Exotic Solutions of the Conformal Scalar Curvature Equation in $\mathbb{R}^n$

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Abstract

We construct global exotic solutions of the conformal scalar curvature equation $\Delta\, u + [n\,(n-2)/4]\, K u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$, with $K(x)$ approaching 1 near infinity in order as close to the critical exponent as possible.

1. Introduction

We consider a special class of positive solutions of the conformal scalar curvature equation

$$\Delta u + \frac{n\,(n-2)}{4} K u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

Here Δ is the standard Laplacian on $\mathbb{R}^n$ equipped with Euclidean metric g_o , K a smooth function on $\mathbb{R}^n$, and $n \ge 3$ an integer. The solutions we construct breach a rather natural lower bound and have peculiar asymptotic property.

Equation (1.1) is studied extensively by many authors in connection with the prescribed scalar curvature problem on a Riemannian manifold in general and on $\mathbb{R}^n$ and S^n in particular (on S^2, the Nirenberg problem; cf. [1], [3], [4], [5], [9], [12], [14], [15] [17], [20], [21], [23], [24], [26] and the references within). As in the case of the Yamabe problem, recent studies indicate that the case when K is

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strictly positive affords many interesting and subtle developments.

Assume that K is bounded between two positive constants in $\mathbb{R}^n$. An important feature of equation (1.1) is the asymptotic behavior of $u(x)$ for large $|x|$ (cf. [2], [5], [6], [7], [8], [10], [12], [16], [18], [19], [22]). It is simpler to classify with the help of the Kelvin transformation:

$$(1.2) \qquad y = \frac{x}{|x|^2} \quad \text{and} \quad w\,(y) := |y|^{2-n}\, u\left(y/|y|^2\right) \quad \text{for } \; x,\, y \in \mathbb{R}^n \setminus \{0\}\,.$$

From (1.2), w satisfies the equation

$$(1.3) \qquad \Delta w\,(y) + \frac{n\,(n-2)}{4}\,\bar{K}(y)\, w^{\frac{n+2}{n-2}}(y) = 0 \quad \text{for } \; y \in \mathbb{R}^n \setminus \{0\}\,,$$

where $\bar{K}(y) := K(y/|y|^2)$ for $y \neq 0$ (see, for instance, [18]). w (and u) is said to have fast decay if w has a removable singularity at the origin. Otherwise, it is called a singular solution. In order to have reasonable control on the geometric and analytic behavior of singular solutions, it is crucial to obtain the upper bound or *slow decay*

$$(1.4) \;\; w\,(y) \le C_1\,|y|^{-(n-2)/2} \quad \text{as } \; y \to 0\,, \;\; \text{i.e.,} \;\; u\,(x) \le C_1\,|x|^{-(n-2)/2} \quad \text{for } \; |x| \gg 1\,,$$

where C_1 is a positive constant. The question on slow decay is discussed in depth in [2], [5], [6], [7], [8], [16], [18], [19], [22] (cf. also [27]; note that our definition of slow decay is slightly different from the one in [5] and [8]). Guided by the case when K is equal to a positive constant outside a compact subset of $\mathbb{R}^n$ (see [2], [16]), it is natural to ask whether a singular positive solution u with slow decay also satisfies the lower bound

$$(1.5) \;\; w\,(y) \ge C_2\,|y|^{-(n-2)/2} \quad \text{as } \; y \to 0\,, \;\; \text{i.e.,} \;\; u\,(x) \ge C_2\,|x|^{-(n-2)/2} \quad \text{for } \; |x| \gg 1\,,$$

where C_2 is a positive constant. If the lower bound holds, then the conformal metric $u^{4/(n-2)} g_o$ on $\mathbb{R}^n$ is complete and has bounded (sectional) curvature [8]. The radial Pohozaev number is an essential invariant in the study of equation (1.1) and is given by

$$(1.6) \qquad P(u) := \lim_{R \to \infty} \int_{B_o(R)} [x \cdot \nabla K(x)]\, u^{\frac{2n}{n-2}}(x)\, dx\,,$$

provided the limit exists. Here $B_o(R)$ is the open ball with center at the origin and radius equal to $R > 0$. The following result is shown by Chen and Lin in [6] and [8], mindful of the slightly different notations we use.

Theorem 1.7 (Chen-Lin). *Let u be a positive smooth solution of equation (1.1). Assume that $\lim_{|x|\to\infty} K(x)$ exists and is positive, and there exist positive constants $l \ge (n-2)/2$ and C such that*

$$C^{-1}|x|^{-(l+1)} \le |\bigtriangledown K(x)| \le C\,|x|^{-(l+1)} \quad for \;\; all \;\; |x| \gg 1\,.$$

Then u has slow decay and $P(u)$ exists and is non-positive. u has fast decay if and only if $P(u) = 0$ (the Kazdan-Warner condition). Furthermore, if u is a singular solution, then we also have the lower bound $u\,(x) \ge C_2\,|x|^{-(n-2)/2}$ for all $|x| \gg 1$ and for some positive constant C_2.

More generally, under the condition that $\lim_{|x|\to\infty} K(x)$ exists and is positive, and $|\bigtriangledown K|$ is bounded in $\mathbb{R}^n$, for a positive smooth solution u of equation (1.1) with slow decay, we show in [10] (cf. also [5], [8]) that $P(u) \le 0$ if $P(u)$ exists. Moreover, $P(u) = 0$ if and only if

$$\liminf_{|x|\to\infty} |x|^{\frac{n-2}{2}} u(x) = 0\,. \tag{1.8}$$

In the latter case, the assumption on K is not strong enough to allow us to deduce that u has fast decay.

Definition 1.9. *We call a singular positive solution u of equation (1.1) with slow decay an* **exotic solution** *if (1.8) holds for u. That is, we* **cannot** *find a positive constant C_2 such that $u(x) \ge C_2\,|x|^{-(n-2)/2}$ for all $|x| \gg 1$.*

Then it is necessary that $P(u) = 0$ if $P(u)$ exists. Exotic solutions are rather peculiar because from $P(u) = 0$ one would expect u to have fast decay. Instead, they decay slowly and the conformal metric $u^{4/(n-2)} g_o$ remains to be complete, but the (sectional) curvature is unbounded [8]. Theorem 1.7 leads to the observation that there are no exotic solutions if $|\bigtriangledown K|$ decays to zero near infinity fast enough.

(Local) Exotic solutions are first found by Chen and Lin in [8]. By a scaling and the Kelvin transform, we may consider the equation

$$\Delta u + \bar{K} u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \;\; B_o(1) \setminus \{0\}\,. \tag{1.10}$$

Assume that $\bar{K}$ is radial and non-increasing in $(0, 1]$, and is given by

$$\bar{K}(r) = 1 - A\,r^l + R\,(r) \tag{1.11}$$

for $r > 0$ close to zero. Here $A > 0$ and $0 < l < (n-2)/2$ are constants, and $R(r) = o\,(r^l)$ and $R'(r) = o\,(r^{l-1})$ for $r > 0$ close to zero. Given a positive number α, let $u(r, \alpha)$ be the unique solution of the initial value problem

$$\begin{cases} u''(r) + \frac{n-1}{r}\, u'(r) + \bar{K}(r)\, u^{\frac{n+2}{n-2}}(r) = 0\,, \\ u(0) = \alpha \quad \text{and} \quad u'(0) = 0\,. \end{cases}$$

Chen and Lin [8] show elegantly that there exists a sequence $\alpha_i \to \infty$ such that $u(r, \alpha_i)$ converges to an (local) exotic C^2-solution of equation (1.10) in $B_o(1) \setminus \{0\}$. Subsequently, Lin [22] obtains characterizations of exotic solutions in terms of the asymptotic expansion of $\bar{K}$ near the origin.

The exponent $(n-2)/2$ is found to be critical. For $l \ge (n-2)/2$, theorem 1.7 shows that there are no exotic solutions of equation (1.1). In this paper we construct global exotic solutions of equation (1.1) in $\mathbb{R}^n$. As described above, in [8], an abstract existence argument is used to show the existence of (local) exotic solutions. Our construction is explicit by gluing the Delaunay-Fowler-type solutions. Given any positive number δ, we show that there is an exotic solution of equation (1.1) with $|K - 1| \le \delta^2$ in $\mathbb{R}^n$. Moreover, with regard to the critical exponent $(n-2)/2$, we show that, given any positive function $\varphi(r)$ defined for $r \gg 1$ such that

(1.12)
$$r^{(n-2)/2}\,\varphi(r) \quad \text{is} \quad \text{non-decreasing} \quad \text{for} \quad r \gg 1 \quad \text{and} \quad \lim_{r \to \infty} r^{(n-2)/2} \varphi(r) = \infty\,,$$

(for example, $\varphi(r) = r^{-(n-2)/2} \ln(\ln r)$ for $r \gg 1$,) we construct an exotic solution of equation (1.1) with

$$|K(x) - 1| \le C_3\, \varphi\,(|x|) \quad \text{for all} \quad |x| \gg 1\,, \tag{1.13}$$

where C_3 is a positive constant. The analytic property of exotic solutions resides in a neighborhood of infinity, or, by the Kelvin transformation, on a neighborhood of the origin. Our emphasis on the whole $\mathbb{R}^n$ reflects the geometric viewpoint of conformal deformations of Euclidean space $(\mathbb{R}^n,\ g_o)$. We follow the convention of using $c\,, C\,, C'\,, C_1\,, ...$ to denote positive constants, whose actual values may differ from section to section.

2. Delaunay-Fowler-type Solutions

Introduce polar coordinates (r, θ) in $\mathbb{R}^n$, where $r = |x|$ and $\theta = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$. Let $t = \ln r$ for $r > 0$ and

$$v(t, \theta) = r^{\frac{n-2}{2}} u(r, \theta) \quad \text{for} \;\; r > 0 \;\; \text{and} \;\; \theta \in S^{n-1}. \tag{2.1}$$

By the above transformation, equation (1.1) can be re-written as

$$\frac{\partial^2 v}{\partial t^2} + \Delta_\theta \, v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} \tilde{K} \, v^{\frac{n+2}{n-2}} = 0 \quad \text{in} \;\; \mathbb{R} \times S^{n-1}. \tag{2.2}$$

Here Δ_θ is the Laplacian on the standard unit sphere in $\mathbb{R}^n$ and $\tilde{K}(t, \theta) := K(x)$, where $|x| = e^t$ and $x/|x| = \theta$. For the case $\tilde{K} \equiv 1$ in $\mathbb{R} \times S^{n-1}$, consider radial solutions v of (2.2) and the ODE

$$v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0 \quad \text{in} \;\; \mathbb{R}\,. \tag{2.3}$$

In connection with the study of surfaces of revolution of constant curvature by Delaunay [11] and a class of semilinear differential equations by Fowler [13], positive smooth solutions of equation (2.3) are known as Delaunay-Fowler-type solutions. We refer to [16], [24], [25] for basic properties of the solutions. Equation (2.3) is autonomous and the Hamiltonian energy

$$H(v, v') = (v')^2 - \frac{(n-2)^2}{4} \left[v^2 - v^{2n/(n-2)} \right] \tag{2.4}$$

is constant along solutions of (2.3). For a positive smooth solution v of (2.3), H is a non-positive constant in the interval $[\,-[(n-2)/n]^{n/2}(n-2)/2\,,\, 0\,]$ (see [16]). By shifting the parameter, we may normalize the solution so that

$$v(0) = \max_{t \in \mathbb{R}} v(t)\,. \tag{2.5}$$

Let v_o be a positive solution of equation (2.3) with $H = 0$. Under the normalization, we have

$$v_o(t) = (\cosh\, t)^{(2-n)/2} \quad \text{for} \;\; t \in \mathbb{R}\,. \tag{2.6}$$

We note that, under the transformation in (2.1), v_o corresponds to

$$u_o(x) = \left(\frac{2}{1+|x|^2} \right)^{\frac{n-2}{2}} \quad \text{for} \ \ x \in \mathbb{R}^n, \tag{2.7}$$

which is a solution of equation (1.1) when $K \equiv 1$ in $\mathbb{R}^n$. In particular, u_o is smooth near 0, which corresponds to $s \to -\infty$ for v_o. The other extreme is when $H = -[(n-2)/n]^{n/2}(n-2)/2$, and the corresponding solution v is a constant function given by $v(t) = [(n-2)/n]^{(n-2)/4}$ for $t \in \mathbb{R}$.

For $H \in (-[(n-2)/n]^{n/2}(n-2)/2\,,\, 0\,)$, the solution can be indexed by the parameter $\varepsilon = \min\limits_{t \in \mathbb{R}} v(t)$, which is called the *neck-size* of the solution, or the Fowler parameter. We have $\varepsilon \in (0,\ [(n-2)/n]^{(n-2)/4})$ and

$$H = H(\varepsilon) = \frac{(n-2)^2}{4} \left[\varepsilon^{2n/(n-2)} - \varepsilon^2 \right] . \tag{2.8}$$

Denote the normalized positive solution by v_ε, where $0 < \varepsilon < [(n-2)/n]^{(n-2)/4}$. It is known that v_ε is periodic with period T_ε. Moreover, we always have [16]

$$\varepsilon \le v_\varepsilon(t) \le v_\varepsilon(0) < 1 \quad \text{for} \ \ t \in \mathbb{R}\,. \tag{2.9}$$

The following result is essentially proved in [24] (cf. also [16]).

Lemma 2.10. *T_ε, the period of v_ε, is monotone in ε for $\varepsilon \in (0,\ [(n-2)/n]^{(n-2)/4})$. We have $T_\varepsilon \to 2\pi/\sqrt{n-2}$ as $\varepsilon \to [(n-2)/n]^{(n-2)/4}$ and $T_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. Furthermore, there exists a positive constant C, independent on ε, such that*

$$-\frac{4}{n-2} \ln (C\,\varepsilon) \ \le \ T_\varepsilon \ \le \ -\frac{4}{n-2} \ln \left(C^{-1}\,\varepsilon \right) \quad \textit{as} \ \ \varepsilon \to 0^+\,. \tag{2.11}$$

It is also known that v_ε converges uniformly in compact subsets of $\mathbb{R}$ to the constant solution as $\varepsilon \to [(n-2)/n]^{(n-2)/4}$, and to $v_o(t) = (\cosh t)^{(2-n)/2}$ as $\varepsilon \to 0^+$ [16]. For applications in section 3, we study the order of the latter convergence in more detail. As H is constant along solutions, we have

$$H(v_\varepsilon\,, v_\varepsilon') = -\frac{(n-2)^2}{4} \left(\varepsilon^2 - \varepsilon^{\frac{2n}{n-2}} \right) = -\frac{(n-2)^2}{4} \left[v_\varepsilon^2(0) - v_\varepsilon^{\frac{2n}{n-2}}(0) \right]$$

for $\varepsilon \in (0,\, [(n-2)/n]^{(n-2)/4})$. Thus we obtain

$$v_\varepsilon^2(0)\,[1 - v_\varepsilon^{\frac{4}{n-2}}(0)] = \varepsilon^2(1 - \varepsilon^{\frac{4}{n-2}})\,. \tag{2.12}$$

As $v_\varepsilon(0) > \varepsilon$ when $\varepsilon \to 0^+$, it follows from (2.12) that $v_\varepsilon(0) \to 1$ and $\varepsilon \to 0^+$. Furthermore,

$$1 - v_\varepsilon^{\frac{4}{n-2}}(0) = O\,(\varepsilon^2)\,.$$

We have

$$v_\varepsilon(0) = [1 + O(\varepsilon^2)]^{(n-2)/4} = 1 + O(\varepsilon^2) \quad \text{as} \;\; \varepsilon \to 0^+. \tag{2.13}$$

Hence there exists a positive constant C_n which depends on n only, such that

$$|v_\varepsilon(0) - 1| \le C_n\, \varepsilon^2 \quad \text{for} \;\; \varepsilon > 0 \;\; \text{small}\,. \tag{2.14}$$

We use the following well-known inequalities a number of times; they can be derived by simple integration methods. For positive constants c and $\alpha \ge 1$, we have

$$|\, x^\alpha - y^\alpha \,| \le C\, |x - y| \quad \text{for} \;\; 0 \le x,\, y \le c\,, \tag{2.15}$$

where $C = C(\alpha, c)$ is a positive constant; moreover, for $\beta > 0$,

$$(1 + z)^\beta = 1 + O\,(|z|) \quad \text{as} \;\; z \to 0\,. \tag{2.16}$$

With v_o given by (2.6), it follows from (2.9) and (2.15) that

$$\left| v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t) \right| \le c_n\, |v_\varepsilon(t) - v_o(t)|\,, \tag{2.17}$$

where c_n is a positive constant depending on n only. Using equation (2.3) we have

$$\begin{aligned} |v_\varepsilon''(t) - v_o''(t)| &\le \frac{(n-2)^2}{4}|v_\varepsilon(t) - v_o(t)| + \frac{n(n-2)}{4}\left| v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t) \right| \\ &\le \left[\frac{(n-2)^2}{4} + \frac{n(n-2)}{4}\, c_n \right] |v_\varepsilon(t) - v_o(t)| \\ &= \bar{C}_n |v_\varepsilon(t) - v_o(t)|\,, \end{aligned} \tag{2.18}$$

where $\bar{C}_n$ is the positive constant defined in the formula. We claim that

$$|v_\varepsilon''(t) - v_o''(t)| \le 2C_n\, \bar{C}_n\, \varepsilon^2 \quad \text{for} \;\; t \in [0, \rho]\,, \tag{2.19}$$

where $\rho = 1/(2C_n\,\bar{C}_n)$. Here C_n and C'_n are the positive constants in (2.14) and (2.18), respectively. Without loss of generality, we may assume that $\rho < C_n$. By (2.14) and (2.18), the bound holds on a neighborhood of 0. Suppose that it holds on $[0, \sigma]$ for some positive number σ less than ρ. As $v'_\varepsilon(0) = v'_o(0) = 0$, we have

$$|v'_\varepsilon(t) - v'_o(t)| \le 2C_n\,\bar{C}_n \varepsilon^2\,\sigma \le \varepsilon^2 \quad \text{for} \ \ t \in [0, \sigma]\,.$$

Hence

$$|\,v_\varepsilon(t) - v_o(t)| \le (C_n + \sigma)\,\varepsilon^2 < 2C_n\,\varepsilon^2 \quad \text{for} \ \ t \in [0, \sigma]\,. \tag{2.20}$$

By (2.18) we have

$$|\,v''_\varepsilon(\sigma) - v''_o(\sigma)| < 2C_n\bar{C}_n\,\varepsilon^2\,.$$

Using an connectedness argument, we obtain (2.19) as claimed. A similar bound holds in $[-\rho, 0]$. Upon integration we obtain the following lemma.

Lemma 2.21. *Let v_ε and v_o be the solutions of equation (2.3) discussed above. There exists positive constants ρ and C_o which depend on n but not on (small enough positive) ε, such that*

$$|\,v_\varepsilon(t) - v_o(t)|\, \le\, C_o\,\varepsilon^2\,, \qquad |\,v'_\varepsilon(t) - v'_o(t)|\, \le\, C_o\,\varepsilon^2 \quad \text{and} \ \ v_\varepsilon(t) \ge 1/2 \tag{2.22}$$

for $t \in [-\rho,\, \rho]$ and $\varepsilon > 0$ close to 0.

3. Gluing Solutions

We follow the notations used in section 2 and consider (2.1) and equation (2.2). Slow decay for a positive smooth solution u of equation (1.1) corresponds to $v(s, \theta) \le C$ for $s \gg 1$, $\theta \in S^{n-1}$ and a positive constant C. Moreover, u is an (global) exotic solution if and only if there exists a sequence $\{(s_i,\, \theta_i)\} \subset \mathbb{R} \times S^{n-1}$ such that $\lim\limits_{i\to\infty} s_i = \infty$ and $\lim\limits_{i\to\infty} v(s_i,\, \theta_i) = 0$, and, when the variable t is changed into r via $t = \ln r$, u is smooth across the origin. Let ϕ_1 be a smooth function on $\mathbb{R}$ such that $0 \le \phi \le 1$ in $\mathbb{R}$ and

$$\phi_1(t) \ = \ \begin{cases} 1 & \text{for} \quad t \le -\rho\,, \\ 0 & \text{for} \quad t \ge \rho\,. \end{cases}$$

We also require that

$$|\,\phi_1'(t)|\,\le\,2/\rho \quad \text{and} \quad |\,\phi_1''(t)|\,\le\,2/\rho^2 \quad \text{for } \; t\in(-\rho\,,\,\rho)\,. \tag{3.1}$$

Let $\phi_2 = 1-\phi_1$ in $\mathbb{R}$. Define

$$v = \phi_1\, v_o + \phi_2\, v_\varepsilon \quad \text{in } \; \mathbb{R}\,, \tag{3.2}$$

where $\varepsilon > 0$ is close to zero. It follows that

$$\begin{aligned} & -v''(t) \,+\, \frac{(n-2)^2}{4} v(t) \\ = \;& \frac{n(n-2)}{4}\left[\phi_1\, v_o^{\frac{n+2}{n-2}}(t) \,+\, \phi_2\, v_\varepsilon^{\frac{n+2}{n-2}}(t)\right] + \phi_1'(t)\,[v_\varepsilon'(t) - v_o'(t)] \\ & \quad + \phi_1''(t)\,[v_\varepsilon(t) - v_o(t)] \end{aligned} \tag{3.3}$$

for $t\in\mathbb{R}$. We also have

$$\begin{aligned} & \phi_1(t)\, v_o^{\frac{n+2}{n-2}}(t) + \phi_2(t)\, v_\varepsilon^{\frac{n+2}{n-2}}(t) \\ = \;& \phi_1(t)\, v_o^{\frac{n+2}{n-2}}(t) + \phi_2(t)\, v_o^{\frac{n+2}{n-2}}(t) + \phi_2(t)\left[v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right] \\ = \;& [\phi_1(t) v_o(t) + \phi_2(t)\, v_o(t)]^{\frac{n+2}{n-2}} + \phi_2(t)\left[v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right] \\ = \;& \{v(t) + \phi_2(t)\,[v_o(t) - v_\epsilon(t)]\}^{\frac{n+2}{n-2}} + \phi_2(t)\left[v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right] \end{aligned}$$

for $t\in[-\rho,\rho]$. We obtain

$$\begin{aligned} & \left|\left[-v''(t) + \frac{(n-2)^2}{4}\, v(t)\right]\left[\frac{n(n-2)}{4}\, v^{\frac{n+2}{n-2}}(t)\right]^{-1} - 1\right| \\ \le \;& \left|\left\{1 + \frac{\phi_2(t)}{v(t)}\,[v_o(t) - v_\epsilon(t)]\right\}^{\frac{n+2}{n-2}} - 1\right| \\ & \quad + \frac{4}{n(n-2)}\, v^{-\frac{n+2}{n-2}}(t)\, \left\{\phi_2(t)\left|v_\varepsilon^{\frac{n+2}{n-2}}(t) \,-\, v_o^{\frac{n+2}{n-2}}(t)\right|\right. \\ & \qquad \left. + \,|\,\phi_1'(t)|\,|v_\varepsilon'(t) - v_o'(t)| + |\,\phi_1''(t)|\,[v_\varepsilon(t) - v_o(t)]\right\} \end{aligned} \tag{3.4}$$

for $t\in[-\rho,\rho]$. It follows from lemma 2.21, (2.16), (2.17), (3.1) and (3.4) that v satisfies the equation

$$v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} K v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \; \mathbb{R}\,, \tag{3.5}$$

where K is a smooth function on $\mathbb{R}$ such that

$$(3.6)\quad |K(t)-1| = \left| \left[-v''(t) + \frac{(n-2)^2}{4}\, v(t) \right] \left[\frac{n(n-2)}{4}\, v^{\frac{n+2}{n-2}}(t) \right]^{-1} - 1 \right| \le C_1\, \varepsilon^2$$

for $t \in [-\rho,\, \rho]$, and $K \equiv 1$ in $\mathbb{R} \setminus [-\rho,\, \rho]$. Here C_1 is a positive constant that depends on n only, so far as $\varepsilon > 0$ is close to zero.

Let $\{\varepsilon_i\}$ be a decreasing sequence of small positive numbers such that $\lim\limits_{i\to\infty} \varepsilon_i = 0$. Denote the period of v_{ε_i} by T_{ε_i} for $i = 1, 2, \ldots$. With ε_1 small enough, we may assume that $T_{\varepsilon_1} \gg \rho$. We construct a positive smooth function by first gluing v_o and v_{ε_1} on $[-\rho,\, \rho]$ as described above and call the resulting positive smooth function v_1. Note that $v_1 = v_{\varepsilon_1}$ in $\mathbb{R}^+ \setminus (0, \rho)$. As $v_{\varepsilon_1}(t + T_{\varepsilon_1}) = v_{\varepsilon_1}(t)$ for $t \in \mathbb{R}$ and v_{ε_1} and v_{ε_2} are close to v_o near $[-\rho,\, \rho]$, we let

$$\tilde v_{\varepsilon_2}(t) = v_{\varepsilon_2}(t - T_{\varepsilon_1}) \quad \text{for } \ t \in \mathbb{R}\,,$$

and glue $\tilde v_{\varepsilon_2}$ and v_1 (that is, v_{ε_1}) on $[T_{\varepsilon_1} - \rho,\, T_{\varepsilon_1} + \rho]$ in a process similar to the one described above. Call the resulting function v_2. We continue to glue the solutions on the intervals

$$[T_{\varepsilon_1} + T_{\varepsilon_2} - \rho,\, T_{\varepsilon_1} + T_{\varepsilon_2} + \rho]\,, \ldots,\, [\sum_{k=1}^{i} T_{\varepsilon_k} - \rho,\, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho]\,, \ldots$$

by $v_{\varepsilon_3}\,, \ldots, v_{\varepsilon_{i+1}}\,, \ldots$, respectively, after shifting appropriately. In particular, in the $(i+1)$-th step, let

$$\tilde v_{\varepsilon_i}(t) = v_{\varepsilon_i}\left(t - \sum_{k=1}^{i-1} T_{\varepsilon_k} \right) \quad \text{and} \quad \tilde v_{\varepsilon_{i+1}}(t) = v_{\varepsilon_{i+1}}\left(t - \sum_{k=1}^{i} T_{\varepsilon_k} \right) \quad \text{for } \ t \in \mathbb{R}\,,$$

and glue $\tilde v_{\varepsilon_{i+1}}$ with $\tilde v_{\varepsilon_i}$ on the interval $[\sum\limits_{k=1}^{i} T_{\varepsilon_k} - \rho,\, \sum\limits_{k=1}^{i} T_{\varepsilon_k} + \rho]$. Finally we obtain a positive smooth function v on $\mathbb{R}$ which satisfies the equation

$$(3.7)\qquad v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} K v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ \mathbb{R}$$

for some smooth function K such that

$$(3.8)\qquad |K(t) - 1| \le C_2\, \varepsilon_1^2 \quad \text{for } \ t \in \mathbb{R}\,,$$

where C_2 is a positive constant depending on n only. We may choose $\varepsilon_1 > 0$ as small as we like. We also have

(3.9)

$$v\left(\sum_{k=1}^{i} T_{\varepsilon_k} - T_{\varepsilon_i}/2\right) = v_i(T_{\varepsilon_i}/2) = \varepsilon_i \to 0 \quad \text{and} \quad v\left(\sum_{k=1}^{i} T_{\varepsilon_k}\right) \to 1^- \quad \text{as } i \to \infty\,.$$

As $v(t) = v_o(t)$ for $t \le -\rho$, by (2.6) and (2.7), the corresponding solution u related to v by (2.1) is smooth across the origin. Thus v corresponds to an exotic solution u of equation (1.1) through (2.1).

Given a positive function $\varphi(r)$ defined for $r \gg 1$ which satisfies (1.12), let $\psi(t) = \varphi(e^t)$. It follows that ψ is defined for $t \gg 1$ and

$$e^{(n-2)t/2}\,\psi(t) \tag{3.10}$$

is non-decreasing for $t \gg 1$ and unbounded from above. Let

$$\varpi(t) = \ln\left[e^{(n-2)t/2}\,\psi(t)\right] \qquad \text{for} \;\; t \gg 1\,. \tag{3.11}$$

We have $\lim\limits_{t\to\infty} \varpi(t) = \infty$. Choose a decreasing sequence of numbers $\{\varepsilon_i\}$ such that ε_1 is small enough and the corresponding periods T_{ε_i} of v_{ε_i} satisfy the relation

$$\varpi(T_{\varepsilon_i}) \ge \frac{n-2}{2}\sum_{k=1}^{i-1} T_{\varepsilon_k} \qquad \text{for} \;\; i = 2,\, 3, ... \tag{3.12}$$

By gluing the solutions v_o, v_{ε_i}, $i = 1,\, 2, ...,$ as described above, we obtain a positive smooth function v which satisfies equation (3.7) for a smooth function K. Suppose that

$$t \notin [-\rho,\, \rho] \cup [T_{\varepsilon_1} - \rho,\, T_{\varepsilon_1} + \rho] \cup \cdots \cup \left[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho,\, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho\right] \cup \cdots,$$

then $K(t) = 1$. Suppose that

$$t \in \left[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho,\, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho\right] \qquad \text{for} \quad \text{some} \;\; i \in \mathbb{N}\,.$$

According to the construction above and lemma 2.10, we have

$$\begin{aligned}
|K(t) - 1| \;&\le\; C_3\,{\varepsilon_i}^2 \le C_4 \exp\left(-\frac{n-2}{2}\,T_{\varepsilon_i}\right) \\
&=\; C_4 \exp\left(-\frac{n-2}{2}\,T_{\varepsilon_i} - \varpi(t) + \varpi(t)\right) \\
&\le\; C_3 \exp\left(-\frac{n-2}{2}\sum_{k=1}^{i} T_{\varepsilon_k}\right)\left[e^{(n-2)t/2}\,\psi(t)\right] \le C_4 \exp\left(\frac{n-2}{2}\rho\right)\psi(t)\,,
\end{aligned}$$

where C_3 and C_4 are positive constants that depend on n only. Hence we obtain $|K(t) - 1| \leq C_5\, \psi(t)$ for $t \gg 1$ and for a positive constant C_5. The corresponding solution u is an exotic solution of equation (1.1) which satisfies (1.13). We note that $K(t)$ in this case is not monotonic for large t.

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