# Hilbert's Axioms

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## Undefined Terms

- Points
- Lines
- Planes
- Lie on, contains
- Between
- Congruent

## Axioms

I. ***Axioms of Incidence***

**Postulate I.1.**
For every two points $A$, $B$ there exists a line $a$ that contains each of the points $A$, $B$.
**Postulate I.2.**
For every two points $A$, $B$ there exists no more than one line that contains each of the points $A$, $B$.
**Postulate I.3.**
There exists at least two points on a line. There exist at least three points that do not lie on a line.
**Postulate I.4.**
For any three points $A$, $B$, $C$ that do not lie on the same line there exists a plane $\alpha$ that contains each of the points $A$, $B$, $C$. For every plane there exists a point which it contains.
**Postulate I.5.**
For any three points $A$, $B$, $C$ that do not lie on one and the same line there exists no more than one plane that contains each of the three points $A$, $B$, $C$.
**Postulate I.6.**
If two points $A$, $B$ of a line $a$ lie in a plane $\alpha$ then every point of $a$ lies in the plane $\alpha$.
**Postulate I.7.**
If two planes $\alpha$, $\beta$ have a point $A$ in common then they have at least one more point $B$ in common.
**Postulate I.8.**
There exist at least four points which do not lie in a plane.

II. ***Axioms of Order***

**Postulate II.1.**
If a point $B$ lies between a point $A$ and a point $C$ then the points $A$, $B$, $C$ are three distinct points of a line, and $B$ then also lies between $C$ and $A$.
**Postulate II.2.**
For two points $A$ and $C$, there always exists at least one point $B$ on the line $AC$ such that $C$ lies between $A$ and $B$.
**Postulate II.3.**
Of any three points on a line there exists no more than one that lies between the other two.
**Postulate II.4.**
Let $A$, $B$, $C$ be three points that do not lie on a line and let $a$ be a line in the plane $ABC$ which does not meet any of the points $A$, $B$, $C$. If the line $a$ passes through a point of the segment $AB$, it also passes through a point of the segment $AC$, or through a point of the segment $BC$.

III. ***Axioms of Congruence***

**Postulate III.1.**
If $A$, $B$ are two points on a line $a$, and $A'$ is a point on the same or on another line $a'$ then it is always possible to find a point $B'$ on a given side of the line $a'$ such that $AB$ and $A'B'$ are congruent.
**Postulate III.2.**
If a segment $A'B'$ and a segment $A''B''$ are congruent to the same segment $AB$, then segments $A'B'$ and $A''B''$ are congruent to each other.
**Postulate III.3.**
On a line $a$, let $AB$ and $BC$ be two segments which, except for $B$, have no points in common. Furthermore, on the same or another line $a'$, let $A'B'$ and $B'C'$ be two segments which, except for $B'$, have no points in common. In that case if $AB \approx A'B'$ and $BC \approx B'C'$, then $AC \approx A'C'$.
**Postulate III.4.**
If $\square ABC$ is an angle and if $B'C'$ is a ray, then there is exactly one ray $B'A'$ on each "side" of line $B'C'$ such that $\square A'B'C' \cong \angle ABC$. Furthermore, every angle is congruent to itself.
**Postulate III.5. (*SAS*)**
If for two triangles $ABC$ and $A'B'C'$ the congruences $AB \approx A'B'$, $AC \approx A'C'$ and $\square BAC \approx \square B'A'C'$ are valid, then the congruence $\square ABC \approx \square A'B'C'$ is also satisfied.

IV. ***Axiom of Parallels***

**Postulate IV.1.**
***Let* a *be any line and* A *a point not on it. Then there is at most one line in the plane that contains* a *and* A *that passes through* A *and does not intersect* a.**

V. ***Axioms of Continuity***

**Postulate V.1. (*Archimedes Axiom*)**
If *AB* and *CD* are any segments, then there exists a number *n* such that *n* copies of *CD* constructed contiguously from *A* along the ray *AB* will pass beyond the point *B*.

**Postulate V.2. (*Line Completeness*)**
An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence (Axioms I-III and V-1) is impossible.

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## Sources:


1. *Foundations of Geometry*, D. Hilbert (trans. L. Unger), Open Court Publ.
2. *Roads to Geometry*, E. Wallace & S. West, Prentice-Hall (Sect 2.4 & App. B)


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