Local properties of the Kazdan-Warner problem on prescribing scalar curvature on S^n

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Abstract

We study the prescribing scalar curvature problem on S^n $(n \ge 3)$ by considering the functional which sends a positive smooth function u to the scalar curvature K of the conformal metric g. The null space of the linearization is described by the (stationary) Schrödinger equation

$$\Delta_g \, \Phi + \frac{K}{n-1} \, \Phi = 0 \, .$$

Obstructions (in terms of eigenvalues) exist for the Schrödinger equation to process non-trivial solutions. Together with eigenvalue estimates for conformal metrics, the results are applied to study existence of solutions of the prescribing scalar curvature problem for symmetric functions on S^n. Somehow surprisingly, the Schrödinger operator is capable of producing a large class of explicit functions which satisfy the Kazdan-Warner type condition. This includes homogeneous harmonic polynomials of higher order, and functions arising from non-uniqueness.

KEY WORDS: prescribing scalar curvature, Schrödinger's operator, eigenvalue.

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1. Introduction

As expounded in the superb book by A. Besse [7], Riemannian functionals are pivotal in understanding the structure of scalar curvature functions on manifolds. For conformal deformations on $(S^n,\ g_1)$, $n \ge 3$, it is natural to consider the *conformal scalar curvature functional*

$$
\begin{aligned}
(1.1) \qquad \mathcal{K}:\ C^\infty_+(S^n) &\ \to\ C^\infty(S^n) \\
u &\ \mapsto\ \mathcal{K}(u) := \frac{c_n R_{g_1}\, u - \Delta_{g_1} u}{c_n\, u^{\frac{n+2}{n-2}}}\,.
\end{aligned}
$$

Here g_1 is the standard metric on the unit sphere S^n, with scalar curvature $R_{g_1} = n(n-1)$, $C^\infty_+(S^n)$ the collection of positive smooth functions on S^n, and the constant c_n takes the value $(n-2)/\,[4\,(n-1)]$. Thus $K = \mathcal{K}(u)$ is the scalar curvature of the conformal metric $g := u^{\frac{4}{n-2}} g_1$. It follows from (1.1) that

$$
(1.2) \qquad \Delta_{g_1} u - c_n\, n(n-1)\, u + c_n\, K\, u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ S^n.
$$

The famed Kazdan-Warner problem (analogous to the Nirenberg problem on S^2) can be paraphrased as: what kind of smooth function can be in the image of $\mathcal{K}$, (denoted by $\mathcal{K}\rangle$), and how to characterize them? The problem has the fine quality of being easy to grasp, yet deep, making it into an attractive topic. Substantial achievements are obtained by advancing insights gained from the Yamabe problem (variational method and blow-up analysis), moving plane methods, as well as nonlinear functional analysis (perturbation and Leray-Schauder degree counting methods), and more, cf. [1], [2], [5], [4], [9], [10], [16], [12], [13], [18], [24], [26], [27], [31], [32], [34], [35], [36], [39], [40], [45] and [46] for recent works on the problem. Nevertheless, a complete solution of the problem remains elusive, partly because little ideas are available to apprehend the structure of $\mathcal{K}\rangle$.

There are two basic and crucial facts that we know about $K \in \mathcal{K}\rangle$.

(I) $K(x) > 0$ *for some* $x \in S^n$. Indeed, from (1.2) and Green's identity we have

$$
(1.3) \qquad \int_{S^n} K\, u^{\frac{2n}{n-2}}\, dV_{g_1} = \int_{S^n} \left[\, n(n-1)\, u^2 + c_n^{-1} \| \bigtriangledown_1 u\|^2 \,\right] dV_{g_1} > 0\,.
$$

Already, it is known that any smooth function in S^n that is positive somewhere is in the closure of $\mathcal{K}\rangle$ in $C^{1,\,\alpha}(S^n)$ $(0 < \alpha < 1)$, see [15] [31] [33].

(II) *The balance formula,* which states that for any conformal Killing vector field X on $(S^n,\ g_1)$,

$$\int_{S^n} X(K)\, dV_g \left(= \int_{S^n} X(\mathcal{K}(u))\, u^{\frac{2n}{n-2}}\, dV_{g_1} \right) = 0\,. \tag{1.4}$$

It is first discovered by Kazdan and Warner, generalized by Bourguignon and Ezin, and extended to a wider sense by Schoen (cf. [8] and [43]). Here $X(K) = \langle X,\ \bigtriangledown_g K \rangle_g$ is the directional derivative. A comprehensive description of conformal Killing vector fields on S^n can be found in the fine article of Han and Li [21].

Guided by (I) and (II), a function $\Psi \in C^\infty(S^n)$ is said to fulfill the

(III) *Kazdan-Warner type condition* if Ψ is positive somewhere and there exists *a* function $f \in C^\infty_+(S^n)$ such that

$$\int_{S^n} X(\Psi)\, f^{\frac{2n}{n-2}}\, dV_{g_1} = 0 \tag{1.5}$$

for all conformal Killing vector field X on S^n. (In the above, some authors prefer not to put the power on f. Nevertheless, this clarifies our overall presentation.)

Condition (1.5) says nothing on whether $\mathcal{K}\,(f) =$ or $\neq \Psi$ [cf. (D) in section 3, which links this simple observation to functions that satisfy (1.5)]. However, for a long time people wonder whether (III) is also a sufficient condition for $\Psi \in \mathcal{K}\rangle$. Only quite recently counter-examples are found on rotationally symmetric functions [14], and on non-rotationally symmetric functions in 3 and 4 dimensions [21].

In this article we consider the local structure of $\mathcal{K}\rangle$. It appears to be natural to ask if $K \in \mathcal{K}\rangle$, then whether small perturbations of K are still inside $\mathcal{K}\rangle$? Shedding lights on the problem is the *linearized map* $\mathcal{K}'_u$, or the derivative, of the functional $\mathcal{K}$ at u. When $u \equiv 1$ and hence $K \equiv n(n-1)$, the null space of $\mathcal{K}'_1$ is determined by the equation $\Delta_{g_1} \phi + n\, \phi = 0$. That is, ϕ is an eigenfunction of the Laplacian with eigenvalue n. For non-constant K, the null space of $\mathcal{K}'_u$ can be described by the (stationary) Schrödinger equation

$$\Delta_g\, \Phi + \frac{K}{n-1}\, \Phi = 0 \quad \text{in } \ S^n. \tag{1.6}$$

We define the operator $\mathcal{L}_g$ by

$$\mathcal{L}_g \, \Phi := \Delta_g \, \Phi + \frac{K}{n-1} \, \Phi \, .$$

(Note that this is different from the conformal Laplacian.)

Much of the local property of $\mathcal{K}\rangle$ is determined by whether (1.6) has non-trivial solutions or not. For if the kernel of $\mathcal{K}'_u$ is empty, then $K \in \mathcal{K}\rangle$ is an *interior point*, and hence small perturbations of K are still inside $\mathcal{K}\rangle$. On the other hand, if Φ is a non-trivial solution of (1.6), the function $K + \varepsilon\Phi$ cannot find solutions near u . More precisely, we obtain the following quantitative result.

Theorem A. *Given $u \in C^\infty_+(S^n)$ and $K = \mathcal{K}(u)$, assume that (1.6) has a non-trivial smooth solution Φ . Given any $\varepsilon > 0$, there exists a positive constant c with the property that we can choose $c \to \infty$ as $\varepsilon \to 0^+$, such that*

$$K + \varepsilon \, \Phi \ \not\in \ \mathcal{K} \left(B_{c \, \varepsilon} \, (u) \right) .$$

Here $B_{c\,\varepsilon} \, (u) := \{ w \in C^\infty_+(S^n) \ | \ \sup_{S^n} |u - w \, | < c\varepsilon \}$. Furthermore, c depends only on n, ε, $K_{\max}$, $K_{\min}$, $u_{\max}$, $\|\Phi\|_{L^2}$ and $\mathrm{Vol}_g \, (S^n)$.

In addition, assume that u , $\tilde{u} \in C^\infty_+(S^n)$ with $\mathcal{K}(u) = \mathcal{K}(\tilde{u})$ (i.e., the conformal metrics have the same scalar curvature). If

$$\mathcal{K}'_u(\phi) = \mathcal{K}'_{\tilde{u}}(\phi) = 0$$

for a function $\phi \in C^\infty(S^n) \setminus \{0\}$, then $u \equiv \tilde{u}$ in S^n.

For the first eigenfunctions ϕ, it is well-known via the Kazdan-Warner condition that functions of the form $n \, (n-1) + \varepsilon \, \phi \not\in \mathcal{K}\rangle$ $(\phi \not\equiv 0 \,)$. It follows from the Obata theorem ([41], cf. [9]) that constant positive functions are *not* interior points in $\mathcal{K}\rangle$. Interestingly, we do not know whether there are other non-trivial solutions of (1.6) having this property.

Our study leads to a rather unexpected finding. The Schrödinger operator $\mathcal{L}_g$ transcends its local origin and carries global implications. We consider in section 3 the infinitesimal version of the balance formula (1.4):

$$\int_{S^n} X(\mathcal{L}_g(\psi))\, dV_g = \frac{n}{2(n-1)} \int_{S^n} X(K)\, \psi\, dV_g\,, \tag{1.7}$$

which holds for all conformally Killing vector fields X of $(S^n,\ g)$ [equivalently, of $(S^n,\ g_1)$]. In particular, when $g = g_1$, hence $K = n(n-1)$, the right hand side of (1.7) is equal to zero. That is,

$$\int_{S^n} X(\Psi)\, dV_{g_1} = 0 \qquad \text{for} \ \ \Psi = \Delta_{g_1} \psi + n\, \psi\,, \quad \phi \in C^\infty(S^n)\,. \tag{1.8}$$

As a consequence, all smooth functions that are positive somewhere and orthogonal to the first eigenspace of $(S^n,\ g_1)$ satisfy the Kazdan-Warner type condition (and hence they are potential candidates to be inside $\mathcal{K}\rangle$). More specifically, this category includes homogeneous harmonic polynomials P^k_Δ on $\mathbb{R}^{n+1}$ (restricted to S^n) of order k $(k > 1)$. We observe that some of these functions do not satisfy the *flatness conditions* that are required in certain existence theorems (cf. [19], [16]). Outside the standard metric g_1, non-uniqueness of the conformal scalar curvature equation (1.2) can also be used to produce functions fulfilling the Kazdan-Warner type condition.

For the Schrödinger equation to possess non-trivial solutions, there are conditions in terms of the eigenvalues of Δ_g. It is a tautology to say that if λ is not an eigenvalue of Δ_g, then the equation $\Delta_g\, \phi + \lambda\, \phi = 0$ has no non-trivial solutions. Our result allows $\dfrac{K}{n-1}$ to vary slightly between two consequent eigenvalues.

Theorem B. *Let K be a positive smooth function on the compact Riemannian manifold $(N,\, h)$. Assume that, either*

(i) $\lambda_1 > \dfrac{K_{\max}}{n-1}\left[1 + \dfrac{\lambda_1\,(n-1)^2}{\mathrm{Vol}_h(N)}\left(\displaystyle\int_N \frac{|\,\nabla_h\, K|^2}{K^4}\, dV_h\right)\right]$; *or*

(ii) $\lambda_{m-1} < \dfrac{K}{n-1} < \lambda_m$ *in* N $(m \geq 2)$, $\displaystyle\int_N \|\,\nabla_h\, K\|^2_h\, dV_h \ \leq\ \gamma^2\, \mathrm{Vol}_h(N)$ *and* $(K_{\max} - K_{\min})^2 \leq \gamma^2$, *where γ is a suitable positive constant.*

Then the equation $\Delta_h\, \phi + \dfrac{K}{n-1}\, \phi = 0$ does not have any non-trivial smooth solutions. In addition, γ depends only on $K_{\max}$, $K_{\min}$, the eigenvalues λ_α of the

Laplacian Δ_h, and their multiplicities ($1 \le \alpha \le m$).

A conformal metric g of g_1 has its first positive eigenvalue λ_1 and scalar curvature K satisfying $\lambda_1 \le \dfrac{K_{\max}}{n-1}$. Furthermore, if the Ricci curvature of g is positive, then we also have $\dfrac{K_{\min}}{n-1} \le \lambda_1$ (section 5). On the other hand, the standard metric on the projective space $\mathbf{P}^n$ has scalar curvature $n\,(n-1)$ and first positive eigenvalue $2(n+1)$. Hence condition (i) in theorem is satisfied. Thus, on $\mathbf{P}^n$, positive constant functions are interior points in the space of conformal scalar curvature functions. Evidently, symmetric functions on S^n of the form $K(-x) = K(x)$ that are close to a positive constants are inside $\mathcal{K}\rangle$. We extend the result to other closed manifolds and also to conformal scalar curvature functions nearby. By the resolution of the Yamabe problem ([3], [42], [48]; see also [25]), any metric on a compact manifold can be conformally deformed into a metric of constant scalar curvature.

Theorem C. *Let $(N,\ h)$ be a compact Riemannian n-manifold $(n \ge 3)$ with constant scalar curvature $\kappa > 0$ and first positive eigenvalue λ_1, such that*

$$\lambda_1 \ \ge\ (1+\delta^2)\,\frac{\kappa}{n-1} \qquad \textit{for a positive number } \delta\,.$$

Given positive numbers a, b and c with

$$1 \le \left(\frac{b}{a}\right)^n \le 1 + \frac{\delta^2}{2} \quad \textit{and} \quad \left(\frac{c}{b}\right)^4 \le \frac{\kappa^2\,\delta^2}{2\,\lambda_1\,(n-1)^2}\left(1+\frac{\delta^2}{2}\right)^{-\frac{10}{n}}, \tag{1.9}$$

assume that the conformal metric $u^{\frac{4}{n-2}}\,h$ has scalar curvature K satisfying

$$a^2 \le K \le b^2 \qquad \textit{and} \qquad \|\,\nabla_h\,K\|_h \le c^2 \qquad \textit{in } \ N\,,$$

and u satisfying the flatness condition (5.13). Then K is an interior point in the space of conformal scalar curvature functions of $(N,\ h)$ with the $C^{\,\alpha}$ topology.

In section 6, we demonstrate how to construct non-trivial solutions of Schrödinger equation (1.6). Here we encounter the Poincaré-Hopf theorem on vector fields. Because of this, the construction only works in even dimensions.

2. Kernel of $\mathcal{K}'_u$

Let $u \in C^\infty_+(S^n)$. With the help of (1.1), we find the linearization of $\mathcal{K}$ at u :

$$(2.1) \qquad \begin{aligned} \mathcal{K}'_u(\phi) &= -\frac{\Delta_{g_1}\phi - c_n\, R_{g_1}\,\phi}{c_n\, u^{\frac{n+2}{n-2}}} + \left(\frac{n+2}{n-2}\right)\frac{\Delta_{g_1}u - c_n\, R_{g_1}\,u}{c_n\, u^{\frac{2n}{n-2}}}\,\phi \\ &= -\frac{\Delta_{g_1}\phi - c_n\, R_{g_1}\,\phi}{c_n\, u^{\frac{n+2}{n-2}}} - \left(\frac{n+2}{n-2}\right)\left(\frac{\phi}{u}\right) K\,. \end{aligned}$$

The first interesting case is when $u \equiv 1$ in S^n. We have

$$\begin{aligned} \mathcal{K}'_1(\phi) &= -\,c_n^{-1}\Delta_{g_1}\phi + n(n-1)\phi - \left(\frac{n+2}{n-2}\right) n(n-1)\,\phi \\ &= -\,\frac{4(n-1)}{n-2}\,(\Delta_{g_1}\phi + n\,\phi)\,. \end{aligned}$$

Thus the null space of $\mathcal{K}'_1$, denoted by E_1, is the space of first eigenfunctions of the Laplacian for $(S^n,\ g_1)$. For a smooth function ψ, by the Fredholm alternative theorem, the equation

$$\Delta_{g_1}\phi + n\,\phi = \psi \quad \text{ in } \ S^n$$

is solvable if and only if ψ is orthogonal E_1. Furthermore, the Obata theorem ([41]; see also [9]) states that any conformal metric of g_1 with constant scalar curvature is, possibly after a rescaling, isometric to g_1. In this case the local object has global implications. More precisely, we know that for $\phi_1 \in E_1 \setminus \{0\}$, $\bigtriangledown_1\phi_1$ is a non-trivial conformal Killing vector field. It follows that the function $n(n-1)+\varepsilon\,\phi_1$ does not satisfy the Kazdan-Warner type condition, and hence it cannot be in $\mathcal{K}\rangle$. Indeed, the presence of first eigenfunctions appears to be the only obstacle toward satisfying the Kazdan-Warner type condition (cf. section 3).

For a generic function $u \in C^\infty_+(S^n)$, the null space of $\mathcal{K}'_u$ is described by

$$(2.2)\ \ \mathcal{K}'_u(\phi) = 0 \iff \Delta_{g_1}\phi + c_n\left(\frac{n+2}{n-2}\right)\left[K u^{\frac{4}{n-2}} - \frac{n(n-1)(n-2)}{n+2}\right]\phi = 0\,.$$

Since $R_{g_1} = n(n-1)$, we have

$$\begin{aligned} K u^{\frac{4}{n-2}} - \frac{n(n-1)(n-2)}{n+2} &= -\,\frac{\Delta_{g_1}u - c(n)R_{g_1}\,u}{c_n\,u} \\ = \ -\frac{\Delta_{g_1}u}{c_n\,u} + n(n-1)\left[1 - \frac{n-2}{n+2}\right] &= -\frac{\Delta_{g_1}u}{c_n\,u} + \frac{4n(n-1)}{n+2}\,. \end{aligned}$$

Thus (2.2) can be rewritten as

$$\Delta_{g_1} \phi + n\,\phi = \left[\left(\frac{n+2}{n-2} \right) \frac{\Delta_{g_1} u}{u} \right] \phi\,. \tag{2.3}$$

Lemma 2.4. *Let $u \in C^\infty_+(S^n)$ and $\phi \in C^\infty(S^n)$. We have $\mathcal{K}'_u(\phi) = 0$ if and only if*

$$\Delta_g \Phi + \frac{K}{n-1} \Phi = 0\,, \qquad \textit{where} \;\; \Phi = \frac{\phi}{u}\,. \tag{2.5}$$

Here $g = u^{\frac{4}{n-2}}\, g_1$ and K is the scalar curvature of g.

Proof. For $v \in C^\infty(S^n)$, we have (cf. [44])

$$\Delta_g\, v - c_n\, K v = u^{-\,\frac{n+2}{n-2}} \left[\, \Delta_{g_1} (uv) - c_n\, n(n-1)\,(uv) \right]\,.$$

Thus

$$\Delta_g \left(\frac{\phi}{u} \right) - c_n K \left(\frac{\phi}{u} \right) = \frac{\Delta_{g_1}\phi - c_n n(n-1)\,\phi}{u^{\frac{n+2}{n-2}}}\,. \tag{2.6}$$

As ϕ is in the null space, it follows from (1.2) and (2.3) that

$$\frac{\Delta_g\, \phi - c_n n(n-1)\,\phi}{u^{\frac{n+2}{n-2}}} = -c_n \left(\frac{n+2}{n-2} \right) K \left(\frac{\phi}{u} \right)\,. \tag{2.7}$$

Combining (2.6) and (2.7) we obtain

$$\Delta_g \left(\frac{\phi}{u} \right) - c_n K \left(\frac{\phi}{u} \right) = -c_n \left(\frac{n+2}{n-2} \right) K \left(\frac{\phi}{u} \right) \quad \Longrightarrow \quad \Delta_g \Phi + \frac{K}{n-1} \Phi = 0\,,$$

where $\Phi = \phi/u$. Likewise, any solution Φ to the (stationary) Schrödinger equation equation $\Delta_g \Phi + \dfrac{K}{n-1}\Phi = 0$ provides a solution to $\mathcal{K}'_u(\phi) = 0$ via $\phi = u\,\Phi$. □

Define the Schrödinger operator

$$\mathcal{L}_g := \Delta_g + \frac{K}{n-1}\,. \tag{2.8}$$

Lemma 2.9. *$\mathcal{L}_g$ is a uniformly elliptic (formally) self-adjoint operator. The solution space $\mathcal{S}_o := \{\Phi \in C^2(S^n) \mid \mathcal{L}_g \, \Phi = 0\}$ is finite dimensional and each non-zero C^2 solution must change sign. Moreover, for $\Phi \in \mathcal{S}_o \setminus \{0\}$, $\Phi^{-1}(0)$ forms an $n-1$ dimensional manifold, except on a closed set of lower dimension.*

Proof. Consider the eigenvalue problem

$$\mathcal{L}_g \, f + \lambda \, f = 0\,.$$

If zero is one of the eigenvalues of $\mathcal{L}_g$, then the equation $\mathcal{L}_g \, \Phi = 0$ has non-trivial C^2 solutions. Standard elliptic theory (see §8.12 in [20]) says that the eigenspace is finite dimensional, and each non-trivial C^2 solution must change sign. This can also be seen from

$$\Phi > 0 \implies \frac{\Delta_g \Phi}{\Phi} + \frac{K}{n-1} = 0 \implies \int_{S^n} \frac{K}{n-1} \, dV_g = - \int_{S^n} \frac{|\bigtriangledown \Phi|^2}{\Phi^2} \, dV_g < 0\,,$$

which contradicts (1.3). The manifold structure on $\Phi^{-1}(0)$ is shown in [17] by S.-Y. Cheng, using a result of Lipman Bers [6]. □

It is helpful to extend (1.1) to *any* Riemannian metric h on a compact n-manifold N with scalar curvature K. Define the corresponding functional $\mathcal{R}$ by

$$\begin{array}{rrcl} (2.10) \qquad & \mathcal{R}: \; C^\infty_+(N) & \to & C^\infty(N) \\ & u & \mapsto & \mathcal{R}(u) = \dfrac{c_n K \, u - \Delta_h \, u}{c_n \, u^{\frac{n+2}{n-2}}}\,. \end{array}$$

Thus $\mathcal{R}(u)$ is the scalar curvature of the conformal metric $u^{\frac{4}{n-2}} \, h$. The linearization of $\mathcal{R}$ at $u \equiv 1$ is found by

$$\begin{aligned} \mathcal{R}'_1(\phi) &= - \left[\frac{\Delta_h \, \Phi - c_n \, K \, \Phi}{c_n \, u^{\frac{n+2}{n-2}}} + \frac{n+2}{n-2} \, \mathcal{R}(u) \left(\frac{\phi}{u} \right) \right]_{u \equiv 1} \\ &= - \frac{1}{c_n} \left[\Delta_h \, \Phi - c_n \, K \Phi + \frac{n+2}{n-2} \, K \, \Phi \right] \\ &= - \frac{1}{c_n} \left[\Delta_h \, \Phi + \frac{K}{n-1} \, \Phi \right]\,. \end{aligned}$$

It follows that the kernel of $\mathcal{R}'_1$ is described by

$$\Delta_h \, \Phi + \frac{K}{n-1} \, \Phi = 0\,,$$

which is exactly the same form as (2.5).

Initially $\mathcal{R}$ is defined on $C^\infty_+(N)$, the space of positive smooth functions on N, but can be extended to Hölder's spaces like $C^{2,\alpha}_+(N)$, or to suitable Sobolev spaces which are Hilbert spaces (cf. [7]).

Given a non-trivial solution Φ of (2.5), as the operator $\mathcal{L}_g$ is formally self-adjoint, Φ is also in the co-kernel. Intuitively, for small positive numbers ε, $K+\varepsilon\,\Phi$ is not in the image of a reasonable small open neighborhood under the map $\mathcal{K}$. The following derivation leads to the local non-existence formally. In addition, it gives information on the "radius" of local non-existence.

Theorem 2.11. *Given $u \in C^\infty_+(S^n)$ and $K = \mathcal{K}(u)$, let Φ be a non-trivial smooth solution of the equation $\mathcal{L}_g\Phi = 0$. Given any $\varepsilon > 0$, there exists a positive constant c with the property that we can choose $c \to \infty$ as $\varepsilon \to 0^+$, such that*

$$K + \varepsilon\,\Phi \not\in \mathcal{K}\left(B_{c\,\varepsilon}\,(u)\right).$$

Here $B_{c\,\varepsilon}\,(u) := \{w \in C^\infty_+(S^n) \mid \sup_{S^n} |u - w\,| < c\varepsilon\,\}$. In addition, c depends only on n, ε, $\max u$, $\max |K|$, $\|\Phi\|_{L^2}$ and $\mathrm{Vol}_g\,(S^n)$.

Proof. As $(u+f)^{\frac{4}{n-2}}\,g_1 = (1+f/u)^{\frac{4}{n-2}}\left(u^{\frac{4}{n-2}}g_1\right) = (1+f/u)^{\frac{4}{n-2}}\,g$, we can move the 'center' to $g = u^{\frac{4}{n-2}}g_1$ and consider the functional $\mathcal{R}$ as defined in (2.10). Set $K_\nu := K + \varepsilon\,\Phi$. Suppose that there exists $\psi \in C^\infty(S^n)$ such that $\mathcal{R}(1+\psi) = K_\nu$ and

$$|\psi| \le c\,\varepsilon \quad \text{ in } \; S^n.$$

The first condition we impose on c is that $c\,\varepsilon < 1$ so that $1+\psi > 0$ in S^n. Let

$$\Psi := \Delta_g\,\psi + \frac{K}{n-1}\,\psi\,. \tag{2.12}$$

We have

$$\begin{aligned} K_\nu \;&=\; \mathcal{R}(1+\psi) = -\,\frac{\Delta_g(1+\psi) - c_n K\,(1+\psi)}{c_n\,(1+\psi)^{\frac{n+2}{n-2}}} \\ &=\; K\left[1 + \frac{n+2}{n-2}\,\psi\right]\,(1+\psi)^{-\frac{n+2}{n-2}} - \frac{\Psi}{c_n(1+\psi)^{\frac{n+2}{n-2}}}\,. \end{aligned} \tag{2.13}$$

It follows from (2.13) that

$$(K+\varepsilon\,\Phi)\,(1+\psi)^{\frac{n+2}{n-2}}=K\left[1+\frac{n+2}{n-2}\,\psi\right]-\frac{\Psi}{c_n}\,. \tag{2.14}$$

By the Green's identity and (2.12) we obtain

$$\int_{S^n}\Psi\,\Phi\,dV_g=\int_{S^n}(\Delta_g\,\psi+\frac{K}{n-1}\,\psi)\,\Phi\,dV_g=\int_{S^n}(\Delta_g\,\Phi+\frac{K}{n-1}\,\Phi)\,\psi\,dV_g=0\,, \tag{2.15}$$

and

$$\int_{S^n}K\,\Phi\,dV_g=(n-1)\int_{S^n}\Delta\Phi\,dV_g=0\,. \tag{2.16}$$

Multiple both sides of (2.14) by $\varepsilon\,\Phi$ we have

$$(K\,\varepsilon\,\Phi+\varepsilon^2\,\Phi^2)\,(1+\psi)^{\frac{n+2}{n-2}}=\varepsilon\,K\,\Phi+\frac{n+2}{n-2}K\,(\varepsilon\,\Phi)\,\psi-\frac{\varepsilon\,\Psi\,\Phi}{c_n}\,.$$

After integrating both sides of the above equation and using (2.15) and (2.16), one obtains

$$\int_{S^n}(K\,\varepsilon\,\Phi+\varepsilon^2\,\Phi^2)\,(1+\psi)^{\frac{n+2}{n-2}}\,dV_g=\frac{n+2}{n-2}\int_{S^n}K\,(\varepsilon\,\Phi)\,\psi\,dV_g\,. \tag{2.17}$$

The key is to show that (2.17) cannot be "balanced" when $\|\psi\|_{C^o}$ is small. We use Taylor's expansion in the following form:

$$(1+\psi)^{\frac{n+2}{n-2}}=1+\left(\frac{n+2}{n-2}\right)\,\psi+R_2\,.$$

Here R_2 is a continuous function on S^n satisfying $\|R_2\|_{C^o}\le C_n\,\|\psi\|^2_{C^o}$, where

$$C_n\;=\;\begin{cases}\frac{4(n+2)}{(n-2)^2}\,\left(\frac{3}{2}\right)^{\frac{6-n}{n-2}} & \text{for}\qquad 3\le n\le 6\,,\\ \frac{4(n+2)}{(n-2)^2}\,2^{-\frac{n-6}{n-2}} & \text{for}\qquad n>6\,.\end{cases}$$

Thus (2.17) becomes

$$\begin{aligned}\int_{S^n}(K\,\varepsilon\,\Phi+\varepsilon^2\,\Phi^2)\,dV_g+\frac{n+2}{n-2}\int_{S^n}(K\,\varepsilon\,\Phi\,\psi+\varepsilon^2\,\Phi^2\,\psi)\,dV_g\\ +\int_{S^n}R_2\,(K\,\varepsilon\,\Phi+\varepsilon^2\,\Phi^2)\,dV_g\;=\;\frac{n+2}{n-2}\int_{S^n}K\,(\varepsilon\,\Phi)\,\psi\,dV_g\,.\end{aligned}$$

That is,

$$\varepsilon^2 \int_{S^n} \Phi^2 \left[1 + R_2 + \frac{n+2}{n-2}\,\psi \right] dV_g = -\,\varepsilon \int_{S^n} R_2\, K\, \Phi\, dV_g\,, \tag{2.18}$$

where (2.16) is used. We further impose the restriction

$$\|\psi\|_{C^o} \;\le\; c\,\varepsilon \;\le\; \left[2 \left(C_n + \frac{n+2}{n-2} \right) \right]^{-1}$$

(which implies $c\,\varepsilon < 1$) so that

$$\left| 1 + R_2 + \frac{n+2}{n-2}\,\psi \right| \ge \frac{1}{2} \quad \text{in } \; S^n.$$

Thus

$$\varepsilon^2 \int_{S^n} \Phi^2 \left[1 + R_2 + \frac{n+2}{n-2}\,\psi \right] dV_g \ge \frac{\varepsilon^2}{2}\, \|\Phi\|^2_{L^2}\,.$$

On the other hand,

$$\begin{aligned} -\int_{S^n} R_2\, K\, \Phi\, dV_g \;&\le\; \left(\int_{S^n} \Phi^2\, dV_g \right)^{\frac{1}{2}} \left(\int_{S^n} |R_2|^2\, |K|^2\, dV_g \right)^{\frac{1}{2}} \\ &\le\; C_n\, \|\Phi\|_{L^2}\, (\max |K|)\, [\, \mathrm{Vol}_g\, (S^n)\,]^{\frac{1}{2}}\, c^2\, \varepsilon^2 \end{aligned}$$

$$\Longrightarrow \quad \|\Phi\|_{L^2} \;\le\; 2\, C_n\; (\max |K|)\, [\, \mathrm{Vol}_g\, (S^n)\,]^{\frac{1}{2}}\, c^2\, \varepsilon\,.$$

Hence if

$$c < \max \left\{ \frac{\|\Phi\|_{L^2}}{\left\{ 2\, \varepsilon\, C_n\; (\max |K|)\, [\, \mathrm{Vol}_g\, (S^n)\,]^{\frac{1}{2}} \right\}^{\frac{1}{2}}}\;,\;\; \left[2 \left(C_n + \frac{n+2}{n-2} \right) \right]^{-1} \frac{1}{\varepsilon} \right\},$$

then (2.18) leads to a contradiction. Moreover, we are free to choose $c \to \infty$ as $\varepsilon \to 0^+$. ☐

The canonical operator $\mathcal{L}_{g_1} = \Delta_{g_1} + n$ has not only local but also global implications, which $\mathcal{L}_g$ seems to be void of (save the case when g is isometric to g_1). The following may be considered as a partial remedy.

Theorem 2.19. *Assume that* $u\,,\ \tilde{u} \in C^\infty_+(S^n)$ *with* $\mathcal{K}(u) = \mathcal{K}(\tilde{u})\,.$ *If* $\mathcal{K}'_u(\phi) = \mathcal{K}'_{\tilde{u}}(\phi) = 0$ *for a function* $\phi \in C^\infty(S^n) \setminus \{0\}\,,$ *then* $u \equiv \tilde{u}$ *in* S^n.

Proof. Let $K := \mathcal{K}(u) = \mathcal{K}(\tilde{u})\,.$ We have

$$\begin{aligned} &\Delta_{g_1} u - c_n\, n(n-1)\, u + c_n K u^{\frac{n+2}{n-2}} = 0\,, \\ &\Delta_{g_1} \tilde{u} - c_n\, n(n-1)\, \tilde{u} + c_n K \tilde{u}^{\frac{n+2}{n-2}} = 0\,. \end{aligned} \tag{2.20}$$

From (2.3) we have

$$\phi\, \tilde{u}\, \Delta_{g_1} u = \phi\, u\, \Delta_{g_1}\, \tilde{u}\,. \tag{2.21}$$

It follows from (2.20) and (2.21) that

$$K\, \phi\, \tilde{u}\, u^{\frac{n+2}{n-2}} = K\, \phi\, u\, \tilde{u}^{\frac{n+2}{n-2}} \quad \Longrightarrow \quad K\, \phi\, u^{\frac{4}{n-2}} = K\, \phi\, \tilde{u}^{\frac{4}{n-2}}\,. \tag{2.22}$$

As $\Phi = \phi/u$ satisfies the equation $\Delta_g\, \Phi + \dfrac{K}{n-1}\, \Phi = 0$ in S^n, by lemma 2.9, the set

$$\mathcal{N} := \{\, x \in S^n \mid \Phi(x) = 0 = \phi(x)\,\}$$

forms an $(n-1)$-dimensional submanifold in S^n, except on a closed set of lower dimension. Let

$$\mathcal{M} := \{\, x \in S^n \mid K(x) = 0\,\}\,.$$

Outside $\mathcal{N} \cup \mathcal{M}$, we have $u = \tilde{u}$ by (2.22). Consider the following cases.

(i) Assume that $x_o \in \mathcal{N} \setminus \mathcal{M}\,.$ There is a sequence $\{x_i\}_{i=1}^\infty$ such that $\lim\limits_{i\to\infty} x_i = x_o$ and $\phi(x_i) \neq 0\,.$ As $K(x_o) \neq 0\,,$ we may also assume that $K(x_i) \neq 0$ for $i \gg 1\,.$ Hence $\phi(x_i)\, u^{\frac{4}{n-2}}(x_i) = \phi(x_i)\, \tilde{u}^{\frac{4}{n-2}}(x_i)\,,$ that is, $u(x_i) = \tilde{u}(x_i)$ for $i \gg 1\,.$ By continuity, $u(x_o) = \tilde{u}(x_o)$ as well.

(ii) Assume that $y_o \in \mathcal{M}$ is *not* an interior point of $\mathcal{M}\,.$ It follows that there is a sequence $\{y_i\}_{i=1}^\infty$ such that $\lim\limits_{i\to\infty} y_i = y_o$ and $K(y_i) \neq 0\,.$ The argument above shows that $u(y_i) = \tilde{u}(y_i)$ and by continuity, $u(y_o) = \tilde{u}(y_o)$

(iii) Let $y \in \mathcal{M}$ be an interior point of $\mathcal{M}$ and let $U \subset \mathcal{M}$ be the maximal connected open set (in the topology of S^n) in $\mathcal{M}$ which contains $y\,.$ As K has to be positive somewhere, $\partial U \neq \emptyset$. By the above argument, we have $u|_{\partial U} = \tilde{u}|_{\partial U}$. So (2.20) leads to

$$\Delta_{g_1} (u - \tilde{u}) = c_n \, n(n-1) \, (u - \tilde{u}) \quad \text{ in } \; U \, , \qquad \text{with } \; u - \tilde{u} = 0 \quad \text{ on } \; \partial U$$

(as $K = 0$ in U) . The maximum principle implies that $u - \tilde{u} = 0$ in U . In particular, $u(y) = \tilde{u}(y)$. ☐

3. Functions satisfying the Kazdan-Warner type condition

For a metric g on S^n with scalar curvature K, consider the one parameter of functions

$$u_\varepsilon := 1 + \varepsilon \, \psi \, .$$

Here $\psi \in C^\infty(S^n)$, and ε is small enough so that u_ε remains positive in S^n. Let $R_\varepsilon := \mathcal{R}(u_\varepsilon) = \mathcal{R}(1 + \varepsilon \, \psi)$ and $g_\varepsilon := u_\varepsilon^{\frac{4}{n-2}} \, g$. It follows from the balance formula (1.4) that

$$0 = \int_{S^n} X(R_\varepsilon) \, dV_{g_\varepsilon} = \int_{S^n} \langle X \, , \; \bigtriangledown_g \, R_\varepsilon \rangle_g \, u_\varepsilon^{\frac{2n}{n-2}} \, dV_g \, , \tag{3.1}$$

where X is a conformal Killing vector field with respect to the metric g . As in (2.1), we obtain

$$\frac{\partial R_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon = 0} = \frac{\partial}{\partial \varepsilon} \left[- \, \frac{\Delta_g \, (1 + \varepsilon \, \psi) - c_n K \, (1 + \varepsilon \, \psi)}{c_n \, (1 + \varepsilon \, \psi)^{\frac{n+2}{n-2}}} \right]_{\varepsilon = 0} = - \, \frac{1}{c_n} \, \left[\Delta_g \, \psi + \frac{K}{n-1} \, \psi \right] \, .$$

Differentiating both sides of equation (3.1) and letting $\varepsilon = 0$, we obtain

$$- \frac{1}{c_n} \, \int_{S^n} \langle X \, , \; \bigtriangledown_g \Psi \rangle_g \; dV_g + \frac{2n}{n-2} \, \int_{S^n} \langle X \, , \; \bigtriangledown_g \, K \rangle_g \, \psi \, dV_g = 0 \, ,$$

where

$$\Psi := \Delta_g \, \psi + \frac{K}{n-1} \, \psi \, . \tag{3.2}$$

That is,

$$\int_{S^n} X(\Psi) \; dV_g = \frac{n}{2(n-1)} \, \int_{S^n} X(K) \, \psi \, dV_g \, . \tag{3.3}$$

Consider the set

$$\mathcal{P} := \{\psi \in C^\infty(S^n) \mid \mathcal{L}_g(\psi) \ \text{ is positive somewhere and } \int_{S^n} X(K)\,\psi\, dV_g = 0\}\,.$$

Thus for any $\psi \in \mathcal{P}$, $\Psi = \mathcal{L}_g(\psi)$ satisfies the Kazdan-Warner type condition. We discuss $X(K) \equiv 0$ and $X(K) \not\equiv 0$ separately.

(A) An immediate example for $X(K) \equiv 0$ is when $g = g_1$ so that $K = n(n-1)$. In this case we just need $\mathcal{L}_g(\psi)$ to be positive somewhere. Given a smooth function Ψ , the equation

$$\Delta_{g_1}\psi + n\,\psi = \Psi$$

has a solution ψ if and only if Ψ is orthogonal to the first eigenspace E_1. In case $\Psi \not\equiv 0$, then either Ψ or $-\Psi$ (or both) satisfies the Kazdan-Warner type condition. (So does $C + \Psi$ for C large enough.)

(B) *Homogeneous harmonic polynomials.* In particular, consider order k homogeneous harmonic polynomials P^k_Δ on $\mathbb{R}^{n+1}$, where k is a positive integer *bigger* than one. It is well known that

$$\Delta_{g_1} P^k_\Delta + k(k+n-1)P^k_\Delta = 0 \quad \text{in } \ S^n\,. \tag{3.4}$$

(3.4) can be rewritten as

$$\Delta_{g_1} P^k_\Delta + nP^k_\Delta = [n - k(k+n-1)]\, P^k_\Delta\,.$$

It follows that for $k > 1$,

$$\int_{S^n} X(P^k_\Delta)\, dV_{g_1} = 0 \tag{3.5}$$

for all conformal Killing vector field on $(S^n,\ g_1)$. As $\int_{S^n} P^k_\Delta\, dV_{g_1} = 0$, P^k_Δ is positive somewhere in S^n. Hence for $k \ge 2$, P^k_Δ satisfies the Kazdan-Warner type condition.

We observe that some of these functions do not satisfy the usual flatness conditions required in a class of existence theorems. For instance, in S^n with $n \ge 4$, consider the order 2 homogeneous harmonic polynomials

$$x_i\, x_j\,, \quad x_i^2 - x_j^2 \quad \text{for } \ 1 \le i \le j \le n+1\,.$$

We can write

$$\begin{aligned} x_1\, x_2 &= \cos\theta\, \sin\theta = \frac{1}{2}\, \sin 2\theta := P(\theta) \\ x_1^2 - x_2^2 &= \cos^2\theta - \sin^2\theta = \cos 2\theta := Q(\theta) \end{aligned}$$

for $0 \le \theta \le \pi$. Observe that $P'(\theta) = \cos 2\theta$ changes signs on the (non-empty) region where $P > 0$. Chen and Li [16] ask whether this property is a sufficient condition for rotationally symmetric functions to be in $\mathcal{K}\rangle$. They show that this is the case under a flatness condition [16]. (We note that $Q > 0$ on $(0,\ \pi/4)$ and $(3\pi/4,\ \pi)$, while Q' is negative on the first region and positive on the second.)

In the present cases, P and Q have *second* derivatives which do not vanish at the maximal points. Although the functions have the symmetry $P(x) = P(-x)$, $Q(x) = Q(-x)$, they do not satisfy the condition $\bigtriangledown^2 P = 0$ at the maximal points, and hence when $n \ge 4$ the result of Escobar and Schoen [19] cannot be applied as well. It would be interesting (excluding the simplest cases) to know whether or not the functions discussed in (A) in general, and P^k_Δ in particular, are in $\mathcal{K}\rangle$.

(C) Let $\phi_1 \in E_1$, the first eigenspace of $(S^n,\ g_1)$. In case $X = \bigtriangledown_{g_1}\phi_1$, which is known to be a conformal vector field, it is easy to show that $\int X(\Psi)\, dV_{g_1} = 0$, where $\Psi = \Delta_{g_1}\psi + n\psi$. Indeed, we have $\Psi\,(\Delta_{g_1}\phi_1 + n\,\phi_1) = 0$, and $\int_{S^n} \Psi\, \phi_1\, dV_{g_1} = 0$, where we use integration by parts. It follows that

$$\begin{aligned} \int_{S^n} \Psi\, \Delta_{g_1}\phi_1\, dV_{g_1} &= 0 \implies \int_{S^n} \langle \bigtriangledown_{g_1}\phi_1\,,\ \bigtriangledown_{g_1}\Psi\rangle_{g_1}\, dV_{g_1} = 0 \\ \implies \int_{S^n} X(\Psi)\, dV_{g_1} &= 0 \qquad (\text{with } \ X = \bigtriangledown_{g_1}\phi_1)\,. \end{aligned}$$

It can be shown that if Ψ satisfies the Kazdan-Warner condition (1.5), then for any conformal transformation T on $(S^n,\ g_1)$, $\Psi \circ T$ also satisfies condition (1.5). Hebey [22] shows that for any smooth function f on S^n with $f_{\max} > 0$, there is a first eigenfunction ϕ_1 of $(S^n,\ g_1)$ and a conformal transformation T such that $(f - \phi_1 \circ T) \in \mathcal{K}\rangle$.

(D) *Relation with non-uniqueness.* When $X(K) \not\equiv 0$, there are two natural subcases where we can find functions ψ to annihilate the right hand side of (3.3).

Both are linked to non-uniqueness.

Assume that the conformal metric $\tilde{g} = \tilde{u}^{\frac{4}{n-2}}\, g$ also has scalar curvature K. We take $\psi = \tilde{u}^{\frac{2n}{n-2}}$ in (3.3), and find that

$$\int_{S^n} X(\Psi)\, dV_g = \int_{S^n} X(K)\, \psi\, dV_g = \int_{S^n} X(K)\, \tilde{u}^{\frac{2n}{n-2}}\, dV_g = \int_{S^n} X(K)\, dV_{\tilde{g}} = 0\,,$$

where

$$\Psi = \mathcal{L}_g\,(\psi) = \mathcal{L}_g\,\left(\tilde{u}^{\frac{2n}{n-2}}\right)\,. \tag{3.6}$$

As $u > 0$ in S^n, by lemma 2.9, $\mathcal{L}_g\left(\tilde{u}^{\frac{2n}{n-2}}\right) \not\equiv 0$. Hence either Ψ or $-\Psi$ satisfies the Kazdan-Warner type condition. (When K is positive in S^n, the maximum principle implies that Ψ is positive somewhere.)

The above relation can also be explored in the Kazdan-Warner type condition. For $K \in C^\infty(S^n)$, assume that K satisfies the Kazdan-Warner type condition. That is, K is positive somewhere and there is a function $f \in C^\infty_+(S^n)$ such that

$$\int_{S^n} X(K)\, f^{\frac{2n}{n-2}}\, dV_{g_1} = 0 \tag{3.7}$$

for all conformal Killing vector field X of $(S^n,\, g_1)$. Assume also that $K \in \mathcal{K}\rangle$ but $\mathcal{K}(f) \neq K$. Hence there exists $u \in C^\infty_+(S^n) \setminus \{f\}$ such that $\mathcal{K}(u) = K$. With $g = u^{\frac{4}{n-2}}\, g_1$, it follows from (3.7) that

$$\int_{S^n} X(K)\, \left(\frac{f}{u}\right)^{\frac{2n}{n-2}}\, dV_g = 0\,.$$

Taking $\psi := \left(\dfrac{f}{u}\right)^{\frac{2n}{n-2}}$ in (3.3), we obtain

$$\int_{S^n} X(\Psi)\, dV_g = \int_{S^n} X(K)\, \psi\, dV_g = 0\,,$$

where $\Psi := \mathcal{L}_g\left((f/u)^{\frac{2n}{n-2}}\right)$. As $\psi > 0$, it follows from lemma 2.9 that ψ is not a solution of the Schrödinger equation. That is, $\Psi \not\equiv 0$. Hence either Ψ or $-\Psi$ satisfies the Kazdan-Warner type condition.

For example, let K be a non-constant smooth function on S^3 with the symmetry

$$K(-x) = -K(x) \quad \text{for} \quad \text{all} \;\; x \in S^3 .$$

It follows that

$$\int_{S^3} X(K)\, dV_{g_1} = 0$$

for all conformal Killing vector fields X on S^3. If K is positive somewhere, then it satisfies the Kazdan-Warner condition with $f \equiv 1$ in S^3. By a result of Escobar and Schoen [19], there is $u \in C_+^\infty(S^3)$ such that $\mathcal{K}(u) = K$. Let $\Psi := \mathcal{L}_g \left(u^{-6} \right)$. It follows that either Ψ or $-\Psi$ satisfies the Kazdan-Warner condition.

(E) In general, let $K = \mathcal{K}(u)$ and $g = u^{\frac{4}{n-2}} g_1$. Consider a function $\psi \in C^\infty(S^n)$ such that

$$\int_{S^n} X(K)\, \psi\, dV_g = 0 \tag{3.8}$$

for all conformal Killing vector fields X of $(S^n,\; g_1)$. Note that ψ needs *not* to be positive. From (3.3) we have

$$\int_{S^n} X(\mathcal{L}_g\, \psi)\; dV_g = 0\,. \tag{3.9}$$

If in addition the function $\Psi := \mathcal{L}_g\, \psi$ is positive somewhere (which is automatic if the minimum value of ψ is negative and occurs in the region where $K < 0$), then it satisfies the Kazdan-Warner condition.

The set of all conformal Killing vector fields on

$$S^n := \left\{ (x_1, \cdots,\; x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

with the standard metric forms a linear space of dimension $(n+1)(n+2)/2$. A base $\{ X_i \mid 1 \le (n+1)(n+2)/2 \}$ can be found by taking $X_i = \bigtriangledown_{g_1} x_i$ for $1 \le i \le n+1$, and the remaining being generators of the rotations. Thus (3.8) is equivalent to

$$\int_{S^n} X_i(K)\, \psi\, u^{\frac{2n}{n-2}} dV_{g_1} = 0 \quad \text{for} \;\; 1 \le i \le \frac{(n+1)(n+2)}{2}\,.$$

In [49], the Kazdan-Warner type condition is considered on open manifolds with nonnegative Ricci curvature outside a compact set.

4. Empty Kernel

We seek conditions to warrant empty kernel for $\mathcal{L}_g$. The study can be extended to any closed Riemannian n-manifolds $(N\,,\,h)$ with $n \ge 3$. Consider a solution $\phi \in C^\infty(N) \setminus \{0\}$ of the Schrödinger equation

$$\Delta_h\,\phi + \frac{K}{n-1}\,\phi = 0 \quad \text{in} \;\; N\,. \tag{4.1}$$

It follows that

$$\int_N |\,\nabla_n\ \phi|^2\,dV_h = \int_N \frac{K}{n-1}\,|\phi|^2\,dV_h\,. \tag{4.2}$$

Denote by λ_1 the first positive eigenvalue of the Laplacian Δ_h. For $f \in C^\infty(N)$ with $\displaystyle\int_N f\,dV_h = 0$, from the *Rayleigh quotient* we obtain

$$\int_N |\,\nabla_h\ f|^2\,dV_h \ge \lambda_1 \int_N |f|^2\,dV_h\,. \tag{4.3}$$

As $\displaystyle\int_N \phi\,dV_h$ may not vanish, we cannot apply (4.3) directly on ϕ. Observe that when $K > 0$ in N, (4.1) implies that

$$\begin{aligned} &(n-1)\int_N \frac{\Delta_h\phi}{K}\,dV_h + \int_N \phi\,dV_h = 0 \\ \Longrightarrow\quad &\int_N \phi\,dV_h = -(n-1)\int_N \frac{\langle\nabla_h K\,,\ \nabla_h\phi\rangle_h}{K^2}\,dV_h\,. \end{aligned} \tag{4.4}$$

We combine this with (4.3) to obtain the following.

Theorem 4.5. *Assume that K is a positive smooth function on N with maximal value $K_{\max}$. If*

$$\lambda_1 > \frac{K_{\max}}{n-1}\left[1 + \frac{\lambda_1\,(n-1)^2}{\mathrm{Vol}_h(N)}\left(\int_N \frac{|\,\nabla_h\,K|^2}{K^4}\,dV_h\right)\right],$$

then equation (4.1) has no non-trivial smooth solutions.

Proof. Suppose that (4.1) has a non-trivial smooth solution ϕ. We normalize ϕ so that $\displaystyle\int_N \phi^2\,dV_h = 1$. Set

$$\tau := \frac{\int_N \phi\,dV_h}{\mathrm{Vol}_h\,(N)} = -\,\frac{n-1}{\mathrm{Vol}_h\,(N)}\int_N \frac{\langle\nabla_h K\,,\ \nabla_h\phi\rangle_h}{K^2}\,dV_h \qquad [\,\text{by}\;\;(4.4)\,]\,. \tag{4.6}$$

Hence

$$\int_N (\phi - \tau)\, dV_h = \int_N \phi\, dV_h - \tau\, \mathrm{Vol}_h(N) = 0\,.$$

It follows from (4.3) that

$$\int_N |\bigtriangledown_h\, \phi|^2\, dV_h \geq \lambda_1 \int_N |\phi - \tau|^2\, dV_h\,. \tag{4.7}$$

Applying Hölder's inequality in (4.6) we obtain

$$\begin{aligned} \tau^2 &\leq \frac{(n-1)^2}{[\mathrm{Vol}_h\,(N)]^2} \left(\int_N \frac{|\bigtriangledown_h K|^2}{K^4}\, dV_h \right) \left(\int_N |\bigtriangledown_h \phi|^2\, dV_h \right) \\ &= \frac{(n-1)^2}{[\mathrm{Vol}_h\,(N)]^2} \left(\int_N \frac{|\bigtriangledown_h K|^2}{K^4}\, dV_h \right) \left(\int_N \frac{K}{n-1}\, \phi^2\, dV_h \right)\,, \end{aligned} \tag{4.8}$$

where (4.2) is also involved. Combining (4.2), (4.7) and (4.8) we have

$$\begin{aligned} &\int_N \frac{K}{n-1}\, |\phi|^2\, dV_h \\ \geq\ & \lambda_1 \int_N (\phi^2 - 2\tau\, \phi + \tau^2)\, dV_h \\ =\ & \lambda_1 \int_N \phi^2\, dV_h + \lambda_1 \left(-2\tau \int_N \phi\, dV_h + \tau^2\, \mathrm{Vol}_h\,(N) \right) \\ =\ & \lambda_1 - \lambda_1\, \tau^2\, \mathrm{Vol}_h\,(N) \\ \geq\ & \lambda_1 - \frac{\lambda_1\,(n-1)^2}{\mathrm{Vol}_h\,(N)} \left(\int_N \frac{|\bigtriangledown_h K|^2}{K^4}\, dV_h \right) \left(\int_N \frac{K}{n-1}\, \phi^2\, dV_h \right)\,. \end{aligned} \tag{4.9}$$

That is,

$$\left[1 + \frac{\lambda_1\,(n-1)^2}{\mathrm{Vol}_h\,(N)} \left(\int_N \frac{|\bigtriangledown_h K|^2}{K^4}\, dV_h \right) \right] \left(\int_N \frac{K}{n-1}\, \phi^2\, dV_h \right)\ \geq\ \lambda_1\,.$$

In case

$$\lambda_1 > \frac{K_{\max}}{n-1} \left[1 + \frac{\lambda_1\,(n-1)^2}{\mathrm{Vol}_h\,(N)} \left(\int_N \frac{|\bigtriangledown_h K|^2}{K^4}\, dV_h \right) \right]$$

(recall that $\int_N \phi^2\, dV_h = 1$), then we have a contradiction. □

In particular, if $K = \kappa$ is a positive constant and $\lambda_1 > \frac{\kappa}{n-1}$, then equation (4.1) has no non-trivial solutions. This occurs in the projective space $\mathbf{P}^n$ with the canonical metric, where $\lambda_1 = 2(n+1)$ and $K = n(n-1)$. The conclusion remains valid in a C^1-neighborhood $u \equiv 1$. Moser shows that any function on $\mathbf{P}^2$ that is positive somewhere is the scalar curvature of a conformal metric [38]. Escobar and Schoen [19] generalizes the result to $\mathbf{P}^3$. By theorem 4.5, functions K satisfying $K(-x) = K(x)$ for all $x \in S^n$ (hence can be descended to $\mathbf{P}^n$) and being sufficiently C^1-close to $n(n-1)$ are inside $\mathcal{K}\rangle$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_m < \cdots$ be the positive eigenvalues of $(N,\ h)$. By exploring the above argument further, we obtain the following "ladder-type" obstruction.

Theorem 4.10. *Assume that* $\dfrac{K}{n-1}$ *lies between two consequent eigenvalues* λ_{m-1} *and* λ_m *($m \ge 2$). That is,*

$$\lambda_{m-1} < \frac{K}{n-1} < \lambda_m \quad \text{in} \;\; N\,. \tag{4.11}$$

There exists a positive constant γ *such that if*

$$\int_N \| \bigtriangledown_h K\|_h^2 \, dV_h \;\le\; \gamma^2 \, \mathrm{Vol}_h(N) \quad \text{and} \;\; (K_{\max} - K_{\min})^2 \le \gamma^2\,, \tag{4.12}$$

then equation (4.1) does not have any non-trivial smooth solutions. In addition, γ *depends only on the eigenvalues* λ_α *and its multiplicity* $\mathcal{M}_\alpha$ *($1 \le \alpha \le m$),* $K_{\min}$ *and* $K_{\max}$.

Proof. For the eigenspace with eigenvalue λ_α (multiplicity $\mathcal{M}_\alpha$), let $\{\phi_{i\,,\alpha}\}_{i=1}^{\mathcal{M}_\alpha}$ be an orthonormal basis. By Green's identity, we have

$$\int_N (\bigtriangledown_h \phi_{i\,,\alpha}) \cdot (\bigtriangledown_h \phi_{j\,,\beta})\, dV_h = \lambda_\alpha \int_N \phi_{i\,,\alpha}\, \phi_{j\,,\beta}\, dV_h = \delta^i_j \, \delta^\alpha_\beta\,.$$

To obtain a contradiction, suppose that (4.1) has a non-trivial smooth solution ϕ, normalized in the form $\int_N \phi^2 \, dV_h = 1$. Define τ as in (4.6). Set

$$Q = \phi - \sum_{\alpha=1}^{m-1} \sum_{i=1}^{\mathcal{M}_\alpha} \left(\int_N \phi\, \phi_{i\,,\alpha}\, dV_h \right) \phi_{i\,,\alpha} - \tau\,. \tag{4.13}$$

It is direct to check that

$$\int_N Q \, dV_h = 0 \quad \text{and} \quad \int_N Q\, \phi_{i\,,\alpha}\, dV_h = 0$$

for $1 \le i \le \mathcal{M}_\alpha$ and $1 \le \alpha \le m-1$. Hence

$$\int_N |\,\bigtriangledown_h\, Q|^2\, dV_h \;\ge\; \lambda_m \int_N Q^2\, dV_h\,. \tag{4.14}$$

We have

$$\begin{aligned}
&\int_N |\,\bigtriangledown_h\, Q|^2\, dV_h \\
= \;& \int_N |\,\bigtriangledown_h\, \phi|^2\, dV_h + \sum_{\alpha=1}^{m-1} \sum_{i=1}^{\mathcal{M}_\alpha} \left(\int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right)^2 \left(\int_N |\,\bigtriangledown_h\, \phi_{i\,,\,\alpha}|^2\, dV_h \right) \\
& \quad - 2 \sum_{\alpha=1}^{m-1} \sum_{i=1}^{\mathcal{M}_\alpha} \left(\int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right) \left(\int_N (\bigtriangledown_h \phi \cdot \bigtriangledown_h \phi_{i\,,\,\alpha})\, dV_h \right) \\
& \quad + 2 \sum_{(i\,,\,\alpha) \ne (j\,,\,\beta)} \left(\int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right) \left(\int_N \phi\, \phi_{j_\beta}\, dV_h \right) \left(\int_N (\bigtriangledown_h \phi_{i\,,\,\alpha} \cdot \bigtriangledown_h \phi_{j\,,\,\beta})\, dV_h \right) \\
= \;& \int_N |\,\bigtriangledown_h\, \phi|^2\, dV_h \,-\, \sum_{\alpha=1}^{m-1} \sum_{i=1}^{\mathcal{M}_\alpha} \lambda_\alpha \left(\int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right)^2.
\end{aligned}$$

We also obtain

$$\int_N Q^2\, dV_h = \int_N \phi^2\, dV_h - \tau^2\, \mathrm{Vol}_h\,(S^n) - \sum_{\alpha=1}^{m-1} \sum_{i=1}^{\mathcal{M}_\alpha} \left(\int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right)^2 .$$

Thus

(4.15)

$$\int_N |\,\bigtriangledown_h\, \phi|^2\, dV_h + \lambda_m\, \tau^2\, \mathrm{Vol}_h\,(S^n) + \sum_{\alpha=1}^{m-1} \sum_{i=1}^{\mathcal{M}_\alpha} (\lambda_m - \lambda_\alpha) \left(\int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right)^2 \;\ge\; \lambda_m$$

(recall that $\int \phi^2\, dV_h = 0$). We seek to estimate the cross term in (4.15). Let $K_o = \left(\int_N K\, dV_h \right) / \,\mathrm{Vol}(N,\ h)$. By Green's identity, we have

$$\begin{aligned}
& \int_N \frac{K}{n-1} \phi\, \phi_{i\,,\,\alpha}\, dV_h = -\int_N (\Delta_h\, \phi)\, \phi_{i\,,\,\alpha}\, dV_h = \lambda_\alpha \int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \\
\Rightarrow \;& \int_N \frac{K_o}{n-1} \phi\, \phi_{i\,,\,\alpha}\, dV_h + \int_N \left(\frac{K - K_o}{n-1} \right) \phi\, \phi_{i\,,\,\alpha}\, dV_h = \lambda_\alpha \int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \\
\Rightarrow \;& \left(\frac{K_o}{n-1} - \lambda_\alpha \right) \int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h = \int_N \left(\frac{K_o - K}{n-1} \right) \phi\, \phi_{i\,,\,\alpha}\, dV_h \\
\Rightarrow \;& \left| \frac{K_o}{n-1} - \lambda_\alpha \right| \left| \int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right| \;\le\; \max \left| \frac{K_o - K}{n-1} \right| \left| \int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right| \\
\Rightarrow \;& \left| \frac{K_o}{n-1} - \lambda_\alpha \right| \left| \int_N \phi\, \phi_{i\,,\,\alpha}\, dV_h \right| \;\le\; \max \left| \frac{K_o - K}{n-1} \right| \left(\int_N \phi^2\, dV_h \right)^{\frac{1}{2}} \left(\int_N \phi_{i\alpha}^2\, dV_h \right)^{\frac{1}{2}} .
\end{aligned}$$

We arrive at

$$\left|\int_N \phi\,\phi_{i\,,\,\alpha}\,dV_h\right| \;\le\; \max\left|\frac{K_o - K}{n-1}\right|\left|\frac{K_o}{n-1} - \lambda_\alpha\right|^{-1}. \tag{4.16}$$

Define $\Gamma_1 := \dfrac{K_{\min}}{n-1} - \lambda_{m-1}$ and $\Gamma_2 := \lambda_m - \dfrac{K_{\max}}{n-1}$. From the conditions on K, we observe that

$$\frac{K_o}{n-1} \ne \lambda_\alpha \quad \text{for } \ \alpha = 1, \cdots,\ m-1\,, \quad \text{and } \ \Gamma_1\,,\ \Gamma_2 > 0\,.$$

It follows from (4.15) and (4.16) that

$$\begin{aligned}\int_N |\bigtriangledown_h \phi|^2\,dV_h &+ \lambda_m\,\tau^2\,\mathrm{Vol}_h\,(S^n) \\ &+ \left(\max\left|\frac{K_o - K}{n-1}\right|\right)^2\left[\sum_{i=1}^{\mathcal{M}_\alpha}\mathcal{M}_\alpha\,(\lambda_m - \lambda_\alpha)\left|\frac{K_o}{n-1} - \lambda_\alpha\right|^{-2}\right] \;\ge\; \lambda_m\,.\end{aligned} \tag{4.17}$$

With (4.2), (4.8), (4.11) and (4.16), (4.17) becomes

$$\begin{aligned}&\frac{K_{\max}}{(n-1)}\left[1 + \frac{\lambda_m\,(n-1)^2}{\lambda_{m-1}^4\mathrm{Vol}_h\,(N)}\left(\int_N |\bigtriangledown_h K|^2\,dV_h\right)\right] \\ &\quad + \left|\frac{K_o - K_{\max}}{n-1}\right|^2\left[\frac{\mathcal{M}_{m-1}(\lambda_m - \lambda_{m-1})}{\left(\frac{K_o}{n-1} - \lambda_{m-1}\right)^2} \;+\; \sum_{\alpha=1}^{m-2}\frac{\mathcal{M}_\alpha(\lambda_m - \lambda_\alpha)}{(\lambda_{m-1} - \lambda_\alpha)^2}\right] \;\ge\; \lambda_m\,.\end{aligned}$$

Together with (4.12) we obtain

$$\begin{aligned}&\frac{K_{\max}}{(n-1)}\left(1 + \frac{\lambda_m\,(n-1)^2\,\gamma^2}{\lambda_{m-1}^4}\right) \\ &\quad + \frac{\gamma^2}{(n-1)^2}\left[\frac{\mathcal{M}_{m-1}(\lambda_m - \lambda_{m-1})}{\Gamma_1^2} \;+\; \sum_{\alpha=1}^{m-2}\frac{\mathcal{M}_\alpha(\lambda_m - \lambda_\alpha)}{(\lambda_{m-1} - \lambda_\alpha)^2}\right] \;\ge\; \lambda_m\,.\end{aligned}$$

Finally, we arrive at

$$\begin{aligned}&\left[\frac{\lambda_m\,(n-1)\,\gamma^2\,K_{\max}}{\lambda_{m-1}^4}\right] \\ &\quad + \frac{\gamma^2}{(n-1)^2}\left[\frac{\mathcal{M}_{m-1}(\lambda_m - \lambda_{m-1})}{\Gamma_1^2} \;+\; \sum_{\alpha=1}^{m-2}\frac{\mathcal{M}_\alpha(\lambda_m - \lambda_\alpha)}{(\lambda_{m-1} - \lambda_\alpha)^2}\right] \;\ge\; \Gamma_2 \;> 0\,.\end{aligned} \tag{4.18}$$

Thus if γ is small enough, we have a contradiction. Moreover, γ can be estimated in terms of quantities in (4.18). □

5. Eigenvalue estimates for conformal metrics

An exquisite result of J. Hersch [23] states that

$$\frac{\lambda_1}{2} \le \frac{\omega_2}{\text{Area}\,(S^2,\ g_s)} \tag{5.1}$$

for any metric g_s on S^2. Together with the Gauss-Bonnet theorem, one obtains

$$\lambda_1 \le K^s_{\max}\,, \tag{5.2}$$

where K^s is the scalar curvature of the metric g_s . Ma and Wu generalize (5.1) to higher dimensions ($n \ge 3$) for *conformal metrics* and show that

$$\frac{\lambda_1}{n} \le \left(\frac{\omega_n}{\text{Vol}_g\,(S^n)} \right)^{\frac{2}{n}}, \tag{5.3}$$

where $g = u^{\frac{4}{n-2}}\, g_1$. Here $\omega_n = \text{Vol}_{g_1}\,(S^n)$. We assert that

$$\lambda_1 \le \frac{K_{\max}}{n-1} \tag{5.4}$$

for any conformal metric g on S^n ($n \ge 3$), with $K = \mathcal{K}(u)$.

For $u \in C^\infty(S^n) \setminus \{0\}$, consider the quotient

$$Q(u) := \frac{\int_{S^n} \left(|\bigtriangledown u|^2 + c_n\, K\, u^2 \right) dV_g}{\left(\int_{S^n} |u|^{\frac{2n}{n-2}}\, dV_g \right)^{(n-2)/n}}\,. \tag{5.5}$$

Let

$$\mu(S^n) = \inf\, \{ Q(u) \mid u \in C^\infty(S^n) \setminus \{0\} \}\,.$$

It is known that $\mu(S^n)$ is a conformal invariant and its value can be found at the standard metric g_1 with $u = 1$ [44]. That is,

$$\mu(S^n) = c_n\, n(n-1)\, \omega_n^{\frac{2}{n}}\,.$$

For the metric g, taking $u = 1$, we obtain

$$\begin{aligned} & \frac{c_n \int_{S^n} K\, dV_g}{(\int_{S^n} dV_g)^{(n-2)/n}} \ge \mu(S^n) \\ \Longrightarrow \quad & K_{\max}\, [\, \text{Vol}_g\,(S^n)\,]^{\frac{2}{n}} \ \ge\ n(n-1)\, \omega_n^{\frac{2}{n}} \\ \Longrightarrow \quad & \frac{K_{\max}}{n(n-1)} \ge \left(\frac{\omega_n}{\text{Vol}_g\,(S^n)} \right)^{\frac{2}{n}}. \end{aligned}$$

Hence we have (5.4). It can also be shown that equality in (5.4) holds if and only if g is isometric to g_1.

For lower bounds on λ_1, Lichnerowicz demonstrates that if $(N,\ h)$ is a compact Riemannian n-manifold without boundary, then

$$\mathrm{Ric}_N \ge \kappa\,(n-1) \quad \Longrightarrow \quad \lambda_1(N) \ge n\kappa\,, \tag{5.6}$$

where $\kappa > 0$ is a constant (cf. [44]). (Obata adds that equality in (5.6) holds if and only if $(N,\ h)$ is isometric to the n-sphere of constant sectional curvature κ.) In (5.6), we may take $\kappa = \dfrac{\min\ K}{n(n-1)}$ and hence

$$\lambda_1 \ge \frac{\min\ K}{n-1}\,.$$

Combining (5.4) and (5.6) we obtain the following.

Proposition 5.7. *In S^n $(n \ge 3)$, for any conformal metric g of g_1, we have*

$$\lambda_1 \le \frac{\max\ K}{n-1}\,.$$

If in addition g has positive Ricci curvature, then

$$\frac{\min\ K}{n-1} \le \lambda_1\,.$$

We consider estimating first eigenvalue under a conformal change of metric. As above, denote by $(N,\ h)$ a compact Riemannian n manifold and $\tilde{h} = u^{\frac{4}{n-2}} h$. The transformation of the Ricci curvature under conformal transformation can be written neatly in terms of $w = u^{-\frac{2}{n-2}}$. We have

$$\mathrm{Ric}_{\tilde{h}} = \frac{K}{n}\,h + \frac{n-2}{w}\left(\mathrm{Hess}_h\,(w) + \frac{\Delta_h w}{n}\,h\right)\,.$$

Because of the Hessian term, it is not effective to estimate $\mathrm{Ric}_{\tilde{h}}$. The following discussion on *floating level* method explores the metric structure of the Rayleigh quotient and requires no lower bounds on Ricci curvature.

For the first positive eigenvalue λ_1 of $(N,\ h)$, let F be an eigenfunction with $\int_N F^2\, dV_h = 1$. Consider the conformal metric $\tilde{h} = u^{\frac{4}{n-2}} h$. Define the floating level by

$$\iota := -\,\frac{\int_N F\, dV_{\tilde{h}}}{\int_N \, dV_{\tilde{h}}}\,. \tag{5.8}$$

As a result,

$$\int_N (F + \iota)\, dV_{\tilde{h}} = 0\,.$$

We observe that, in the original metric h, $\int_N F\, dV_h = 0$. Thus we have

$$\begin{aligned} \int_N (F+\iota)^2\, dV_h &= \int_N (F^2 + 2\iota\, F + \iota^2)\, dV_h \\ = \quad \int_N (F^2 + \iota^2)\, dV_h &= 1 + \iota^2\, \mathrm{Vol}_h\,(S^n)\,. \end{aligned}$$

Let $\tilde{\lambda}_1$ be the first eigenvalue of $(N,\ \tilde{h})$. Using the Rayleigh quotient, we obtain

$$\begin{aligned} \tilde{\lambda}_1 \ &\le \ \frac{\int_N |\,\nabla_{\tilde{h}}\,(F+\iota)|^2_{\tilde{h}}\, dV_{\tilde{h}}}{\int_N (F+\iota)^2\, dV_{\tilde{h}}} \\ &= \ \frac{\int_N |\,\nabla_h\, F|^2_h\, u^2\, dV_h}{\int_N (F+\iota)^2\, u^{\frac{2n}{n-2}}\, dV_h} \\ &\le \ \left[\frac{(\max\ u)^2}{(\min\ u)^{\frac{2n}{n-2}}} \right] \ \frac{\int_N |\,\nabla_h\, F|^2_h\, dV_h}{\int_N (F+\iota)^2\, dV_h} \\ &\le \ \left[\frac{(\max\ u)^2}{(\min\ u)^{\frac{2n}{n-2}}} \right] \left(\frac{\int_N |\,\nabla_h\, F|^2_h\, dV_h}{1 + \iota^2\, \mathrm{Vol}_h\,(S^n)} \right) \\ &= \ \frac{\lambda_1}{1 + \iota^2\, \mathrm{Vol}_h\,(S^n)} \ \left[\frac{(\max\ u)^2}{(\min\ u)^{\frac{2n}{n-2}}} \right]\,. \end{aligned}$$

That is,

$$\frac{\tilde{\lambda}_1}{\lambda_1} \le \frac{1}{1 + \iota^2\, \mathrm{Vol}_h\,(S^n)} \ \left[\frac{(\max\ u)^2}{(\min\ u)^{\frac{2n}{n-2}}} \right]\,, \tag{5.9}$$

and equality holds if and only if $u \equiv 1$.

The *reverse* of the above is given by

$$\frac{\lambda_1}{\tilde{\lambda}_1} \le \frac{1}{1 + \varsigma^2\,\mathrm{Vol}_{\tilde{h}}\,(S^n)} \left[\frac{[\max\ u^{-1})]^2}{[\min\ u^{-1}]^{\frac{2n}{n-2}}} \right] , \tag{5.10}$$

where the floating level ς of $(N,\ \tilde{h})$ is defined similarly. Hence we have

$$[1 + \varsigma^2\,\mathrm{Vol}_{\tilde{h}}\,(S^n)] \left[\frac{(\min\ u)^2}{(\max\ u)^{\frac{2n}{n-2}}} \right] \le \frac{\tilde{\lambda}_1}{\lambda_1} \le \frac{1}{1 + \iota^2\,\mathrm{Vol}_h\,(S^n)} \left[\frac{(\max\ u)^2}{(\min\ u)^{\frac{2n}{n-2}}} \right] . \tag{5.11}$$

Or simply as

$$\frac{(\min\ u)^2}{(\max\ u)^{\frac{2n}{n-2}}} \le \frac{\tilde{\lambda}_1}{\lambda_1} \le \frac{(\max\ u)^2}{(\min\ u)^{\frac{2n}{n-2}}} . \tag{5.12}$$

We further assume that u is *flat* on a maximum point x_M and a minimum point x_m , meaning that,

$$\Delta_h\, u(x_M) = 0 = \Delta_h\, u(x_m) , \tag{5.13}$$

where Δ_h is the Laplacian of $(N,\ h)$. Assume also that the scalar curvature of h is equal to a positive constant κ . If $K > 0$, then it follows from an analog of equation (1.2) that

$$\begin{gathered} (\max\ u)^{\frac{4}{n-2}} \le \frac{\kappa}{K_{\min}} , \qquad (\min\ u)^{\frac{4}{n-2}} \ge \frac{\kappa}{K_{\max}} \\ \Longrightarrow \frac{K_{\min}}{\kappa} \left(\frac{K_{\min}}{K_{\max}} \right)^{\frac{n-2}{2}} \le \frac{\tilde{\lambda}_1}{\lambda_1} \le \frac{K_{\max}}{\kappa} \left(\frac{K_{\max}}{K_{\min}} \right)^{\frac{n-2}{2}} . \end{gathered} \tag{5.14}$$

Theorem 5.15. *Let $(N,\ h)$ be a compact Riemannian n-manifold $(n \ge 3)$ with constant scalar curvature $\kappa > 0$ and first positive eigenvalue λ_1 . Assume that*

$$\lambda_1 > (1 + \delta^2)\, \frac{\kappa}{n-1} \tag{5.16}$$

for some positive number δ . Given positive constants a, b and c which satisfy

$$1 \le \left(\frac{b}{a} \right)^n \left[1 + \frac{(n-1)^2\, \lambda_1}{\kappa^2} \left(\frac{b}{a} \right)^{10-n} \left(\frac{c}{b} \right)^4 \right] \le 1 + \delta^2 \tag{5.17}$$

(for instance, $b/a \approx 1$ and $c \approx 0$), assume that the conformal metric $\tilde{h} = u^{\frac{4}{n-2}}\, h$ has scalar curvature K satisfying

$$a^2 \le K \le b^2 \quad \text{and} \quad \| \bigtriangledown_h K\|_h \le c^2 \quad \text{in } N\,, \tag{5.18}$$

and u satisfying the flatness condition (5.13). Then $K = \mathcal{R}(u)$ is an interior point in $\mathcal{R}\rangle$ with the C^{α} topology.

Proof. From (5.14) and (5.18) we have

$$\tilde{\lambda}_1 \ge \lambda_1 \frac{K_{\max}}{\kappa} \left(\frac{K_{\min}}{K_{\max}}\right)^{\frac{n}{2}} \ge \left(\frac{K_{\max}}{n-1}\right) \frac{(n-1)\,\lambda_1}{\kappa} \left(\frac{a}{b}\right)^n . \tag{5.19}$$

In order to apply theorem 4.5, we need

$$\frac{(n-1)\,\lambda_1}{\kappa} \left(\frac{a}{b}\right)^n > 1 + \frac{\tilde{\lambda}_1\,(n-1)^2}{\mathrm{Vol}_{\tilde{h}}(N)} \left(\int_N \frac{\| \bigtriangledown_{\tilde{h}} K\|_{\tilde{h}}^2}{K^4}\, dV_{\tilde{h}}\right) .$$

By (5.16), we have

$$\frac{(n-1)\,\lambda_1}{\kappa} \left(\frac{a}{b}\right)^n > (1+\delta^2) \left(\frac{a}{b}\right)^n .$$

Thus we only need

$$(1+\delta^2) \left(\frac{a}{b}\right)^n \ge 1 + \frac{\tilde{\lambda}_1\,(n-1)^2}{\mathrm{Vol}_{\tilde{h}}(N)} \left(\int_N \frac{\| \bigtriangledown_{\tilde{h}} K\|_{\tilde{h}}^2}{K^4}\, dV_{\tilde{h}}\right) . \tag{5.20}$$

Using (5.14) and (5.18) we obtain

$$\tilde{\lambda}_1 \le \frac{\lambda_1\, b^2}{\kappa} \left(\frac{a}{b}\right)^{n-2} .$$

It follows that

$$\begin{aligned}
& 1 + \frac{\tilde{\lambda}_1\,(n-1)^2}{\mathrm{Vol}_{\tilde{h}}(N)} \left(\int_N \frac{| \bigtriangledown_{\tilde{h}} K|^2}{K^4}\, dV_{\tilde{h}}\right) \\
\le\ & 1 + \frac{(n-1)^2\,\lambda_1}{\kappa}\, b^2 \left(\frac{a}{b}\right)^{n-2} \frac{1}{\mathrm{Vol}_{\tilde{h}}(N)} \int_N \frac{\| \bigtriangledown_{\tilde{h}} K\|_{\tilde{h}}^2}{K^4}\, dV_{\tilde{h}} \\
\le\ & 1 + \frac{(n-1)^2\,\lambda_1}{\kappa}\, b^2 \left(\frac{a}{b}\right)^{n-2} \frac{1}{a^8\, \mathrm{Vol}_{\tilde{h}}(N)} \int_N \| \bigtriangledown_h K\|_h^2\, u^{-\frac{4}{n-2}}\, dV_{\tilde{h}} \\
\le\ & 1 + \frac{(n-1)^2\,\lambda_1}{\kappa}\, b^2 \left(\frac{a}{b}\right)^{n-2} \frac{c^4}{a^8} \frac{1}{(\min\ u)^{\frac{4}{n-2}}} \\
\le\ & 1 + \frac{(n-1)^2\,\lambda_1}{\kappa}\, b^2 \left(\frac{a}{b}\right)^{n-2} \left(\frac{c^4}{a^8}\right) \left(\frac{b^2}{\kappa}\right) .
\end{aligned}$$

Hence (5.20) holds if we have

$$(5.21)\qquad (1+\delta^2)\left(\frac{a}{b}\right)^n \ge 1+\frac{(n-1)^2\,\lambda_1}{\kappa^2}\left(\frac{a}{b}\right)^{n-2}\left(\frac{b^4\,c^4}{a^8}\right).$$

After simplification, (5.21) gives

$$1+\delta^2 \ \ge \left(\frac{b}{a}\right)^n + \frac{(n-1)^2\,\lambda_1}{\kappa^2}\left(\frac{b}{a}\right)^{10}\left(\frac{c}{b}\right)^4 ,$$

which is (5.17). □

It can be seen that condition (1.11) in theorem C implies (5.17) by using $1+\delta^2/2$ and $\delta^2/2$ to bound respectively the two terms on the left hand side of (5.17). On the projective space $\mathbf{P}^n$ with the canonical metric, the first positive eigenvalue of the Laplacian is equal to $2(n+1)$. Thus we may take δ in theorem 5.15 to be any positive number small than $\sqrt{(n+2)/n}$.

Example 5.22. We demonstrate that, for certain conformal metrics of g_1 on S^n, the first positive eigenvalue can become arbitrarily small even though the scalar curvature is very close to $n(n-1)$. Consider a bean of $2m+1$ unit n-spheres in $\mathbb{R}^{n+1}$, arranging along the x_{n+1}-axis. The middle sphere has center at the origin, and the sphere in the bean touches the adjacent sphere(s) at the pole(s). By slight modifications near the points of intersection, the space, denoted by $(\mathcal{S}_m\,,\ g_{b_m})$, is conformal to $(S^n,\ g_1)$ and has scalar curvature as close to $n(n-1)$ as we like (see [28] and [29], cf. also [**?**]). Indeed, by using the Delaunay-Fowler type solutions, the deviations of the scalar curvature from $n(n-1)$ occurs only near the points of intersection and the conformal deformation can be made symmetric. It follows from the arrangement that $\int_{\mathcal{S}_m} x_{n+1}\,dV_{g_{b_m}} = 0$. In addition,

$$\int_{\mathcal{S}_m} x_{n+1}^2\,dV_{g_{b_m}} \ \ge\ [1^2+3^2+\cdots+(2m-1)^2]\,\omega_n \ =\ \frac{m(4m^2-1)}{3}\,\omega_n\,.$$

On the other hand

$$\begin{aligned}\int_{\mathcal{S}_m} \|\bigtriangledown x_{n+1}\|^2_{g_{b_m}}\,dV_{g_{b_m}} \ &\approx\ \sum_{k=-m}^{m}\int_{S^n}\|\bigtriangledown_1\ (x_{n+1}+2k)\|_1^2\,dV_1\\ &=\ (2m+1)\int_{S^n}\|\bigtriangledown x_{n+1}\|_1^2\,dV_{g_1}\,.\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1(\mathcal{S}_m) \le \frac{\int_{\mathcal{S}} \| \bigtriangledown x_{n+1}\|^2_{g_{b_m}}\, dV_{g_{b_m}}}{\int_{\mathcal{S}} x^2_{n+1}\, dV_{g_{b_m}}} & \approx \frac{1}{m(2m-1)} \frac{\int_{S^n} \| \bigtriangledown_1 x_{n+1}\|^2_1\, dV_{g_1}}{\int_{S^n} dV_{g_1}} \\
& \le \frac{1}{m(2m-1)} \frac{\int_{S^n} \| \bigtriangledown_1 x_{n+1}\|^2_1\, dV_{g_1}}{\int_{S^n} x^2_{n+1}\, dV_{g_1}} \\
& \le \frac{n}{m(2m-1)}\,.
\end{aligned}$$

Although K can be made as close to $n\,(n-1)$ as we like, $\lambda_1(\mathcal{S}_m)$ can become very small when m is large. As a comparison, a result of Li-Yau [30] (improved by Yang and Zhong in [50]) states that if $\text{Ric}_h \ge 0$, then $\lambda_1 \ge \pi^2/d^2$, where d is the diameter of $(N\,,\ h)$.

6. Constructing solutions

Lemma 6.1. *Given a non-trivial first eigenfunction ϕ of $(S^n,\ g_1)$, let $u \in C^\infty_+(S^n)$ be such that*

$$(6.2)\qquad (\bigtriangledown_1 \ln u)\cdot \bigtriangledown_1 \phi = -\frac{2}{n-2}\,\phi\, \| \bigtriangledown_1 \ln u\|^2 \quad \text{in } S^n\,.$$

Denote by K the scalar curvature of the conformal metric $g = u^{\frac{4}{n-2}}\, g_1$. Then $\Phi = u^{\frac{4}{n-2}}\,\phi$ is a (non-trivial) solution of the equation

$$(6.3)\qquad \Delta_g \Phi + \frac{K}{n-1}\,\Phi = 0 \quad \text{in } S^n.$$

In (6.2), and throughout this section, the dot product is respect to the standard spherical metric g_1.

Proof. Consider the function $\psi = u^{\frac{n+2}{n-2}}\,\phi$. We have

$$\begin{aligned}
\bigtriangledown_1 \left(u^{\frac{n+2}{n-2}}\,\phi\right) &= (\bigtriangledown_1 \phi)\, u^{\frac{n+2}{n-2}} + \left(\frac{n+2}{n-2}\right) u^{\frac{4}{n-2}} \phi \bigtriangledown_1 u\,, \\
\Delta_{g_1} \psi &= \Delta_{g_1} \left(u^{\frac{n+2}{n-2}}\,\phi\right) = u^{\frac{n+2}{n-2}}\, \Delta_{g_1} \phi + \left(\frac{n+2}{n-2}\right) \frac{\Delta_{g_1} u}{u}\, u^{\frac{n+2}{n-2}}\,\phi \\
&\quad + 2\left(\frac{n+2}{n-2}\right) u^{\frac{4}{n-2}} \bigtriangledown_1 \phi \cdot \bigtriangledown_1 u
\end{aligned}$$

$$
\begin{aligned}
& + \left(\frac{n+2}{n-2}\right)\left(\frac{4}{n-2}\right) u^{\frac{4}{n-2}-1}\phi \,\| \bigtriangledown_1 u\|^2 \\
= \;& -n\psi + \left(\frac{n+2}{n-2}\right) \frac{\Delta_{g_1} u}{u}\psi \\
& + 2\left(\frac{n+2}{n-2}\right) u^{\frac{4}{n-2}-1} \left(u \bigtriangledown_1 \phi + \frac{2}{n-2}\phi \bigtriangledown_1 u\right) \cdot \bigtriangledown_1 u
\end{aligned}
$$

$$
\Longrightarrow \quad \Delta_{g_1} \psi \;+\; \left[n - \left(\frac{n+2}{n-2}\right) \frac{\Delta_{g_1} u}{u}\right] \psi = 2\left(\frac{n+2}{n-2}\right) u^{\frac{2}{n-2}} \bigtriangledown_1 \left(u^{\frac{2}{n-2}} \phi\right) \cdot \bigtriangledown_1 u \,.
$$

Thus if

$$
\bigtriangledown_1 \left(u^{\frac{2}{n-2}} \phi \right) \cdot \bigtriangledown_1 u = 0 \quad \text{in } \; S^n \,, \tag{6.4}
$$

then

$$
\Delta_{g_1} \psi + \left[n - \left(\frac{n+2}{n-2}\right) \frac{\Delta_{g_1} u}{u}\right] \psi = 0 \,.
$$

It follows from (2.3) and lemma 2.4 that $\Phi = \psi/u = u^{\frac{4}{n-2}} \phi$ satisfies (6.3). As $u > 0$ in S^n, (6.4) can be restated as

$$
\bigtriangledown_1 \left(u^{\frac{2}{n-2}} \phi \right) \cdot \bigtriangledown_1 \left(u^{\frac{2}{n-2}} \right) = 0 \quad \text{in } \; S^n \,, \tag{6.5}
$$

Let $U := u^{\frac{2}{n-2}}$. Rewrite (6.5) as

$$
\phi \| \bigtriangledown_1 U\|^2 = -U \left(\bigtriangledown_1 U \cdot \bigtriangledown_1 \phi \right) \quad \Longleftrightarrow \quad \phi \| \bigtriangledown_1 \ln U\|^2 = -(\bigtriangledown_1 \ln U)\cdot(\bigtriangledown_1 \phi)\,. \tag{6.6}
$$

The last condition in (6.6) is equivalent to (6.2). □

Thus we seek a smooth function $\chi := \ln U$ such that

$$
\phi \,\| \bigtriangledown_1 \chi \,\|^2 = - \bigtriangledown_1 \chi \cdot \bigtriangledown_1 \phi \,. \tag{6.7}
$$

Let $S^n \subset \mathbb{R}^{n+1}$ be given by

$$
x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\,.
$$

It is known that x_i , $1 \le i \le n+1$, are first eigenfunctions of $(S^n,\, g_1)$. We take $\phi = x_{n+1}$. It follows that $\bigtriangledown_1 \phi = \bigtriangledown_1 x_{n+1} = \sqrt{1 - x_{n+1}^2}$.

As g_1 is the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$, we have

$$
\bigtriangledown_1 \chi \cdot \bigtriangledown_1 x_{n+1} = \| \bigtriangledown_1 \chi \| \sqrt{1 - x_{n+1}^2} \, \cos\theta \,,
$$

where θ is the angle between them, provided that the vectors are non-zero. Thus (6.7) can be rewritten as

$$x_{n+1} \| \bigtriangledown_1 \chi \| = -\sqrt{1 - x_{n+1}^2} \, \cos\theta \,, \qquad \text{provided } \; \| \bigtriangledown_1 \chi \| \neq 0 \,. \tag{6.8}$$

Observe that $\bigtriangledown_1 x_{n+1}$ "points upward" .

For $x \in S^n$, we set

$$\chi(x) = \chi(x_1, ..., \, x_{n+1}) \; = \begin{cases} 1 & \text{for} \quad \frac{3}{4} \le x_{n+1} \le 1 \,, \\ 2 & \text{for} \quad -1 \le x_{n+1} \le \frac{1}{4} \,. \end{cases}$$

The $(n-1)$-spheres S_ρ are defined by

$$S_\rho : \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1 - \rho^2 \quad \text{for} \quad -1 < \rho < 1 \,.$$

Consider $x \in S^n$ with $\dfrac{1}{4} \le x_{n+1} \le \dfrac{3}{4}$. $\bigtriangledown_1 x_{n+1}$ is perpendicular to the $(n-1)$-spheres $S_{\frac{1}{4}}$ and $S_{\frac{3}{4}}$. In order for χ to satisfy (6.7) , we make $\bigtriangledown_1 \chi(x)$ almost perpendicular to $\bigtriangledown_1 x_{n+1}$ when x is close to $S_{\frac{3}{4}}$ and $S_{\frac{1}{4}}$. In a uniform way on S_ρ with ρ being closed to 1/4 or 3/4 , this is possible only when $n-1$ is odd (that is, n is even) because of the following.

Theorem (Poincaré-Hopf). *The index of a smooth vector field with finitely many zeros on a compact, oriented manifold N is the same as the Euler characteristic of N .*

If the Euler characteristic of N is zero, one can construct a nowhere vanishing vector field on N (see chapter 11 in the book by Spivak [47]). Thus it is possible to define a non-vanishing vector field on S^{n-1} if and only if $n-1$ is odd. The change of χ from $S_{\frac{3}{4}}$ to $S_{\frac{1}{4}}$ is achieved first by making $\bigtriangledown_1 \chi$ almost tangential to S_ρ when $\dfrac{3}{4} \succeq \rho$ (so that $\cos\theta \approx 0$) . $\bigtriangledown_1 \chi$ then gradually picks up component in the $\bigtriangledown_1 x_{n-1}$ direction (the twisted part), and again becomes almost tangential to $S_{\rho'}$ when $\rho' \succeq \dfrac{1}{4}$.

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