bigtriangledown K_o(y)| \le C_7$ for all $y \in \mathbb{R}^n$. From (4.3) we have

$$\begin{aligned}
&\left(I_{\rho/\lambda}\, \frac{\Lambda}{2n}\right) \lambda^2 - C_6\, \lambda^{\frac{5}{2}} \le \int_{\mathbb{R}^n \setminus B_o(\rho)} |y \cdot \bigtriangledown K_o|\, u_o^{\frac{2n}{n-2}}\, dy \qquad (4.12)\\
\Longrightarrow\quad & \left(I_{\rho/\lambda}\, \frac{\Lambda}{2n}\right) \lambda^2 \le C_6\, \lambda^{\frac{5}{2}} + C_8\, \lambda^3 \\
\Longrightarrow\quad & \frac{\Lambda\, I_{20}}{2n} \le C_9\, \lambda^{\frac{1}{2}} \qquad \left(\text{provided} \quad \text{that} \ \ \lambda \le \min\left\{\frac{\rho}{20}\,,\ 1\right\}\right) \\
\Longrightarrow\quad & \lambda > \left[\frac{\Lambda\, I_{20}}{2n\, C_9}\right]^2.
\end{aligned}$$

Hence we may choose

$$c_1 = \left[\frac{\Lambda\, I_{20}}{2n\, C_9} \right]^2 .$$

From (4.12), we conclude that (4.2) cannot hold for $\lambda \le c_1$ and $\delta \le c_2$. The case $\Delta\, K_o(0) < 0$ is similar. □

Remark 4.13. Fixing ρ in (4.2), we observe that when λ is small enough, then (4.2) implies that

$$\int_{\mathbb{R}^n \setminus B_0(\rho)} u_o^{\frac{2n}{n-2}}\, dy \;\le\; \lambda^2 \int_{B_0(\rho)} u_o^{\frac{2n}{n-2}}\, dy\,.$$

(Compare with the calculations in the introduction following (1.4).) It follows that (4.2), when projected back to S^n, also implies inequality of the form (3.2).

§ 5. Third order restriction

Theorem 5.1. *Let K_o be as in (1.2). Assume that $\| \bigtriangledown K_o(0)\| = \Delta\, K_o(0) = 0$ and $\bigtriangledown (\Delta\, K_o)(0) \neq 0$. Given any positive numbers C and ρ , there exist positive constants c_3 and c_4, such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequalities*

$$\left\| \left(\frac{u_o}{u_{\lambda,\,0}} \right)^{\frac{2n}{n-2}} - 1 \right\|_{C^o\,(B_o(\rho))} \le \delta\, \lambda\,, \qquad \int_{\mathbb{R}^n \setminus B_o(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le C\, \lambda^3 \tag{5.2}$$

cannot take place for $\lambda \le c_3$ and $\delta \le c_4$.

Proof. We proceed as in the proof of theorem 4.1, and observe the effect of translations. Take a point $p = (p_1,\, \cdots,\, p_n) \in \mathbb{R}^n$ such that

$$\gamma := p \cdot \bigtriangledown (\Delta\, K_o)(0) > 0\,. \tag{5.3}$$

Consider the translation

$$K_p(y) := K_o(y - p) \quad \text{ and } \quad u_p := u_o(y - p) \quad \text{ for } \; y \in \mathbb{R}^n. \tag{5.4}$$

It follows that u_p satisfies the equation

$$\Delta\, u_p + c_n\, K_p\, u_p^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ \mathbb{R}^n. \tag{5.5}$$

In addition, u_p has similar decay property as expressed in (1.2). We also have

$$\left\| \left(\frac{u_p(y)}{u_{\lambda,\,0}(y-p)} \right)^{\frac{2n}{n-2}} - 1 \right\|_{C^o(B_p(\rho))} \le \delta\,\lambda\,, \qquad \int_{\mathbb{R}^n \setminus B_p(\rho)} u_p^{\frac{2n}{n-2}}\, dy \le C\,\lambda^3. \tag{5.6}$$

Let

$$u_{\lambda\,,\,p}(y) := u_{\lambda,\,0}(y-p) = \left(\frac{\lambda}{\lambda^2 + \|y-p\|^2} \right)^{\frac{n-2}{2}} \qquad \text{for } \ y \in \mathbb{R}^n.$$

One obtains

$$\begin{aligned} & \int_{B_p(\rho)} (y \cdot \bigtriangledown K_p)\, u_p^{\frac{2n}{n-2}}\, dy \\ = \ & \int_{B_p(\rho)} [(y-p)\cdot \bigtriangledown K_p]\, u_p^{\frac{2n}{n-2}}\, dy + \int_{B_p(\rho)} (p \cdot \bigtriangledown K_p)\, u_p^{\frac{2n}{n-2}}\, dy \\ \ge \ & \int_{B_p(\rho)} [(y-p)\cdot \bigtriangledown K_p]\, u_{\lambda\,,\,p}^{\frac{2n}{n-2}}\, dy - \int_{B_p(\rho)} |\,(y-p)\cdot \bigtriangledown K_p\,| \left| u_p^{\frac{2n}{n-2}} - u_{\lambda\,,\,p}^{\frac{2n}{n-2}} \right| dy \\ & \quad + \int_{B_p(\rho)} (p \cdot \bigtriangledown K_p)\, u_{\lambda\,,\,p}^{\frac{2n}{n-2}}\, dy - \int_{B_p(\rho)} |p \cdot \bigtriangledown K_p| \left| u_p^{\frac{2n}{n-2}} - u_{\lambda\,,\,p}^{\frac{2n}{n-2}} \right| dy\,. \end{aligned}$$

We apply the Taylor expansion

$$\begin{aligned} K_p(y) &= K_p(p) + \frac{1}{2} \sum_{i\,,\,j} (y_i - p_i)\,(y_j - p_j)\, \frac{\partial^2 K_p}{\partial y_i \partial y_j}\,(p) \\ & \qquad + \frac{1}{3!} \sum_{i\,,\,j\,,\,k} (y_i - p_i)\,(y_j - p_j)\,(y_k - p_k)\, \frac{\partial^3 K_p}{\partial y_i \partial y_j \partial y_k}\,(p) + R(y)\,, \\ &= K_o(0) + \frac{1}{2} \sum_{i\,,\,j} (y_i - p_i)\,(y_j - p_j)\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) \\ & \qquad + \frac{1}{3!} \sum_{i\,,\,j\,,\,k} (y_i - p_i)\,(y_j - p_j)\,(y_k - p_k)\, \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}\,(0) + R(y)\,. \end{aligned}$$

Here R is a smooth function with vanishing derivatives (up to at least third order) at p. As in (4.4),

$$\begin{aligned} (y-p)\cdot \bigtriangledown K_p(y) &= \sum_{i\,,\,j} (y_i - p_i)\,(y_j - p_j)\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) \\ & \quad + \frac{1}{2} \sum_{i\,,\,j\,,\,k} (y_i - p_i)\,(y_j - p_j)\,(y_k - p_k)\, \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}\,(0) \\ & \quad + (y-p)\cdot \bigtriangledown R(y)\,. \end{aligned} \tag{5.7}$$

By the remainder theorem for Taylor's expansions, we have

$$\begin{aligned} |p\cdot\bigtriangledown R(y)| &\le C_1\|y-p\|^3, \\ |(y-p)\cdot\bigtriangledown R(y)| &\le C_2\|y-p\|^4 \qquad \text{for } \ \|y-p\|<\rho\,. \end{aligned} \tag{5.8}$$

From (5.7), we also have

$$|(y-p)\cdot\bigtriangledown K_p(y)| \le C_3\|y-p\|^2 \quad \text{for } \ \|y-p\|<\rho\,. \tag{5.9}$$

Likewise,

$$\begin{aligned} & p\cdot\bigtriangledown K_p(y) \\ = \ & \frac{1}{2}\sum_{i\,,\,j}\,[\,p_i\,(y_j-p_j)+p_j\,(y_i-p_i)\,]\,\frac{\partial^2 K_o}{\partial y_i\partial y_j}\,(0) \\ & +\frac{1}{3!}\sum_{i\,,\,j\,,\,k}\,[\,p_i\,(y_j-p_j)\,(y_k-p_k)+(y_i-p_i)\,p_j\,(y_k-p_k) \\ & \qquad +(y_i-p_i)\,(y_j-p_j)\,p_k\,]\,\frac{\partial^3 K_o}{\partial y_i\partial y_j\partial y_k}\,(0)+\,p\cdot\bigtriangledown R(y)\,. \end{aligned} \tag{5.10}$$

By symmetry,

$$\begin{aligned} & \int_{B_p(\rho)}(y_j-p_j)\,u_{\lambda\,,\,p}^{\frac{2n}{n-2}}(y)\,dy=0\,, \\ & \int_{B_p(\rho)}(y_i-p_i)\,(y_j-p_j)\,u_{\lambda\,,\,p}^{\frac{2n}{n-2}}(y)\,dy=0 \quad \text{ for } \ i\ne j\,, \\ & \int_{B_p(\rho)}(y_i-p_i)^2\,u_{\lambda\,,\,p}^{\frac{2n}{n-2}}(y)\,dy=\frac{1}{n}\int_{B_p(\rho)}r^2\,u_{\lambda}^{\frac{2n}{n-2}}(y)\,dy \qquad (\text{here } \ r^2=\|y-p\|^2)\,, \\ & \int_{B_p(\rho)}(y_i-p_i)\,(y_j-p_j)\,(y_k-p_k)\,u_{\lambda\,,\,p}^{\frac{2n}{n-2}}(y)\,dy=0 \quad \text{ for } \ 1\le i,\ j,\ k\le n\,. \end{aligned}$$

Similar to the proof of theorem 4.1, $\Delta\,K_p(p)=\Delta\,K_o(0)=0$ and (5.8) imply that

$$\begin{aligned} \int_{B_p(\rho)}[(y-p)\cdot\bigtriangledown K_p]\,u_{\lambda\,,\,p}^{\frac{2n}{n-2}}\,dy &= \int_{B_p(\rho)}[(y-p)\cdot\bigtriangledown R]\,u_{\lambda\,,\,p}^{\frac{2n}{n-2}}\,dy \\ &\le C\,\lambda^4\,\omega_{n-1}\int_0^{\arctan(\rho/\lambda)}\sin^{n+3}\phi\,\cos^{n-5}\phi\,d\phi\,. \end{aligned}$$

In the above, if $n=3$, we have

$$\int_0^{\arctan(\rho/\lambda)}\sin^{n+3}\phi\,\cos^{n-5}\phi\,d\phi\le\int_0^{\arctan(\rho/\lambda)}\sec^2\phi\,d\phi=\rho/\lambda\,.$$

Whiles $n \geq 4$, the situation is akin to (4.9). Hence

$$\int_{B_p(\rho)} [(y-p)\cdot \bigtriangledown K_p]\, u_{\lambda,\,p}^{\frac{2n}{n-2}}\, dy \leq C\,\lambda^3\,. \tag{5.11}$$

Assuming that (5.2) holds, it follows from (5.10) that

$$\begin{aligned}
(5.12) \qquad & \int_{B_p(\rho)} [\, p\cdot \bigtriangledown K_p(y)\,]\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \\
\geq\ & \frac{1}{3!} \sum_{i\,,\,j\,,\,k} \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}(0)\ \times \\
& \quad \times \int_{B_p(\rho)} [\, p_i\,(y_j - p_j)\,(y_k - p_k) + (y_i - p_i)\, p_j\,(y_k - p_k) \\
& \qquad\qquad + (y_i - p_i)\,(y_j - p_j)\, p_k\,]\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \\
& \qquad\qquad - C \int_{B_p(\rho)} \|y - p\|^3\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(y - p)\, dy \\
=\ & \frac{1}{3!} \left[\sum_i \frac{\partial^3 K_o}{\partial y_i^3}(0) \int_{B_p(\rho)} 3\, p_i\,(y_i - p_i)^2\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(all three indices equal)} \\
& \qquad \left. + \sum_{i \neq j} \frac{\partial^3 K_o}{\partial y_i \partial y_j^2}(0) \int_{B_p(\rho)} 3\, p_i\,(y_j - p_j)^2\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \right] \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(exactly two indices equal)} \\
& \qquad - C \int_{B_p(\rho)} \|y - p\|^3\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(y - p)\, dy \\
=\ & \frac{1}{2} \left[\sum_i p_i \frac{\partial^3 K_o}{\partial y_i^3}(0) + \sum_{i \neq j} p_i\, \frac{\partial^3 K_o}{\partial y_i \partial y_j^2}(0) \right] \int_{B_p(\rho)} (y_j - p_j)^2\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \\
& \qquad - C \int_{B_p(\rho)} \|y - p\|^3\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \\
=\ & \frac{1}{2}\, [p \cdot \bigtriangledown(\Delta K_o)\,(0)] \int_{B_p(\rho)} (y_i - p_i)^2\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \\
& \qquad\quad - C \int_{B_p(\rho)} \|y - p\|^3\, u_{\lambda,\,p}^{\frac{2n}{n-2}}(p)\, dy \\
\geq\ & \frac{\gamma}{2n}\, I_{\rho/\lambda} \left(\frac{\rho}{\lambda} \right) \lambda^2\ - C_4\, \lambda^{\frac{5}{2}}.
\end{aligned}$$

Here $I_{\rho/\lambda}$ is defined as in (4.9). Similarly,

$$\int_{B_p(\rho)} |(y-p)\cdot \bigtriangledown K_p| \left| u_p^{\frac{2n}{n-2}} - u_{\lambda,\,p}^{\frac{2n}{n-2}} \right| dy \ \ \leq\ \ C_5\, \delta\, \lambda^3 \tag{5.13}$$

$$\int_{B_p(\rho)} |p \cdot \bigtriangledown K_p| \left| u_p^{\frac{2n}{n-2}} - u_{\lambda,\,p}^{\frac{2n}{n-2}} \right| dy \ \ \le \ \ C_6\, \delta\, \lambda^2,$$

where we use (5.6) and the estimate $|p \cdot \bigtriangledown K_p| \le C_7\, \|y - p\|$ for $y \in B_p(\rho)$. Using (4.3), (5.10), (5.12) and (5.13) (compare also with the proof of theorem 4.1), we obtain a contradiction when δ and λ are small enough. □

§ 6. Fourth order

One may ask what is likely to happen when $\Delta\, K(0) = \|\bigtriangledown K(0)\| = 0$? (Interesting examples include *homogeneous harmonic polynomials* of higher degrees, see the next section.) The method expounded in theorem 4.1 can be used to search for algebraic relations on higher order derivatives of K, cf. [20]. Here we continue with the fourth order condition.

Theorem 6.1. *For $n \ge 5$, Let K_o be as in (1.2). Assume that $\| \bigtriangledown K_o(0)\| = \Delta\, K(0) = 0$, and*

$$\Upsilon := 3 \left(\sum_{i=1}^{n} \frac{\partial^4 K_o}{\partial y_i^4}\,(0) \right) + \Delta^2\, K_o(0) \ne 0\,. \tag{6.2}$$

Given positive constants C and ρ, there exist positive numbers c_5 and c_6 such that for any positive smooth solution u_o of equation (1.3) in the form of (1.2), the concentration inequalities

$$\left\| \left(\frac{u_o}{u_{\lambda,\,0}} \right)^{\frac{2n}{n-2}} - 1 \right\|_{B_o(\rho)} \le \delta\, \lambda^2\,, \qquad \int_{I\!R^n \setminus B_o(\rho)} u_o^{\frac{2n}{n-2}}\, dy \le C\, \lambda^5 \tag{6.3}$$

cannot hold for $\lambda \le c_5$ and $\delta \le c_6$.

Proof. We explore the main ideas in the proof of theorem 4. Starting with the case that $\Upsilon > 0$, consider the following Taylor expansion

$$\begin{aligned} K_o(y) \ \ &= \ \ K_o(0) + \frac{1}{2} \sum_{i,\,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) + \frac{1}{3!} \sum_{i,\,j,\,k} y_i\, y_j\, y_k\, \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}\,(0) \\ &\qquad + \frac{1}{4!} \sum_{i,\,j,\,k,\,l} y_i\, y_j\, y_k\, y_l\, \frac{\partial^4 K_o}{\partial y_i \partial y_j \partial y_k \partial y_l}\,(0) + R_5(y)\,, \end{aligned}$$

which implies that

$$
\begin{aligned}
y \cdot \bigtriangledown K_o(y) \;\; &= \;\; \sum_{i\,,\,j} y_i\, y_j\, \frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) + \frac{1}{2!} \sum_{i\,,\,j\,,\,k} y_i\, y_j\, y_k\, \frac{\partial^3 K_o}{\partial y_i \partial y_j \partial y_k}\,(0) \\
&\qquad + \frac{1}{3!} \sum_{i\,,\,j\,,\,k\,,\,l} y_i\, y_j\, y_k\, y_l\, \frac{\partial^4 K_o}{\partial y_i \partial y_j \partial y_k \partial y_l}\,(0) + \, y \cdot \bigtriangledown R_5(y)\,.
\end{aligned}
$$

Here

$$
|y \cdot \bigtriangledown R_5(y)| \le C_1\, |y|^5 \quad \text{for} \;\; |y| < \rho\,. \tag{6.4}
$$

As in the proof of theorem 4.1, by symmetry and the fact that $\Delta K_o(0) = 0$, we have

$$
\begin{aligned}
&\sum_{i\,,\,j} \left(\frac{\partial^2 K_o}{\partial y_i \partial y_j}\,(0) \int_{B_o(\rho)} y_i y_j\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy \right) = 0\,, \\
&\int_{B_o(\rho)} y_i y_j y_k\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for} \;\; 1 \le i, \;\; j, \;\; k \le n\,, \\
&\int_{B_o(\rho)} y_i y_j y_k y_l\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for} \;\; i, \;\; j, \;\; k, \;\; l \;\; \text{being} \quad \text{all} \quad \text{distinct}\,, \\
&\int_{B_o(\rho)} y_i^2 y_j y_k\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for} \;\; j \ne k\,, \\
&\int_{B_o(\rho)} y_i^3 y_j\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy = 0 \quad \text{for} \;\; i \ne j\,.
\end{aligned}
$$

Assuming that (6.3) holds, it follows as in (4.7) that

$$
\begin{aligned}
&\int_{B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy \\
\ge \;\; &\frac{1}{3!} \left[\sum_i \frac{\partial^4 K_o}{\partial y_i^4}\,(0) \int_{B_o(\rho)} y_i^4\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy + \sum_{i \ne j} \frac{\partial^4 K_o}{\partial y_i^2 \partial y_j^2}\,(0) \int_{B_o(\rho)} y_i^2\, y_j^2\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy \right] \\
&\quad - C_2\, \delta \int_{B_o(\rho)} \lambda^2\, |y|^2\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy - C_3 \int_{B_o(\rho)} |y|^5\, u_{\lambda,\,0}^{\frac{2n}{n-2}}\, dy\,.
\end{aligned} \tag{6.5}
$$

To compute the first two integrals in (6.5), let θ be the angle to the y_n-axis. That is, $y_n = |y| \cos\theta$. Set

$$
\begin{aligned}
I_n \;\; &:= \;\; \omega_{n-2} \int_0^\pi \cos^4\theta \sin^{n-2}\theta\, d\theta = \frac{3\,\omega_{n-1}}{n(n+2)}\,, \\
J_n \;\; &:= \;\; \frac{\omega_{n-2}}{n-1} \int_0^\pi \cos^2\theta \sin^n\theta\, d\theta = \frac{\omega_{n-1}}{n(n+2)} \qquad \Longrightarrow \quad I_n = 3\, J_n\,.
\end{aligned}
$$

Here we use the formulas

$$\begin{aligned}\int_0^\pi \sin^l\theta\,\cos^m\theta\,d\theta &= \frac{m-1}{m+l}\int_0^\pi \sin^l\theta\,\cos^{m-2}\theta\,d\theta\,,\\ \omega_{n-1} &= \omega_{n-2}\int_0^\pi \sin^{n-2}\theta\,d\theta\,,\end{aligned}$$

where $l\ge 1\,,\ m\ge 2\,,\ n\ge 3\,,$ and ω_{n-2} is the volume of the standard sphere S^{n-2}. It follows that

$$\begin{aligned}\int_{B_o(\rho)} y_n^4\, u_{\lambda,0}^{\frac{2n}{n-2}}\,dy &= \int_0^\rho \left[\omega_{n-2}\int_0^\pi (r\cos\theta)^4\,(r\sin\theta)^{n-2}\,r\,d\theta\right]\left(\frac{\lambda}{\lambda^2+r^2}\right)^n dr\\ &= \omega_{n-2}\int_0^\rho\left[\int_0^\pi \cos^4\theta\,\sin^{n-2}\theta\,d\theta\right]\left(\frac{\lambda}{\lambda^2+r^2}\right)^n r^{n+3}\,dr\\ &= I_n\int_0^\rho \left(\frac{\lambda}{\lambda^2+r^2}\right)^n r^{n+3}\,dr\\ &= I_n\,\lambda^4\int_0^{\arctan(\rho/\lambda)} \sin^{n+3}\phi\,\cos^{n-5}\phi\,d\phi\\ &\qquad\qquad (\text{using the substitution } r=\lambda\tan\phi)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{B_o(\rho)} y_n^2\,(y_1^2+\cdots+y_{n-1}^2)\,u_{\lambda,0}^{\frac{2n}{n-2}}\,dy\\ =\ &\int_{B_o(\rho)} y_n^2(r^2-y_n^2)\,u_{\lambda,0}^{\frac{2n}{n-2}}\,dy\\ =\ &\int_0^\rho\left[\omega_{n-2}\int_0^\pi (r\cos\theta)^2(r\sin\theta)^2(r\sin\theta)^{n-2}\,r\,d\theta\right]\left(\frac{\lambda}{\lambda^2+r^2}\right)^n dr\\ \Longrightarrow\quad &(n-1)\int_{B_o(\rho)} y_i^2y_j^2\,u_{\lambda,0}^{\frac{2n}{n-2}}\,dy\qquad (i\ne j)\\ &\quad=\omega_{n-2}\int_0^\rho\left(\int_0^\pi \cos^2\theta\,\sin^2\theta\sin^{n-2}\theta\,d\theta\right)\left(\frac{\lambda}{\lambda^2+r^2}\right)^n r^{n+3}\,dr\\ \Longrightarrow\quad &\int_{B_o(\rho)} y_i^2y_j^2\,u_{\lambda,0}^{\frac{2n}{n-2}}\,dy = J_n\int_0^\rho\left(\frac{\lambda}{\lambda^2+r^2}\right)^n r^{n+3}\,dr\\ \Longrightarrow\quad &\int_{B_o(\rho)} y_i^2y_j^2\,u_{\lambda,0}^{\frac{2n}{n-2}}\,dy = J_n\,\lambda^4\int_0^{\arctan(\rho/\lambda)}\sin^{n+3}\phi\,\cos^{n-5}\phi\,d\phi\,,\qquad i\ne j\,.\end{aligned}$$

Setting $\sigma = \Upsilon\, J_n\,,$ we obtain

$$
\begin{aligned}
(6.7)\qquad & \int_{B_o(\rho)} (y \cdot \bigtriangledown K_o)\, u_o^{\frac{2n}{n-2}}\, dy \\
\geq\ & \left(\frac{\sigma}{6}\right) \lambda^4 \int_0^{\arctan(\rho/\lambda)} \sin^{n+3}\phi\, \cos^{n-5}\phi\, d\phi - C_2\, \delta \int_{B_o(\rho)} \lambda^2\, \|y\|^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy \\
& -C_3 \int_{B_o(\rho)} \|y\|^5\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy\,.
\end{aligned}
$$

As in (4.8) and (4.9),

$$
\begin{aligned}
(6.8)\qquad \int_{B_o(\rho)} \lambda^2\, \|y\|^2\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= \lambda^4 \int_0^{\arctan(\rho/\lambda)} \sin^{n+1}\phi\, \cos^{n-3}\phi \leq C_4\, \lambda^4\,, \\
\int_{B_o(\rho)} \|y\|^5\, u_{\lambda,0}^{\frac{2n}{n-2}}\, dy &= \lambda^5 \int_0^{\arctan(\rho/\lambda)} \sin^{n+4}\phi\, \cos^{n-6}\phi \leq C_5\, \lambda^{5-\varepsilon}.
\end{aligned}
$$

Here $n \geq 5$ and $\varepsilon \in (0,\ 1)$ is a positive constant. With (6.3), (6.7) and (6.8), we conclude as in the proof of theorem 4.1 that contradiction arises when λ and δ are small enough. The case $\Upsilon < 0$ is similar. $\square$

§ 7. Homogeneous harmonic polynomials

Here we present some simple functions which satisfy the conditions in theorem 6.1. Let

$$
(7.1)\qquad Q_k(x) = \sum C_{i_1,\cdots,i_k}\, x_{i_1} \cdots x_{i_k}
$$

be a *homogeneous harmonic polynomial* of degree $k \geq 2$ in $\mathbb{R}^{n+1}$. It is shown in [19] that Q_k satisfies the Kazdan-Warner type identity, namely,

$$
\int_{S^n} X(Q_k)\, dV_{g_1} = 0
$$

for any conformal Killing vector field X on S^n.

Assuming that the indices $i_1, \cdots, i_k$ *in* (7.1) *are all smaller than* $n+1$. Consider the stereographic projection $\mathcal{P}$ from

$$
S^n := \{ x = (x_1,\ \cdots,\ x_{n+1}) \in \mathbb{R}^{n+1}\ \mid\ \|x\|^2 = 1 \}
$$

to $\mathbb{R}^n$, with Cartesian coordinates $(y_1, \cdots, y_n)$. It is given by

$$
\begin{aligned}
(7.2) \qquad y_i &= \frac{x_i}{1 - x_{n+1}}, \qquad 1 \le i \le n, \\
x_i &= \frac{2\, y_i}{1 + \|y\|^2}, \qquad 1 \le i \le n, \qquad \text{and} \qquad x_{n+1} = \frac{\|y\|^2 - 1}{\|y\|^2 + 1}.
\end{aligned}
$$

Using $\mathcal{P}$, we transfer Q_k into $\mathbb{R}^n$ and obtain

$$
(7.3) \qquad Q_k(y) = \left(\frac{2}{1 + \|y\|^2} \right)^k \sum C_{i_1, \cdots, i_k}\, y_{i_1} \cdots y_{i_k}.
$$

It follows that $\bigtriangledown Q_k\,(0) = 0$ (as $k \ge 2$). Moreover,

$$
\begin{aligned}
\Delta_y\, Q_k(y) &= \left(\frac{2}{1 + \|y\|^2} \right)^k \Delta_y \left(\sum C_{i_1, \cdots, i_k}\, y_{i_1} \cdots y_{i_k} \right) \\
&\quad - \frac{2^{k+1}\, k}{(1 + \|y\|^2)^{k+1}} \sum C_{i_1, \cdots, i_k}\, [y \cdot \bigtriangledown_y\, (y_{i_1} \cdots y_{i_k})] \\
&\quad + \left[\frac{2^{k+2} k(k+1)\, \|y\|^2}{(1 + \|y\|^2)^{k+2}} - \frac{2^{k+1} k\, n}{(1 + \|y\|^2)^{k+1}} \right] \sum C_{i_1, \cdots, i_k}\, y_{i_1} \cdots y_{i_k} \\
&= 0 - \frac{2^{k+1}\, k^2}{(1 + \|y\|^2)^{k+1}} \sum C_{i_1, \cdots, i_k}\, y_{i_1} \cdots y_{i_k} \\
&\quad + \left[\frac{2^{k+2} k(k+1)\, \|y\|^2}{(1 + \|y\|^2)^{k+2}} - \frac{2^{k+1} k\, n}{(1 + \|y\|^2)^{k+1}} \right] \sum C_{i_1, \cdots, i_k}\, y_{i_1} \cdots y_{i_k} \\
&= 2^{k+1} k \left[\frac{(k + 2 - n)\, \|y\|^2 - (n + k)}{(1 + \|y\|^2)^{k+2}} \right] Q_k(y)
\end{aligned}
$$

$$
\Longrightarrow \qquad \Delta_y\, Q_k(0) = 0.
$$

Likewise, $\bigtriangledown_y\, (\Delta_y\, Q_k)(0) = 0$ and $\Delta_y\, (\Delta_y\, Q_k)(0) = 0$. Here we make use of the fact that Q_k is a harmonic polynomial and x_{n+1} is not present in $Q_k(x)$.

Using above, one can construct the desired functions. For instance, let Q_4 be the homogeneous harmonic polynomial defined by

$$
Q_4(x) := x_1^4 + x_2^4 - 6\, x_1^2\, x_2^2 \qquad \text{for} \quad x \in S^n \subset \mathbb{R}^{n+1}.
$$

Using the stereographic projection, we obtain

$$
Q_4(y) = \left(\frac{2}{1 + \|y\|^2} \right)^4 (y_1^4 + y_2^4 - 6\, y_1^2\, y_2^2).
$$

Let $K_o := 1 + Q_4$ in $\mathbb{R}^n$. We have

$$K_o(0) = 1 > 0\,, \quad \bigtriangledown K_o(0) \;=\; 0\,, \quad \Delta\, K_o(0) = 0\,, \quad \bigtriangledown(\Delta K_o)(0) = 0\,,$$

but $$3\left(\sum_{i=1}^{n} \frac{\partial^4 Q_4}{\partial y_i^4}\,(0)\right) + \Delta_y^2\, Q_4\,(0) \;=\; 3\left(\sum_{i=1}^{n} \frac{\partial^4 Q_4}{\partial y_i^4}\,(0)\right) \neq 0\,.$$

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DEPARTMENT OF MATHEMATICS,
NATIONAL UNIVERSITY OF SINGAPORE,
2 SCIENCE DRIVE 2,
SINGAPORE 117543,
REPUBLIC OF SINGAPORE

matlmc@math.nus.edu.sg