Perspectives on bubble analysis for critically nonlinear PDEs

Dedicated to Professor Man Keung Siu.

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Jan. 2005

Abstract

This article serves as an introduction to the bubbling phenomenon in nonlinear partial differential equations (PDEs). Rooted in the Sobolev embedding theorem, the bounderline noncompactness brings in the fundamental behavior of bubble formation, manifested in the Yamabe problem and the Nirenberg/Kazdan-Warner problem. These problems have to do with conformal transformations. As far as possible, we present the systematic development of the blow-up analysis, and strive to illuminate the principles by constructing examples and providing intuition.

KEY WORDS: blow-up; conformal deformation; scalar curvature; critical Sobolev exponent; nonlinear elliptic differential equation.

2000 AMS MS CLASSIFICATIONS: Primary 35J60; Secondary 53C21.

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Contents.

Introduction.

In the study of nonlinear PDEs, people gradually unearth an interesting feature: solutions can only become large in a specific way that mimics bubbling. The blow-up phenomenon is common to a variety of equations invariant under conformal transformations. A main purpose of this article is to provide an intuitive and rigorous introduction to the study of blow-ups in an important class of PDEs involving the critical Sobolev exponent.

The main object of our discussion is the Nirenberg/Kazdan-Warner equation

$$\Delta_{g_1} u - c_n\, n(n-1)\, u + c_n\, K u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \;\; S^n. \tag{1}$$

Here $n \ge 3$ and $c_n = (n-2)/[4\,(n-1)]$. As a kind of 'mirror image' of the Yamabe equation (cf. § 3), the above equation arises from the famous question on prescribing the scalar curvature function K in the conformal class of the standard metric g_1 on the unit sphere S^n. From the progressive study on the problem (cf. the survey style articles [36] [**?**]), it becomes clear that blow-up is endogenous to the equation. Furthermore, the equation affords different types of blow-ups, all naturally formed, ranging from simple blow-up to towering blow-up and spiky blow-up. See the examples in § 7 e, f & g.

As a first step to understand the intricate bubbling process for equation (1), R. Schoen introduces the concepts of isolated blow-up and simple blow-up (commonly referred as simple isolated blow-up) [59]. In a series of exquisite and thorough papers (cf. [48] [49]), Y.-Y. Li develops the notions and applies them to obtain existence and compactness results. A main issue of this article is to introduce some of the essential tools used in analyzing simple blow-ups.

The study on isolated and simple blow-ups are furthered by C.-C. Chen and C.-S. Lin, and others (cf. [13] [14] [53]). Among other things, Chen and Lin develop the powerful moving plane methods for some *nonconstant* K. In the last section, we briefly highlight the working principles of the method.

The largely untouched spiky blow-up, with new constructions by Taliaferro [66] [67] [68] and the author [45] (independently), remains a topic for future endeavor. The main results in the article can be found in the major works cited above and in the references. Our objective is to provide, as far as possible, selective and clear perspectives on this vast subject.

Conventions. Throughout this article, $n \ge 3$ is an integer. We observe the practice of using C, possibly with sub-indices, to denote various positive constants, which may be rendered differently from line to line according to the contents.

1. Critical Sobolev exponent.

§ 1 a. *The Sobolev embedding theorem.* Typically there are two ways to measure the regularity of a function, namely, the L^p norm, which is global, and the Hölder norms $C^{k,\alpha}$, which is more local in nature. The combination of the two is a creation of Sobolov: for a positive integer k and a positive real number p, the Sobolev space L^p_k contains information on the L^p norms of the derivatives of the function, up to order k. Intuitively, the higher the values of p and k, the more regular the function is.

Consider different values of p and k in L^p_k. One can think of increasing the value of p, and balancing this act by giving up some regularity on the derivative. This is made precise by the *Sobolev embedding theorem*, which states that the inclusion $L^p_k(\mathcal{M}) \hookrightarrow L^q_\ell(\mathcal{M})$ is *continuous* if

$$0 < \frac{1}{p} - \frac{k-\ell}{n} \leq \frac{1}{q}\,. \tag{1.1}$$

Moreover, if the second inequality in (1.1) *is strict, then the embedding is compact.* Here $\mathcal{M}$ is a compact n-dimensional manifold without boundary. See, for examples, [5] and [33].

When $n \geq 3$, $p = 2$, $k = 1$ and $\ell = 0$, the *first* incident for the lost of compactness is when

$$\frac{1}{2} - \frac{1}{n} = \frac{1}{q} \quad \Longrightarrow \quad q = 2^* := \frac{2n}{n-2}\,.$$

2^* is known as the critical Sobolev exponent. Below the critical exponent, the inclusion is more *rigid* (i.e., compact). Once above 2^*, the *flexibility* (noncompactness) shows up. We expect that *both* rigidity and flexibility are manifested in the delicate critical case. (Indeed, we see in § 3 a that the flexibility is governed by the wonderful Kazdan-Warner formula (3.2), while the rigidity is expressed in the bubbling process, cf. § 2 a & b.)

§ 1 b. *A noncompact sequence.* The noncompactness of the inclusion

$$L_1^2(\mathcal{M}) \hookrightarrow L^{2^*}(\mathcal{M})$$

can be illustrated by the following consideration. For simplicity sake we present the discussion in $\mathbb{R}^n$. Similar argument can be brought to any compact manifold by a local chart and a partition of unity. The starting point is the family of functions

$$(1.2) \qquad U_\lambda(y) := \left(\frac{\lambda}{\lambda^2 + |y|^2} \right)^{\frac{n-2}{2}} = \frac{1}{\lambda^{\frac{n-2}{2}}} \left(\frac{1}{1 + |\,\lambda^{-1} y|^2} \right)^{\frac{n-2}{2}} \qquad \text{for } \; y \in \mathbb{R}^n \,,$$

where λ is a (fixed) positive number (eventually $\lambda \to 0^+$). Observe that

$$(1.3) \qquad \lambda^{\frac{n-2}{2}} U_\lambda(\lambda\, y) = U_1(y)\,.$$

Direct calculation shows that

$$(1.4) \qquad \Delta\, U_\lambda + n(n-2)\, U_\lambda^{\frac{n+2}{n-2}} = 0 \quad \text{ in } \; \mathbb{R}^n,$$

and

$$\int_{\mathbb{R}^n} |\,\nabla\; U_\lambda|^2\, dy = n(n-2) \int_{\mathbb{R}^n} U_\lambda^{\frac{2n}{n-2}}\, dy\,.$$

Using (1.3) we find that

$$(1.5) \quad \begin{aligned} \int_{\mathbb{R}^n} U_\lambda^{2^*}(y)\, dy \; &= \; \int_{\mathbb{R}^n} U_\lambda^{2^*}(\lambda\, \tilde y)\, \lambda^n\, d\tilde y = \int_{\mathbb{R}^n} \left[\lambda^{\frac{n-2}{2}} U_\lambda(\lambda\, \tilde y) \right]^{\frac{2n}{n-2}}\, d\tilde y \\ &= \; \int_{\mathbb{R}^n} U_1^{2^*}(\tilde y)\, d\tilde y \qquad\qquad (\text{where } \; y = \lambda\, \tilde y)\,. \end{aligned}$$

In particular, the last term is independent on λ. Likewise,

$$\int_{\mathbb{R}^n} U_\lambda^q\, dy = \lambda^{\frac{n-2}{2}\,(2^* - q)} \int_{\mathbb{R}^n} U_1^q\, dy \to \begin{cases} \;0 \;\text{ as } \; \lambda \to 0^+ & \text{for } \; 0 < q < 2^*\,, \\ \;\infty \; \text{ as } \; \lambda \to 0^+ & \text{for } \; q > 2^*\,. \end{cases}$$

Thus $\{U_\lambda\}$ is bounded in $L_1^2(\mathbb{R}^n)$ for $n \ge 3$. On the other hand,

$$\lim_{\lambda \to 0^+} U_\lambda(y) \; = \left(\lim_{\lambda \to 0^+} \frac{\lambda}{\lambda^2 + |y|^2} \right)^{\frac{n-2}{2}} = \begin{cases} \;0 \; \text{ for } & y \ne 0\,, \\ \;\infty \; \text{ for } & y = 0\,. \end{cases}$$

It follows from (1.5) that $\{U_\lambda\}$ does *not* have a convergent subsequence in $L^{2^*}(\mathbb{R}^n)$.

We remark that (1.3) defines the transformation $u(y) \mapsto \lambda^{\frac{n-2}{2}} u(\lambda y)$. Thus the source of noncompactness is the rescaling λy, which is a *conformal transformation.*

§ 1 c. *Proportion and projection.* The family of functions $\{U_\lambda\}$ provides a simple, yet important, example of the *concentration process* - it can be seen that

$$\int_{\mathbb{R}^n} U_\lambda^{\frac{2n}{n-2}}\, dy \quad \text{is independent on } \lambda \quad \text{(by (1.5))},$$

$$\int_{\mathbb{R}^n \setminus B_o(\rho)} U_\lambda^{\frac{2n}{n-2}}\, dy \le C \left(\frac{\lambda}{\rho}\right)^n \longrightarrow 0 \quad \text{as} \quad \lambda \to 0^+.$$

(We use $B_o(\rho)$ to denote the open ball in $\mathbb{R}^n$ with center at the origin and radius $\rho > 0$. Likewise, $S_o(\rho)$ represents the sphere in $\mathbb{R}^n$ with center at the origin and radius ρ.)

The family of functions $\{U_\lambda\}$ is beautifully formed. Here we observe the *proportionality* of U_λ :

$$\begin{aligned} M := \max_{\mathbb{R}^n} U_\lambda &= \frac{1}{\lambda^{\frac{n-2}{2}}}, \\ U_\lambda\,\big|_{|y|=1} &\approx \lambda^{\frac{n-2}{2}} (\approx 0), \quad \text{and} \\ U_\lambda\,\big|_{|y|=\sqrt{\lambda}} &\approx 1 \quad \text{for } \lambda \approx 0. \end{aligned}$$

It follows that, for $y \ne 0$,

$$M \cdot U_\lambda(y) = \left(\frac{1}{\lambda^2 + |y|^2}\right)^{\frac{n-2}{2}} \to \frac{1}{|y|^{n-2}} \quad \text{as } \lambda \to 0^+. \tag{1.6}$$

The last expression in (1.6) is recognized as a Green's function for the Laplacian.

Proposition 1.7. *For $\lambda > 0$, the metric $4\,U_\lambda^{\frac{4}{n-2}}\, dy^2$ on $\mathbb{R}^n$ is isometric to the standard spherical metric g_1 on $S^n \setminus \{N\}$. Here dy^2 is the Euclidean metric and N the north pole on S^n.*

Proof. Denote by $(y_1, \cdots, y_n)$ the Cartesian coordinates. Consider the stereographic projection $\mathcal{P} : S^n \setminus \{N\} \to \mathbb{R}^n$. That is, for $x = (x_1, \cdots, x_{n+1}) \in S^n \setminus \{N\}$,

$$y = \mathcal{P}(x), \qquad \text{where} \qquad y_i = \frac{x_i}{1 - x_{n+1}}, \qquad 1 \le i \le n.$$

Conversely, $x_i = \dfrac{2\,y_i}{1+r^2}\,,\quad 1 \le i \le n\,,\quad x_{n+1} = \dfrac{r^2-1}{r^2+1}\,,\quad$ where $r = |y|\,.$

In can be seen that $\mathcal{P}$ is a conformal map with respect to the standard metrics on S^n and $\mathbb{R}^n$. The conformal factor can be discerned by using simple trigonometry:

$$d\theta^2 = (1 - x_{n+1}^2)^{-1}\, dx_{n+1}^2 = \frac{(1+r^2)^2}{4r^2} \cdot \left[\frac{4r}{(1+r^2)^2}\right]^2 dr^2 = \left[\frac{4}{(1+r^2)^2}\right] dr^2\,.$$

Here θ is the angle to the x_{n+1} axis. It follows that the map $\mathcal{P}$ is an isometry from $(S^n \setminus \{N\}\,,\ g_1)$ to $\left(\mathbb{R}^n,\ 4\,U_1^{\frac{4}{n-2}}\, dy^2\right)$.

The proposition is then obtained by rescaling the stereographic projection. For a positive number c, let $\Phi_c : S^n \setminus \{N\} \to \mathbb{R}^n$ be defined by

$$\Phi_c(x) = c \cdot \mathcal{P}(x)\,.$$

Being composition of conformal maps, Φ_c is a conformal map. Indeed,

$$y = c\,\mathcal{P}(x) \implies y_i = \frac{c\,x_i}{1 - x_{n+1}} \implies x_{n+1} = \frac{r^2 - c^2}{r^2 + c^2}\,.$$

As above, we compute the conformal factor:

$$d\theta^2 = \frac{(r^2+c^2)^2}{4c^2\,r^2} \cdot \left[\frac{4c^2 r}{(r^2+c^2)^2}\right]^2 dr^2 = 4\left[\frac{c}{c^2+r^2}\right]^2 dr^2\,. \tag{1.8}$$

Take $c = \lambda$. (1.8) shows that if $\mathbb{R}^n$ is equipped with the metric $4\,U_\lambda^{\frac{4}{n-2}}\, dy^2$, then it is isometric to $(S^n \setminus \{N\}\,,\ g_1)$ via Φ_λ. □

It can be seen that the stereographic projection $\mathcal{P}$ extends as a map from S^n to $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. A result tracing back to Liouville states that, for $n \ge 3$, the conformal transformation group of $\hat{\mathbb{R}}^n$ is generated by translations, rotations, scalings and inversions [62].

2. Borderline noncompactness and the Yamabe equation.

§ 2 a. *The Yamabe equation and the subcritical method.* The borderline noncompactness finds its expression in the Yamabe equation. In the 1960s, it is proposed to find metrics of constant curvature by a min-max method as a way to tackle the Poincare conjecture. The first step is to minimize the total scalar curvature functional in the conformal class. Such minimizers has constant scalar curvature. Yamabe asserts that, for any compact Riemannian manifold $(\mathcal{M},\, g)$ with scalar curvature R_g, there is a conformal metric of g whose scalar curvature is constant. This is equivalent to finding a positive smooth solution u of (his name sake) equation

$$\Delta_g\, u - c_n R_g u + \lambda\, u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathcal{M}. \tag{2.1}$$

Here λ is a constant and Δ_g the Laplacian for $(\mathcal{M},\, g)$. The scalar curvature of the conformal metric $u^{\frac{4}{n-2}}\, g$ is given by $\lambda\, c_n^{-1}$.

Yamabe introduces the subcritical equations to overcome the hurdle posed by the borderline noncompactness of the Sobolev embedding. For a number $s \in (2,\ 2^*]$, consider the quotient

$$Q_s(f) := \frac{\int_{\mathcal{M}} [\, |\bigtriangledown f|^2 + c_n\, R_g\, f^2\,]\; dV_g}{\left(\int_{\mathcal{M}} |f|^s\, dV_g\right)^{2/s}}\,, \tag{2.2}$$

where $f \in L_1^2(\mathcal{M}) \setminus \{0\}$. Let

$$\lambda_s = \inf \left\{ Q_s(u) \mid f \in L_1^2(\mathcal{M}) \setminus \{0\} \right\}.$$

When $s = 2^*$, the number is simply denoted by $\lambda(\mathcal{M})$. A nice property about $\lambda(\mathcal{M})$ is that it depends only on the conformal class of g [62].

By using a direct minimization procedure, it can be shown that for $2 < s < 2^*$, there exists a smooth positive function u_s such that its L^s-norm is equal to one, $Q_s(u_s) = \lambda_s$ and u_s satisfies the equation

$$\Delta_g\, u_s - c_n\, R_g\, u_s + \lambda_s\, u_s^{s-1} = 0 \quad \text{in } \mathcal{M}\,. \tag{2.3}$$

(Cf. [39].) The direct method does not work when $s = 2^*$ because the Sobolev embedding $L^2_1(\mathcal{M}) \hookrightarrow L^{2^*}(\mathcal{M})$ is not compact.

Once $\{u_s\}$ is uniformly bounded from above, we can apply established results in PDEs to extract a convergent subsequence, thus regaining compactness. The whole new phenomenon of blow-up takes life when $\{u_s\}$ is *not* uniformly bounded from above. This is possible only when $\lambda(\mathcal{M}) > 0$ [39]. Cf. also [64].

As the blow-up process in the Yamabe equation serves as a foundation for later discussion, we present the essential bubbling construction here, minus some technical details, which can be found in standard texts like [62].

Suppose that the sequence $\{u_s\}$ is not uniformly bounded from above. After extracting a subsequence $\{u_{s_i}\}$, we may assume that

$$M_i := u_{s_i}(\mathbf{x}_i) = \max \ u_{s_i} \to \infty \,, \qquad \mathbf{x}_i \to \mathbf{x}_o \quad \text{as} \quad i \to \infty.$$

Introducing a normal coordinate system centered at $\mathbf{x}_o$, let the coordinates of $\mathbf{x}_i$ be $\mathbf{y}_i$, $i = 1, 2, \cdots$. The idea here is to do the opposite of (1.2) (cf. (1.3)). Consider the rescaled and normalized function

$$v_i(y) := \lambda_i^{\frac{n-2}{2}} u_i(\mathbf{y}_i + \lambda_i \, y) \,, \qquad \text{where} \ \ \lambda_i = M_i^{-\frac{2}{n-2}}.$$

Standard results [31] imply that a subsequence of $\{v_i\}$ converges in $C^2_{\text{loc}}(\mathbb{R}^n)$ to a positive function v which satisfies that equation

$$\text{(2.4)} \qquad \Delta \, v + \lambda(\mathcal{M}) \, v^{\frac{n+2}{n-2}} = 0 \quad \text{in} \ \ \mathbb{R}^n, \qquad v(0) = \max v = 1 \qquad (\lambda(\mathcal{M}) > 0) \,.$$

§ 2 b. *Rigidity.* The solutions of (2.4) is classified by a Liouville type theorem obtained by Gidas, Ni and Nirenberg [29] [30], namely, $v = U_1$. Intuitively, the blow-up in the Yamabe equation 'sucks' up the whole manifold and forms a bubble.

§ 2 c. *The conformal invariant* $\lambda(\mathcal{M})$. A key feature of the Yamabe equation is that we can determine precisely when the sequence $\{u_s\}$ blows up. This is done with the help of $\lambda(\mathcal{M})$. Let us introduce the operator $\mathcal{L}_g := \Delta_g - c_n R_g$, known as the conformal Laplacian. Using integration by parts and density results, (2.3) can be rewritten as

$$\lambda(\mathcal{M}) = \inf_{u \in C^2_+(\mathcal{M})} \frac{-\int_{\mathcal{M}} (\mathcal{L}_g u)\, u \, dV_g}{\left(\int_{\mathcal{M}} u^{\frac{2n}{n-2}} \, dV_g\right)^{(n-2)/n}}. \tag{2.5}$$

It is shown that if blow-up occurs in $\{u_s\}$, then

$$\lambda(\mathcal{M}) = \lambda(S^n) = \frac{n(n-2)}{4} (\mathrm{Vol}\, S^n)^{\frac{2}{n}}.$$

Thus if $\lambda(\mathcal{M}) < \lambda(S^n)$, then there is no blow-up for $\{u_s\}$ and the Yamabe problem has a solution. The key (shown by Schoen) is that *blow-up for* $\{u_s\}$ *occurs only in* S^n, that is $\lambda(\mathcal{M}) = \lambda(S^n)$ implies that $\mathcal{M}$ is conformally equivalent to S^n. The bound $\lambda(\mathcal{M}) \le \lambda(S^n)$ suggests that there is at most one concentration point.

Thus blow-up is rare in the Yamabe problem, and when it happens, it gives no hindrance for solving the problem.

§ 2 d. *Compactness of the solution set.* Consider the collection of positive smooth solutions of the Yamabe equation (i.e., equation (2.1) with $\lambda = \lambda(\mathcal{M})$). One expects blow-up is impossible except in the case of S^n. That is, the solutions should be uniformly bounded from above except when $\mathcal{M}$ is conformally equivalent to S^n. This is shown to be the case for conformally flat manifolds (other than S^n) by Schoen [60]. For general compact manifolds, low dimension cases are considered by Druet [24], Li and Zhang [50] [51] [52], Marques [56] and others. To our knowledge, the whole issue is not completely settled in high dimensions.

§ 2 e. *Supremum of* $\lambda(\mathcal{M})$. When we take the supremum of $\lambda(\mathcal{M})$ among all the metrics on $\mathcal{M}$, we arrive at an invariant, denoted by $\sigma(\mathcal{M})$, of the manifold itself. If $\sigma(\mathcal{M}) > 0$, then $\mathcal{M}$ admits a metric of constant positive scalar curvature. We know that the n-torus T^n ($n \ge 3$) does not admit a metric of positive scalar curvature (see, for example, [38]). So $\sigma(T^n) = 0$. We also know that $\sigma(S^{n-1} \times S^1) = \sigma(S^n) = \lambda(S^n)$ for $n \ge 3$. Dubbed as Schoen's σ-invariant, it is extraordinarily difficult to determine $\sigma(\mathcal{M})$ for general $\mathcal{M}$. We refer to the recent breakthrough by Bray and Neves [9].

§ 2 f. *Sharp Sobolev inequality.* Let h be a smooth function on $\mathcal{M}$. The equation

$$\Delta_g u + h u - u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \ \ \mathcal{M} \tag{2.6}$$

is associated with the sharp Sobolev inequality (cf [23]). Unbounded sequences of positive solutions of the above (which can be called the para-Yamabe equation) are studied in [23], [25] and [65], among others. Roughly speaking, they can be approximated by a superimposition of a base solution and bubble(s).

3. Nirenberg/Kazdan-Warner problem.

Given a compact manifold $\mathcal{M}$ (without boundary) and *any* Riemannian metric g on $\mathcal{M}$ with *nonconstant* scalar curvature K_g, the Yamabe problem asks for *a* conformal metric of g with *constant* scalar curvature.

The above description has a nice 'mirror image', namely:

(I) *Given a fixed metric g on $\mathcal{M}$ with* **constant** *scalar curvature, what kind of* **nonconstant** *functions K can be the scalar curvature of* some *conformal metric of g?*

The most interesting manifold for the 'mirror' problem is arguably S^n equipped with the canonical metric g_1, which (as a contrast) is one of the most uninteresting cases in the Yamabe problem.

(I) is recognized as a primitive version of the famed Nirenberg/Kazdan-Warner problem, which can also be paraphrased as follows:

Find a set of (simple?) criteria which can determine any given function in $C^1(S^n)$ is the scalar curvature function of a conformal metric of g_1 or not.

For $n = 2$, we express the conformal metric as $e^{2f} g_1$. The problem at hand is to determine which $K \in C^1(S^n)$ can afford a solution $f \in C^2(S^2)$ of the equation

$$\Delta_{g_1} f + K e^{2f} - 1 = 0 \quad \text{in } S^2.$$

For $n \geq 3$, if we write the conformal metric as $u^{\frac{4}{n-2}} g_1$ (here $u > 0$), the equation becomes

$$\Delta_{g_1} u - c_n\, n(n-1)\, u + c_n K\, u^{\frac{n+2}{n-2}} = 0 \quad \text{in } S^n. \tag{3.1}$$

(Refer to [5] [40].) In this article, we are mainly interested in $n \geq 3$.

In the Yamabe problem, the basic feature is that a solution exists if there is no blow-up. As a 'mirror image', one may put forward that, in order to solve equation (3.1), blow-up should be an integrated part of the consideration.

Equation (3.1) actually makes sense for $K \in C^o(S^n)$. However, we soon find out that the gradient of K plays an essential role in the problem. (With all this, cf. also [69].)

§ 3 a. *Positivity and a global balance formula.* One quickly observes that in order for (3.1) to process a positive solution, K has to be positive somewhere. Indeed, it follows from (3.1) that

$$\int_{S^n} K\, u^{\frac{2n}{n-2}}\, dV_{g_1} = \frac{1}{c_n} \int_{S^n} | \bigtriangledown u|^2\, dV_{g_1} + n(n-1) \int_{S^n} u^2\, dV_{g_1} > 0\,.$$

A deep relation is revealed when we differentiate K with respect to a *conformal Killing vector field* X – one that generates a family of conformal transformations. In this way we obtain the renowned Kazdan-Warner formula [8]

$$\int_{S^n} X(K)\, u^{\frac{2n}{n-2}}\, dV_{g_1} = 0\,, \qquad \text{where } X(K) = X \cdot \bigtriangledown K\,. \tag{3.2}$$

The collection of all conformal Killing vector field on S^n can be regarded as a linear space of dimension $(n+1)(n+2)/2$, with a basis formed by the generators of the dilations ($n+1$ dimension), denoted by $X_1, \cdots, X_{n+1}$, and of the rotations ($n(n+1)/2$ dimension), denoted by $X_{n+2}, \cdots, X_{\frac{(n+1)(n+2)}{2}}$. Refer to [32].

Simple and elegant, the Kazdan-Warner formula encapsulates a central character of the equation, namely, the averaging of the interaction between $\bigtriangledown K$ and u.

§ 3 b. *Pohozaev identity.* Applying the stereographic projection as in the proof of Proposition 1.7, we obtain

$$\Delta v + c_n \tilde{K} v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{3.3}$$

where

$$\tilde{K}(y) := K(\mathcal{P}^{-1}(y)) \qquad \text{and} \qquad v(y) = u\,(\mathcal{P}^{-1}(y)) \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}} . \tag{3.4}$$

Together with $X = \bigtriangledown_{g_1} x_{n+1}$ in identity (3.2) we have

$$\int_{\mathbb{R}^n} r \frac{\partial \tilde{K}}{\partial r} v^{\frac{2n}{n-2}} \, dy = 0 \,. \tag{3.5}$$

(3.5) can also be obtained by using the divergence theorem on the vector field

$$\begin{aligned} V(y) \;\; &= \;\; (y \cdot \bigtriangledown v(y)) \bigtriangledown v(y) - \frac{|\bigtriangledown v(y)|^2}{2} y + \frac{n-2}{2n} c_n \tilde{K}(y) v^{\frac{2n}{n-2}}(y)\, y \\ & \qquad + \frac{n-2}{2} v(y) \bigtriangledown v(y) \quad \text{ for } \; y \in \mathbb{R}^n. \end{aligned}$$

One can appreciate the wonderful cancelations when computing the divergence of V, which leads to

$$\begin{aligned} & \frac{n-2}{2n} c_n \int_{B_o(r)} \left(y \cdot \bigtriangledown \tilde{K} \right) v^{\frac{2n}{n-2}} \, dy \\ = \;\; & \int_{S_o(r)} \left[(y \cdot \bigtriangledown v) \frac{\partial v}{\partial \nu} - \frac{r}{2} |\bigtriangledown v|^2 + \frac{n-2}{2n} c_n \, r \tilde{K} v^{\frac{2n}{n-2}} + \frac{n-2}{2} v \frac{\partial v}{\partial \nu} \right] dS \,, \end{aligned} \tag{3.6}$$

where ν is the unit outward normal on $S_o(r)$. (3.5) and (3.6) are known as the Pohozaev identities. In particular, (3.6) links information inside the ball to that along the boundary.

4. Nonexistence results.

A function $F \in C^1(S^n)$ is said to fulfill the *Kazdan-Warner condition* (K-W condition) if it is positive somewhere and there is a positive function $w \in C^o(S^n)$ such that

$$\text{(K-W)} \qquad \int_{S^n} X(F)\, w^{\frac{2n}{n-2}}\, dV_{g_1} = 0$$

for *all* conformal Killing vector fields X on S^n. From § 3 a, we know that (K-W) is a necessary condition for the function F to be the scalar curvature function of a conformal metric of g_1. Cf. [47] for a large class of functions satisfying the K-W condition.

§ 4 a. *Rotationally symmetric functions.* It follows from (K-W) that $X(F)$ ($\not\equiv 0$) has to change sign in S^n. Hence axial-symmetric, monotonic and nonconstant functions cannot be the scalar curvature functions of conformal metrics of g_1. Indeed, we have the following.

Proposition 4.1. *Let $F \in C^1(S^n)$ be rotationally symmetric and nonconstant. Then F satisfies the K-W condition if and only if*

(4.2) *F is positive somewhere in S^n and F' changes sign.*

Proof. We may assume, without loss of generality, then F depends on x_{n+1} only. It follows that $X_i(F) = 0$ for $1 \le i \le n$. Here $\{X_i\}$ is the basis described in § 3 a. For $n+2 \le i \le (n+1)(n+2)/2$ and for w depending on x_{n+1} only, we have

$$\int_{S^n} X_i(F)\, w^{\frac{2n}{n-2}}\, dV_{g_1} = 0\,,$$

as we have cancelation at the x_{n+1} level. Therefore, we need only to find a continuous function w, which depends on x_{n+1} only, such that

$$\int_{S^n} X_{n+1}(F)\, w^{\frac{2n}{n-2}}\, dV_{g_1} = \mathrm{Vol}\,(S^{n-1}) \int_{-1}^{1} F'(x_{n+1})\, w^{\frac{2n}{n-2}}(x_{n+1})\, [1-x_{n+1}^2]^{\frac{n}{2}}\, dx_{n+1} = 0\,.$$

This is possible if and only if F' (the derivative of F with respect to x_{n+1}) changes sign. □

W.-X. Chen and C.-M. Li show that for a rotationally symmetric function F, K-W condition is not enough for equation (3.1) to have a positive smooth solution with $K = F$. *Their results implies that for a rotationally symmetric function F, if it is* monotone in the region where it is positive, *then* (3.1) *admits no solution unless F is a positive constant* [18].

Hence it is reasonable to ask whether the condition

(4.3) *F is positive somewhere in S^n and F' changes sign in the region(s) where F is positive*

is enough or not for rotationally symmetric functions F to afford a positive solution for equation (3.1). It turns out that in this case the question on existence depends sensitively on the behaviors near the poles.

Definition 4.4. *$K \in C^1(S^n)$ is called a simple 'down-up' function if K depends on x_{n+1} only and there exists a number $c_o \in (-1,\ 1)$ such that K is nonincreasing in $(-1,\ c_o)$ and nondecreasing in $(c_o,\ 1)$.*

Recall that $x_{n+1} = 1$ corresponds to the north pole $N \in S^n$, and $x_{n+1} = -1$ the south pole $S \in S^n$. The following result is due to Bianchi.

Theorem 4.5 [6]. *Let $K \in C^1(S^n)$ be a simple 'down-up' function. Then any positive smooth solution u of equation(3.1) also depends on x_{n+1} only (i.e., u is also rotationally symmetric).*

Combining with a result of Bianchi and Engell [7], the following *nonexistence* result is obtained ($n \geq 3$).

Theorem 4.6 [7]. *Take two numbers ρ_1 and ρ_2 such that*

(i) $\rho_i > \dfrac{n(n-2)}{n+2}$ *for* $i = 1,\ 2$, *and*

(ii) $\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} \geq \dfrac{2}{n-2}$.

For any two positive numbers C_N and C_S, there exist a positive number ε and a *positive function* $F \in C^\infty(S^n \setminus \{N,\ S\})$ *which depends on* x_{n+1} *only, with*

$$F(x) \;=\; C_N - \varepsilon|x-N|^{\rho_1} \quad \textit{for } x \textit{ in a neighbhorhood of } N\,, \quad \textit{and}$$
$$F(x) \;=\; C_S - \varepsilon|x-S|^{\rho_2} \quad \textit{for } x \textit{ in a neighbhorhood of } S\,,$$

such that equation (3.1) has no positive solution for $K = F$. *Moreover,* $F(x_{n+1})$ *is decreasing in* $(-1,\ c_o)$ *and increasing in* $(c_o,\ 1)$, *where* $c_o \in (-1,\ 1)$.

As far as we know, this remains as the only nonexistence result making use *global and local* properties of the function. Comparing with a result in [Chen-Li; Theorem 1.3], it seems possible to relax condition (i) in Theorem 4.6. As for (ii), there is a sense of sharpness. Indeed, W.-X. Chen and C.-M. Li show that, under certain flatness condition at the poles with order in $(n-2,\ n)$, (4.3) is a necessary and sufficient condition for (3.1) to have a solution [20] .

§ 4 b. *Nonaxial symmetric functions.* Besides rotationally symmetric functions, other symmetric functions can also be checked conveniently for the K-W condition. Specifically, we have the following result, due to Han and Li.

Proposition 4.7 [32]. *Let* $F \in C^1(S^n)$ *be positive somewhere. Assume that* F *satisfies*

(4.8) $$F(\cdots,\ -x_i,\ \cdots) \;=\; F(\cdots,\ x_i,\ \cdots) \quad \textit{for} \;\; 1 \le i \le n\,,$$

(4.9) $$F(x_1,\cdots,\ x_n,\ -x_{n+1}) \;=\; -F(x_1,\cdots,\ x_n,\ x_{n+1})$$

for all $(x_1,\cdots,\ x_{n+1}) \in S^n$. *If* $X_{n+1}(F) = \bigtriangledown_{g_1} x_{n+1} \cdot \bigtriangledown F$ *changes sign, then* F *satisfies the K-W condition.*

Proof. Consider positive functions w satisfying

$$w(\cdots,\ -x_i\,,\ \cdots) = w(\cdots,\ x_i\,,\ \cdots) \quad \text{for} \;\; (x_1,\cdots,\ x_{n+1}) \in S^n, \;\; 1 \le i \le n+1\,.$$

Let $\{X_i\}$ be the basis introduced in § 3 a. From (4.9), we have

$$X_i(F)\,(\cdots,\ -x_i,\ \cdots) = -X_i(F)\,(\cdots,\ x_i,\ \cdots) \quad \text{for} \;\; 1 \le i \le n\,.$$

It follows that

$$\int_{S^n} X_i(F)\, w^{\frac{2n}{n-2}}\, dV_{g_1} = 0 \quad \text{for} \;\; 1 \le i \le n\,.$$

From (4.8) and (4.9), we have

$$F(-x) = -F(x) \quad \text{for all } x \in S^n.$$

Considering the orientation, we conclude that, as vectors in $\mathbb{R}^{n+1}$,

$$\nabla_{g_1} F(-x) = \nabla_{g_1} F(x) \quad \text{for all } x \in S^n.$$

On the other hand, $X_i(-x) = -X_i(x)$ for $n+2 \le i \le (n+1)(n+2)/2$, which leads to $X_i(F)(-x) = -X_i(F)(x)$. Hence

$$\int_{S^n} X_i(F)\, w^{\frac{2n}{n-2}}\, dV_{g_1} = 0 \quad \text{for } n+2 \le i \le (n+1)(n+2)/2\,.$$

Finally, as $X_{n+1}(F)$ *changes sign*, we can select w suitably so that

$$\int_{S^n} X_{n+1}(F)\, w^{\frac{2n}{n-2}}\, dV_{g_1} = 0\,.$$

Hence F satisfies the K-W condition. □

Han and Li construct a smooth function h_o satisfying the symmetries described in (4.8) and (4.9), such that $K_o = 2 + h_o$ is positive and

$$X_{n+1} \cdot \nabla_{g_1} h_o > 0 \quad \text{in } S^n \setminus \{N,\ S\}\,.$$

Hence K_o does not satisfy the K-W condition. By modifying h_o slightly near a small neighborhood of the south pole S, they show that there is a family of smooth positive functions K_ε, indexed by $\varepsilon > 0$ small, such that

(i) $1/2 \le K_\varepsilon \le 5/2\,;$

(ii) $K_\varepsilon \to K_o$ in $C^\infty(S^n)$ as $\varepsilon \to 0^+$;

(iii) K_ε satisfies the symmetric conditions (4.8) and (4.9); and

(iv) in terms of the stereographic projection,

$$X_{n+1}(K_\varepsilon)(\mathcal{P}^{-1}(y)) = 8(y_1^2 - \varepsilon y_2^2) + O(|y|^4) \quad \text{for } y \text{ close to the origin}\,.$$

Because of (iii), K_ε satisfies the K-W condition (by Proposition 4.7).

Theorem 4.10 [32]. *For $n = 3$, let $K \in C^2(S^3)$ satisfy, for some positive constants k_1 and d,*

$$(4.11) \quad K \geq k_1 > 0 \quad \text{in} \;\; S^3, \quad |\Delta_{g_1} K(x)| \geq d \quad \text{whenever} \;\; |\bigtriangledown_{g_1} K(x)| \leq d\,.$$

Then for any constant $\alpha \in (0,\ 1)$, there exists a positive constant C such that for all positive smooth solutions u of equation (3.1),

$$\|u\|_{C^{3,\alpha}(S^3)} \leq C\,.$$

Moreover, C depends only on k_1, d, α, $\|K\|_{C^2(S^3)}$, and the modulo of continuity of $\bigtriangledown^2_{g_1} K$ on S^3.

Using this compactness result as well as the one in four dimension (see § 6 e), they conclude that, for $\varepsilon > 0$ small enough, equation (3.1) cannot afford to have positive smooth solutions with $K = K_\varepsilon$ (otherwise K_o also affords a positive solution of (3.1) with $K = K_o$, which is a contradiction). Note that these functions, although highly symmetric, are not axial symmetric.

Using the stereographic projection, one can check that

$$\bigtriangledown K_\varepsilon(\mathcal{P}^{-1}(y)) \cdot y = 2(y_1^2 - \varepsilon y_2^2) + 4y_2^4.$$

(Cf. [32].) That is, $X_{n+1}(K_\varepsilon)(y)$ becomes only slightly negative and the other part it is positive. As K_ε fulfills the K-W condition, w in (K-W) has to be relatively large near S.

Note that the condition (4.12) is C^2-stable. More specifically, K_ε as in the above satisfies condition (4.11) for some k_1 and d, then the functions in a small open neighborhood of K_ε in $C^2(S^3)$, they all satisfy condition (4.11) with constants $k_1/2$ and $d/2$. Since K_ε affords no solutions of equation (3.1), it follows that in a small open neighborhood of K_ε in $C^2(S^3)$, all the functions also affords no solutions of equation (3.1). Thus the collection of scalar curvature functions is *not* C^2-dense, at least when $n = 3$ or 4. This is pointed out in [49]. It turns out that the collection is always $C^{1,\alpha}$-dense for any $\alpha \in (0,\ 1)$. See § 6 f for more details. This also reveals the sharpness of condition (4.12).

5. $\bigtriangledown K$ and its direction.

§ 5 a. *Gradient vectors.* Let H be a C^1-function on S^n that is positive somewhere. Consider $\bigtriangledown_{g_1} H$ as a vector in $\mathbb{R}^{n+1}$. The collection

$$\mathcal{H}_{\bigtriangledown} := \left\{ \frac{\bigtriangledown_{g_1} H(x)}{\| \bigtriangledown_{g_1} H(x) \|} \;\middle|\; \bigtriangledown_{g_1} H(x) \neq 0\,, \quad x \in S^n \right\} \tag{5.1}$$

forms a subset in S^n.

Proposition 5.2. *If $\mathcal{H}_{\bigtriangledown}$ is nonempty and is contained in an open hemisphere in S^n, then H does not satisfy the K-W condition.*

Proof. Without loss of generality, we may assume that $\mathcal{H}_{\bigtriangledown}$ is contained in the (open) south hemisphere $(x_{n+1} < 0)$. Take a $p \in S^n$ with $\bigtriangledown_{g_1} H(p) \neq 0$. After a rotation about the x_{n+1} axis, we may assume that $p = (0, \cdots, 0\,, a,\, b\,)$ with $a > 0$ and $a^2 + b^2 = 1$. (There is no lost in generality as a rotation of $\mathbb{R}^{n+1}$ does not change the dot product.) Take $X = -(\bigtriangledown_{g_1} x_{n+1})$. We want to show that

$$X(p) \cdot \bigtriangledown_{g_1} H(p) > 0\,. \tag{5.3}$$

Observe that $X(p)$ is in the same *direction* as $(0, \cdots, 0\,, b,\, -a\,)$. Let

$$\bigtriangledown_{g_1} H(p) = (\gamma_1, \cdots, \gamma_n\,,\, \gamma_{n+1}\,)\,, \qquad \text{with} \;\; \gamma_{n+1} < 0\,.$$

As $\bigtriangledown_{g_1} H(p)$ is on the tangent space of the sphere at $p = (0, \cdots, 0\,, a,\, b\,)$, we have

$$\bigtriangledown_{g_1} H(P) \cdot (0, \cdots, 0\,, a,\, b\,) = 0 \quad \Longrightarrow \quad a\,\gamma_n + b\,\gamma_{n+1} = 0\,. \tag{5.4}$$

Furthermore,

$$d := (0, \cdots, 0\,, b,\, -a\,) \cdot (\gamma_1, \cdots, \gamma_n\,,\, \gamma_{n+1}\,) \quad \Longrightarrow \quad b\,\gamma_n - a\,\gamma_{n+1} = d\,. \tag{5.5}$$

The sign of $X(p) \cdot \bigtriangledown_{g_1} H(p)$ is the same as the sign of d. From (5.4) and (5.5) we have

$$(a^2 + b^2)\,\gamma_{n+1} = -a\,d \quad \Longrightarrow \quad d > 0 \qquad (\text{as} \;\; a > 0 \;\; \text{and} \;\; \gamma_{n+1} < 0)\,.$$

Hence (5.3) holds for any $p \in S^n$ with $\bigtriangledown_{g_1} H(p) \neq 0$. It follows that the K-W condition does not hold for H. □

Proposition 5.6. *Let $H \in C^1(S^n)$ be a nonconstant antipodal symmetric function, i.e., $H(-x) = H(x)$ for $x \in S^n$. Then $\mathcal{H}_\bigtriangledown$ is not contained in any closed hemisphere in S^n.*

Proof. Suppose that, on the contrary, $\mathcal{H}_\bigtriangledown$ is contained in a closed hemisphere. After possibly applying a rotation, we may assume, without loss of generality, that

$$\mathcal{H}_\bigtriangledown \subset \{(x_1, \cdots, x_{n+1}) \in S^n \mid x_{n+1} \le 0\} \,.$$

As H is antipodal symmetric, it follows that $x \in \mathcal{H}_\bigtriangledown \implies -x \in \mathcal{H}_\bigtriangledown$. Thus $\mathcal{H}_\bigtriangledown$ is inside the equatorial hyperplane $\{x_{n+1} = 0\}$ as well. Take a point $p \in S^n \setminus \{S\}$. Join p to N by the shorter arc of the great circle (the geodesic). As $\bigtriangledown_{g_1} H$ has no component in the x_{n+1} direction, H remains unchanged along the geodesic. It follows that $H(p) = H(N)$ for all $p \in S^n$. This contradicts that H is not a constant function. $\square$

§ 5 b. *Questions.* From the nonexistence examples in § 4, it seems that for equation (3.1) to process a positive solution with $K = H$, H cannot behave in certain 'simple' way. That is, it should be a bit twisty. The following questions are meant to spur further discussion rather than to fix conjectures.

Question 5.7. *Consider a function $H \in C^1(S^n)$ which depends on x_{n+1} only. We ask: if H has at least three positive maxima with $H'' < 0$ at these points (or at least three positive local minima with $H'' > 0$ at these points), does equation (3.1) always process a positive C^2-solution for $K = H$?*

Question 5.8. *Consider function $H \in C^1(S^n)$. We ask: if there are at least three positive maximal points where the Hessian being nondegenerate (or three positive minimal points where the Hessian being nondegenerate), does equation (3.1) always process a positive C^2-solution for $K = H$?*

6. Existence in cases of symmetry and dimension 3 & 4.

One of the early existence results on the Nirenberg/Kazdan-Warner equations mimics the original thought behind the Yamabe equation [58]. By a rescaling, equation (3.1) can be written as

$$L_{g_1} u - K u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad S^n . \tag{6.1}$$

The problem is cast into a variational setting: to minimize $\langle u, \, Lu \rangle_{L^2}$ subjected to the constraint

$$\int_{S^n} K \, u^{\frac{2n}{n-2}} \, dV_{g_1} = 1 . \tag{6.2}$$

Here $K \in C^\infty(S^n)$ is required to be positive somewhere.

As in the Yamabe problem, one can formulate similar variational problem for a subcritical index $p > 1$, i.e., $p < (n+2)/(n-2)$. Thus we choose a sequence of p increasing to the critical level and show that the corresponding sequence of solutions converges to a solution of the (6.1). Similar to the Yamabe problem (cf. §2 b), a sufficient condition for this approach to work is

$$\left[\max_{S^n} K \right]^{\frac{n-2}{n}} E \, < \, \lambda(S^n) \, , \tag{6.3}$$

where E is the *infimum* of $\langle u, \, L_{g_1} u \rangle_{L^2}$ subject to condition (6.2). See [26].

Condition (6.3) is able to eliminate the possibility of blow-up (which would imply equality in (6.3)). Because E is the infimum, one needs only to produce *one* test function for (6.3) to hold.

§ 6 a. *The test function.* In the paper [26], the main assumption on K is the following flatness condition.

(6.4) *There is a (absolute) maximum point P_o of K at which $K(P_o) > 0$ and all partial derivatives of K of order less than or equal to $(n-2)$ vanish.*

Theorem 6.5 [26]. *Let Γ be a nontrivial discrete group acting on S^n. Suppose that K is invariant under Γ and satisfying* (6.4). *Then equation* (6.1) *has a positive smooth solution.*

When $n = 3$, condition (6.4) is automatically satisfied. In this case theorem 6.5 is a complete generalization of a renowned result of Moser in dimension two. When $n > 3$, the example of Bianchi described in § 4 a indicates that condition (6.4) is really needed: one looks at $\rho_1 = \rho_2 = n - 2$ in Theorem 4.7 and $\Gamma = \{\text{Id}, \ \text{antipodal map}\}$. However, Bianchi's results cannot be used to yield counterexamples when the group Γ has more than 2 elements. Cf. also [1].

Without going into the details of the proof in [26], we discuss specifically the test function, which is global in nature and involves the *Green's function*. Consider the manifold $\mathcal{M} := S^n/\Gamma$. Let $\pi : S^n \to \mathcal{M}$ be the covering map, and $g_{\mathcal{M}} = \pi^*(g_1)$ the metric on $\mathcal{M}$. Without loss of generality, we may assume that $\pi(S) = P_o$. By a rescaling, we may also assume without loss of generality that $K(P_o) = 1$.

The test function is formed by joining a standard solution U_λ with the Green function. Via the stereographic projection, define the function

$$\text{(6.6)} \quad \Phi\,(x) \;=\; \begin{cases} \left(\frac{\lambda}{\lambda^2+|y|^2}\right)^{\frac{n-2}{2}} & \text{for} \quad x \in B_{P_o}(\rho_o)\,, \quad y = \mathcal{P}(\pi^{-1}_{|_{B_{P_o}(\rho_o)}}(x))\,, \\ \varepsilon_o\,[G(x) - \psi_o(x)\alpha(x)] & \text{for} \quad x \in B_{P_o}(2\,\rho_o) \setminus B_{P_o}(\rho_o)\,, \\ \varepsilon_o\, G(x) & \text{for} \quad x \in M \setminus B_o(2\rho_o)\,. \end{cases}$$

Here ε_o is a suitably chosen positive number so that Φ is continuous. G is the Green function of the conformal Laplacian $L_{g_{\mathcal{M}}}$ on $\mathcal{M}$ with pole at P_o. In terms of $y = \mathcal{P}(\pi^{-1}_{|_{B_{P_o}(\rho_o)}}(x))$,

$$G(y) = |y|^{2-n} + B + \alpha(y) \qquad \text{for} \quad |y| \ \text{ small}\,,$$

where $B \geq 0$ is a constant, α a harmonic function with $\alpha(0) = 0$. (ψ_o is a kind of cut-off function. We refer to [26] for its precise definition.)

With such a test function, careful calculation shows that (see [26])

$$\langle \Phi,\, L_{g_{\mathcal M}}\, \Phi\rangle_{L^2} \;\le\; \lambda(S^n)\left(\int_{\mathcal M} K u^{\frac{2n}{n-2}}\, dV_{g_{\mathcal M}}\right)^{\frac{n-2}{n}} - (n-2)\,\mathrm{Vol}\,(S^{n-1})\, B\, \varepsilon_o^2 + c\,\rho_o^{-n}\varepsilon_o^{\frac{2n}{n-2}} + c\,\rho_o\, \varepsilon_o^2\,. \tag{6.7}$$

Here we use the flatness condition to obtain $1 - K(x) \le C\,[\,d_{g_1}(x,\, P_o)]^{n-1}$ for x close to P_o. So the key is to show that $B > 0$. Once this is known, we achieve (6.3) by adjusting ρ_o and ε_o.

§ 6 b. *Positivity of* B. We seek to justify that $B > 0$ if and only if Γ has more than one element. Let $\pi^{-1}(P_o) = \{q_1, ..., q_k\} \subset S^n$, where $k > 1$ if Γ is nontrial. Without lost of generality, we assume that $q_i \ne N$. Applying the stereographic projection, we obtain

$$G \circ \pi(\mathcal{P}^{-1}(y)) = \frac{1}{|y - y_1|^{n-2}} + \sum_{l=2}^{k} \frac{a_l}{|y - y_l|^{n-2}} \qquad \text{for } \; y \in \mathbb{R}^n \setminus \{y_1,\; y_2, \cdots,\; y_k\}\,.$$

Here $y_l = \mathcal{P}(q_l)$, and a_l are positive constants. Thus

$$G \circ \pi(\mathcal{P}^{-1}(y)) = \frac{1}{|y - y_1|^{n-2}} + B + \alpha(y) \tag{6.8}$$

for y close to y_1, where

$$B = \sum_{l=2}^{k} \frac{a_l}{|y_1 - y_l|^{n-2}} > 0\,,$$

$$\alpha(y) = \sum_{l=2}^{k} \frac{a_l}{|y - y_l|^{n-2}} - \sum_{l=2}^{k} \frac{a_l}{|y_1 - y_l|^{n-2}} \quad \Longrightarrow \quad \alpha(y_1) = 0\,.$$

§ 6 c. *Large and small parts.* It is shown in [26] that $\varepsilon_o \approx \lambda^{\frac{n-2}{2}}$. *Hence the test function* Φ *has a large part ('blow-up' part) which is given by*

$$U_\lambda(y) = \left(\frac{\lambda}{\lambda^2 + |y|^2}\right)^{\frac{n-2}{2}} \qquad \textit{for } \; |y| \le \rho_o\,,$$

and a relatively small part ('collapsed' part) given by $\varepsilon_o G$. In other words, Φ concentrates near P_o.

Intuitively, the conformal metric $\Phi^{\frac{4}{n-2}} g_{\mathcal{M}}$ consists of a spherical part, and an 'annex', into which the topology of $M \setminus \{P_o\}$ is cast. We jump a bit to note that the dominating spherical parts together with a small region formed by a Green's function are essential features of this type of blow-up. Geometrically, the Green function in (6.8) with $B > 0$ forms a connecting 'neck' to *link* the bubble to the small part (cf. § 7 b & § 7 c).

§ 6 d. *Dimension* $n = 3$. As appeared in § 6 a, the 'flatness' of the critical points of K seems to be a crucial element in preventing blow-up to occur. When K is smooth, any critical point of K is automatically flat at least up to order 2. Thus one expects the lower dimension cases to be cleaner.

Recall that a Morse function has only nondegenerate critical points (i.e., the Hessian at a critical point is a nondegenerate bilinear form). The index of the critical point is defined to be the number of negative eigenvalues of the Hessian at that point. In [63], the following result is obtained (cf. also [2] [11] [12]).

Theorem 6.9 [63]. *In S^3, let K be a positive Morse function with nonzero Laplacian at its critical points. For $d = 0, 1, 2$, denoted by D_d the number of critical points of K at which $\Delta(-K) > 0$ and at which the Morse index of $-K$ is d. If*

$$D_o - D_1 + D_2 \neq 1\,, \tag{6.10}$$

then equation (3.1) has a positive solution.

The axial symmetric cases considered in § 4 a demonstrate that condition (6.10) is related to the global balance condition (the K-W condition). The above result, although being generalized (cf. [12] [35]), indicates a general trend for existence results, namely, *local conditions on the critical points of K plus a global condition in terms of degree.*

§ 6 e. *Dimension* $n = 4$. In [49], we find existence and compactness results in dimension four. Given $K \in C^2_+(S^4)$, define

$$\begin{aligned} \mathcal{C} &:= \{ p \in S^4 \mid \bigtriangledown_{g_1} K(p) = 0 \} , \\ \mathcal{C}^+ &:= \{ p \in \mathcal{C} \mid \Delta_{g_1} K(p) > 0 \} , \quad \text{and} \quad \mathcal{C}_- := \{ p \in \mathcal{C} \mid \Delta_{g_1} K(p) < 0 \} . \end{aligned}$$

For any positive integer k less than or equal to the cardinality of $\mathcal{C} \setminus \mathcal{C}^+$, and for any k distinct point $p_1, \cdots, p_k \in \mathcal{C} \setminus \mathcal{C}^+$, consider the $k \times k$ matrix $M(p_1, \cdots, p_k)$ defined by

$$M_{ij} = \begin{cases} -\frac{\Delta_{g_1} K(p_i)}{[K(p_i)]^2} & \text{for } i = j , \\ \\ -\frac{36\, G_{p_i}(p_j)}{\sqrt{K(p_i) \cdot K(p_j)}} & \text{for } \imath \neq j . \end{cases}$$

Here G_{p_i} is the Green function of L_{g_1} on S^4 with the pole at p_i. Note that $\text{Vol}\,(S^4) = \dfrac{4}{3} \text{Vol}\,(S^3)$. The form of M_{ij} underlines the complexity of bubble's interaction with K.

Let $\mu(M(p_1, \cdots, p_k))$ be the least eigenvalue of $M(p_1, \cdots, p_k)$. Set

$$\begin{aligned} \mathcal{A} := \{ K \in C^2_+(S^4) \mid \Delta_{g_1} K \neq 0 \ \text{ on } \ \mathcal{C} , \\ \mu(M(p_1, \cdots, p_k)) \neq 0 \quad \text{for} \quad \text{all} \quad p_1, \cdots, p_k \in \mathcal{C}_- , \quad k \geq 2 \} . \end{aligned}$$

$\mathcal{A}$ is an open and dense set in $C^2_+(S^n)$. Define Index: $\mathcal{A} \to \mathbf{Z}$ by the following properties:

(i) If K is a Morse function with $\mathcal{C}_- = \{ p_1, \cdots, p_m \}$, we define

$$\text{Index}\,(K) = \sum_{k=1}^m \left[\sum_{\mu(M(p_{i_1}, \cdots, p_{i_k})) > 0} (-1)^{k - 1 + \sum_{j=1}^k i(p_{i_j})} \right] ,$$

where $i(p_{i_j})$ denotes the Morse index of K at p_{i_j}.

(ii) Extend Index: $\mathcal{A} \to \mathbf{Z}$ as a continuous map with respect to the $C^2(S^4)$ norm.

Theorem 6.11 [49]. *For any $K \in \mathcal{A}$, there exist positive constants $\delta = \delta(K)$ and $C = C(K)$ such that for $K_1 \in C^2(S^4)$ with $\|K_1 - K\|_{C^2(S^4)} < \delta$, and for any positive solution u_1 of equation (3.1) with $K = K_1$, we have*

$$C^{-1} \leq u_1 \leq C \quad \textit{in} \;\; S^4 , \quad \textit{and} \quad \|u_1\|_{C^3(S^4)} < C .$$

In addition, if Index $(K) \neq 1$, *then equation (3.1) has a positive solution.*

In case $K \in \mathcal{A}$ is a Morse function, under the extra condition that

$$[\Delta_{g_1} K(p)] \cdot [\Delta_{g_1} K(q)] \leq 9\, K(p) \cdot K(q) \quad \text{ for } \ p\,,\ q \in \mathcal{C}_-\,,$$

existence of a positive solution is guaranteed if $\displaystyle\sum_{p \in \mathcal{C}_-} (-1)^{i(p)} \neq 1\,,$ which is quite similar to (6.10). See [49].

Theorems 6.9 and 6.11 are obtained by carefully analyzing the blow-up process, some of which are discussed in later chapters. In dimensions 3 & 4, it is possible to show that blow-up does not occur under the above conditions. Thus the solution set is uniformly bounded from above. The conditions on indices are used to show that the operator $u \mapsto u + L^{-1}\left(K u^{\frac{n+2}{n-2}}\right)$ has nonzero Leray-Schauder degree. Cf. [48] [55] .

§ 6 f. $C^{1,\,\alpha}$*- density.* Included in the paper by Schoen and Zhang [63] is the following result, which is also obtained by Y.-Y. Li in [49]. See also [3] [4].

Theorem 6.12. *For* $n \geq 3$ *and* $0 < \alpha < 1$, *the collection of smooth conformal scalar curvature functions is dense in* $C_+^{1,\,\alpha}(S^n)$.

It is noted in [**?**] that the above density theorem cannot be improved to $C^2(S^n)$, at least when $n = 3$ & 4. To see this in $S^3 \subset \mathbb{R}^4$, allow the Cartesian coordinates of $\mathbb{R}^4$ be $(x_1,\, x_2,\, x_3,\, x_4)$, and θ the angle to the x_4-axis. The function

$$K_4\,(x_1,\, x_2,\, x_3,\, x_4) := 2 + x_4 = 2 + \cos\theta \quad \text{ for } \ \theta \in [-\pi,\, \pi]$$

does not satisfies the K-W condition. On the other hand,

$$|\bigtriangledown_{g_1} K_4| = -\sin\theta\,, \quad |\Delta_{g_1} K_4| = \left|(\cos\theta)'' + 2\,\frac{\cos\theta}{\sin\theta}\,(\cos\theta)'\right| = 3\,|\cos\theta|\,.$$

Thus $|\bigtriangledown_{g_1} K_4| = 0$ only when $\theta = \pm\,\pi$. When $\theta = \pm\,\pi$, $|\Delta_{g_1} K_4| = 3$. Hence any C^∞ function K that is sufficiently close to K_4 in $C^2(S^3)$ norm satisfies

$$K > k_1\,, \qquad \min_{x \in S^3,\ |\bigtriangledown_{g_1} K(x)| \leq d} |\Delta_{g_1} K(x)| \ \geq\ d\,,$$

where we can take $k_1 = 1/2$ and $d = 1/10$. By the compactness result in [32], for any solution u of (3.1), we have $\|u\|_{C^{3,\alpha}(S^3)} \le C$. Thus if the density result holds for $C^2_+(S^3)$, then the compactness result [loc. cit.] implies that K_4 also affords a positive solution of equation (3.1) with $K = K_1$, which is a contradiction.

Because the argument makes use of compactness results, for higher dimensions, as far as we know, the C^2- density appears to be an open question. In particular, we ask: *Can the $C^{1,\alpha}$- density in Theorem 6.12 be improved to C^2- density (or even better) once* $\dfrac{n-2}{2} > 2$, *that is, when* $n \ge 7$?

For a smooth function K, the $C^{1,\alpha}$- norm actually gives little information on $|\bigtriangledown_{g_1} K(x) - \bigtriangledown_{g_1} K(y)|$ when x is very close to y. To see this, suppose $f(t)$ is differentiable at 0. We have

$$\lim_{t\to 0} \frac{|f(t) - f(0)|}{|t-0|^\alpha} = |f'(0)| \lim_{t\to 0} |t|^{1-\alpha} = 0\,.$$

It follows that for any constant $c > 0$, we have

$$|f(t) - f(0)| \le c\,|t|^\alpha \quad \text{for} \ \ |t| \ \ \text{small}\,.$$

7. The notion of blow-up and typical examples.

In the n-sphere S^n $(n \ge 3)$ equipped with the standard metric g_1, recall that the *conformal Laplacian* for g_1 takes the form

$$L_{g_1} u := \Delta_{g_1} u - c_n n(n-1)\, u\,. \tag{7.1}$$

As in the Yamabe equation, for a number $p \in \left(1, \frac{n+2}{n-2}\right]$, we consider the equation

$$L_{g_1} u + c_n K_p\, u^p = 0 \quad \text{in} \ \ S^n, \tag{7.2}$$

where $K_p \in C^1(S^n)$. Introduce the conditions (here c and C are fixed positive constants)

(B): $K_{p_i} \in C^1(S^n)$ with $0 < c^2 \le K_{p_i} \le C^2$;

(S): $\{u_{p_i}\} \subset C_+^{2,\alpha}(S^n)$, $p_i \uparrow \frac{n+2}{n-2}$; each u_{p_i} is a solution of (7.2) with $p = p_i$ and $K_p = K_{p_i}$, where $\{K_{p_i}\}$ satisfies **(B)**.

With conditions **(B)** and **(S)**, if the sequence $\{u_{p_i}\}$ is uniformly bounded from above, then a subsequence of $\{u_{p_i}\}$ converges in $C^1(S^n)$. We are interested in the case when the sequence $\{u_{p_i}\}$ is *not* uniformly bounded from above. As $\max K_{p_i} \ge c^2 > 0$, we can exclude the simple case of rescaling with the scaling factor approaching infinity.

Definition 7.3. *Let the sequence $\{u_{p_i}\}$ satisfies* **(S)**. *A point $x_b \in S^n$ is said to be a blow-up point (with respect to the sequence) if there exists a sequence of points $\{\bar{x}_i\} \subset S^n$ such that*

$$\lim_{i\to\infty} u_{p_i}(\bar{x}_i) = \infty \quad \textit{and} \quad \lim_{i\to\infty} \bar{x}_i = x_b\,.$$

$\{u_{p_i}\}$ is called a blow-up sequence if it has at least one blow-up point.

Clearly if the sequence $\{u_{p_i}\}$ which satisfies **(S)** is not uniformly bounded from above, that a subsequence of $\{u_{p_i}\}$ processes a blow-up point.

§ 7 a. *Projection to* $\mathbb{R}^n$. After applying the stereographic projection $\mathcal{P}$ as in proposition 1.7, equation (7.2) can be expressed as

$$\Delta\, v + c_n \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}\Upsilon_p} \tilde{K}_p\, v^p = 0 \quad \text{in } \mathbb{R}^n. \tag{7.4}$$

Here

$$v(y) = \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}} u(\mathcal{P}^{-1}(y))\,, \qquad \tilde{K}_p(y) = K_p(\mathcal{P}^{-1}(y)) \quad \text{for } y \in \mathbb{R}^n, \tag{7.5}$$

and

$$\Upsilon_p := \frac{n+2}{n-2} - p\,. \tag{7.6}$$

In particular, $v(y) = O(|y|^{-(n-2)})$ when $|y| \gg 1$. In case $p = (n+2)/(n-2)$, we have

$$\Delta\, v + c_n\, \tilde{K}_p\, v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n. \tag{7.7}$$

§ 7 b. *Expression in log-cylindrical coordinates.* As concentration happens in a small neighborhood of a point, besides the rescaling (enlargement) discussed in § 2a, another useful perspective is to stretch the radial direction. Let us introduce the log-cylindrical coordinates $(t,\ \theta)$, where

$$t = -\ln|y| \quad \text{and} \quad \theta = \frac{y}{|y|} \quad \text{for } \ y \in \mathbb{R}^n \backslash \{0\}\,.$$

Observe that when $|y|$ is small, t becomes large.

Allow v be as in (7.4). The function

$$w(t,\,\theta) := |y|^{\frac{2}{p-1}}\, v(y) \quad \text{for } \ t \in \mathbb{R} \ \text{ and } \ \theta \in S^{n-1} \tag{7.8}$$

satisfies the equation

$$\frac{\partial^2 w}{\partial t^2} - a_p \frac{\partial w}{\partial t} + \Delta_\theta\, w - b_p\, w + c_n \left(\frac{2}{1+e^{-2t}} \right)^{\frac{n-2}{2}\Gamma_p} \mathcal{K}_p\, w^p = 0 \quad \text{in } \ \mathbb{R} \times S^{n-1}. \tag{7.9}$$

Here Δ_θ is the Laplacian for the standard unit sphere S^{n-1} in $\mathbb{R}^n$, and

$$\mathcal{K}_p(t,\,\theta) := \tilde{K}(y)\,, \quad \text{with } \ |y| = e^{-t} \ \text{ and } \ y/|y| = \theta\,.$$

Moreover,

$$a_p = n - 2 - \frac{4}{p-1}\,, \quad \text{and} \quad b_p = \frac{2}{p-2}\left(n - \frac{2p}{p-1} \right).$$

Cf. [28].

§ 7 c. *Delaunay-Fowler type solutions.* We are interested in *radial* solutions of equation (7.9) with $p = (n+2)/(n-2)$ and $\mathcal{K}_p \equiv 4n\,(n-1)$. In this case, (7.9) is given by the following autonomous O.D.E.

$$w'' - \frac{(n-2)^2}{4}\, w + n(n-2)\, w^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ \mathbb{R}\,. \tag{7.10}$$

For our discussion, equation (7.10) devise a ‘perfect’ way to *stack* bubble.

We call positive smooth solutions of (7.10) Delaunay-Fowler type solutions. By a re-parameterization, we standardize the solutions so that

$$w(0) = \max_{t\in\mathbb{R}} w(t)\,. \tag{7.11}$$

Being periodic, the Delaunay-Fowler type solutions can be indexed by the *neck-size* which is given by $\eta := \inf_{t\in\mathbb{R}} w(t)$. For $\eta = 0$, $w(t) = (2\cosh t)^{(2-n)/2}$ with 'infinite period' (cf. [45] [54]). The relation between η and the period T is described in [54] (see also [45]).

In our presentation, we find it more convenient to index the Delaunay-Fowler type solutions by the period T. Thus for $T \gg 1$, denote by w_T the Delaunay-Fowler type solution with period T, neck-size $\eta_T > 0$, and with standardization (7.11). Furthermore, for a fixed number $D > 0$,

$$w_T(t) \;\to\; (2\cosh t)^{(2-n)/2} \quad \text{in } \; [-D,\, D] \;\text{ as }\; T \to \infty\,. \tag{7.12}$$

Geometrically, as noted in [37], the metrics $w_T^{\frac{4}{n-2}}\,(dt^2 + d\theta^2)$ converge as $T \to \infty$ to a bead of spheres of same radius that are arranged along a fixed axis.

Transforming back to $\mathbb{R}^n$, we let v_T correspond to w_T via (7.8). Observe that, because of (7.8),

$$v_T(y) \le C\,|y|^{-\frac{n-2}{2}} \quad \text{for } \; |y| \gg 1 \;\text{ and }\; |y| \approx 0\,. \tag{7.13}$$

§ 7 d. *Truncation.* One can truncate the Delaunay-Fowler type solution w_T by introducing

$$\bar w_{k,T}\,(t) \;=\; \begin{cases} (2\cosh t)^{(2-n)/2} & \text{for} \quad t \le -D\,, \qquad \text{where } \; 0 < D \le \frac{T}{2}\,, \\ \quad w_T(t) & \text{for} \quad D < t < kT - D\,, \quad \text{where } \; k \in \mathbb{N}\,, \\ [2\cosh\,(t - kT)]^{(2-n)/2} & \text{for} \quad kT + D < t\,. \end{cases}$$

In the above, D is a fixed positive number. The cut-and-paste process takes place in $[-D,\, D]$ and $[T - D,\, T + D]$. Using the limit in (7.12) and the periodicity of w_T, we find that the corresponding scalar curvature function is approaching $4n\,(n-1)$ when $T \to \infty$. Each of the truncated Delaunay-Fowler type solution $\bar w_{k,T}$ can be brought back to S^n via (7.8) and (7.5). Geometrically and intuitively, $k+1$ spheres are stacked up like a tower.

§ 7 e. *Simple examples of blow-up.* Let U_λ be a standard bubble as defined in (1.2). Consider the equation

$$
\begin{aligned}
(7.14)\qquad & \Delta U_\lambda + n(n-2)\, U_\lambda^{\frac{n+2}{n-2}} = 0 \\
\Longrightarrow \qquad & \Delta U_\lambda + c_n \left[4n(n-1) U_\lambda^{\Upsilon} \right] U_\lambda^{\frac{n+2}{n-2}-\Upsilon} = 0 \\
\Longrightarrow \qquad & \Delta U_\lambda + c_n\, K_p\, U_\lambda^p = 0\,, \\
& \text{where } \ p := \frac{n+2}{n-2} - \Upsilon \quad \text{and} \quad \tilde K_p := 4n(n-1)\, U_\lambda^{\Upsilon}\,.
\end{aligned}
$$

Select sequences $\lambda_i \to 0^+$ and $\Upsilon_i \to 0^+$ such that

$$
(7.15)\qquad \lim_{i\to\infty} \lambda_i^{\Upsilon_i} = 1\,.
$$

In this case we may let

$$
v_{p_i}(y) = U_{\lambda_i}(y - y_i)\,, \quad \tilde K_{p_i}(y) = 4n(n-1)\, U_{\lambda_i}^{\Upsilon_i}(y-y_i) \quad \text{where } \ p_i = \frac{n+2}{n-2} - \Upsilon_i\,.
$$

Here $y_i \to y_o \in \mathbb{R}^n$. Using transformation (7.5), we obtain a simple example of a blow-up sequence $\{u_{p_i}\}$ in S^n. Because $\lim\limits_{i\to\infty} \lambda_i^{\Upsilon_i} = 1$, $\{K_{p_i}\}$ so obtained from $\{\tilde K_{p_i}\}$ using (7.5) satisfies condition **(B)**.

§ 7 f. *Towering blow-up.* For an increasing sequence $\{T_i\}$ with $T_1 \gg 1$ and $T_i \to \infty$, consider the truncated Delaunay-Fowler type solution $\bar w_{i,\,T_i}$ as defined in § 7 d. Let its correspondence in $\mathbb{R}^n$ (via (7.8)) be denoted by $\bar v_i$. It satisfies the equation

$$
\Delta\, \bar v_i + c_n\, \tilde K_i\, \bar v_i^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ \mathbb{R}^n.
$$

Following (7.14), we have

$$
\Delta\, \bar v_i + c_n\, (\tilde K_i\, \bar v_i^{\Upsilon_i})\, \bar v_i^{p_i} = 0 \quad \text{in } \ \mathbb{R}^n,
$$

where

$$
(7.16)\qquad \Upsilon_i := \frac{n+2}{n-2} - p_i\,, \quad \text{and} \quad (\text{as} \quad \text{required}) \quad \lim_{i\to\infty} (\max\, \bar v_i)^{\Upsilon_i} = 1\,.
$$

Observe that the scalar curvature function $(\tilde K_i\, \bar v_i^{\Upsilon_i})$ approaches a constant as $i \to \infty$.

After projecting back to S^n via (7.5), we obtains a blow-up sequence $\{\bar u_i\}$, in which more and more spheres are stacked up, towering higher and higher. Contrasting to the simple examples in § 7 e, we have

$$\int_{S^n} \bar u_i^{\frac{2n}{n-2}}\, dV_{g_1} \to \infty \quad \text{ as } \ i \to \infty\,.$$

By virtue of (7.8), we still have

$$\bar u_i(x) \le C\,[d_{g_1}(x,\, S)]^{-\frac{n-2}{2}} \quad \text{ for } \ x \in B_S(1) \setminus \{S\}\,. \tag{7.17}$$

In this connection, compare with the solutions constructed by Chen and Li in [16] with a *fixed* scalar curvature function. Cf. also [27] [57].

§ 7 g. *Spiky blow-ups.* By using the symmetry of the standard bubble under specific Kelvin transformation [44], a method is derived by the author in [45] to offset the 'spheres' in truncated Delaunay-Fowler type solutions. If we continue to offset rounder and rounder bubbles, and use the arrangement in (7.14), we obtain a blow-up sequence with more and more local maxima. Geometrically, the picture is like bubbles are glued in disjoint neighborhoods of a sequence of points $\{x_i\} \in S^n$, with $\lim\limits_{i\to\infty} x_i = x_b$.

In $\mathbb{R}^n$, the corresponding blow-up sequence $\{v_i\}$ converges *outside the origin* to a positive function u_b, which satisfies the equation

$$\Delta v_b + c_n \tilde K_b\, v_b^{\frac{n+2}{n-2}} = 0 \quad \text{ in } \ \mathbb{R}^n \setminus \{0\}\,.$$

In contrast to the examples in the previous two sections, v_b has infinite number of local maxima (hence the word spiky), and *does not* satisfy estimate of the type (7.17) near the origin. Compare also with the construction by Taliaferro in [66], which is refined in [68] & [67]. Cf. [21] for $n = 2$. We remark that one can repeat the argument to construct a blow-up sequence with distinct blow-up points at y_{bj}, so that $\lim\limits_{i\to\infty} y_{bj} = 0$. That is, the blow-up points are not isolated.

8. Isolated blow-up.

As blow-up is endogenous to the Nirenberg/Kazdan-Warner problem, further progress relies on deeper understanding of the bubbling process. § 7 g indicates that the sequence $\{u_i\}$ which satisfies condition (**S**) in § 7 can have infinite number of blow-up points. In this case there is a limit point in S^n, which is again a blow-up point. To understand the analytic behavior of $\{u_i\}$ near the limit blow-up point is a challenging endeavor. But let us consider *isolated* blow-up points first.

Let x_b be a blow-up point of the sequence $\{u_i\}$ which satisfies (**S**). One can say that x_b is an isolated blow-up point if there are no other blow-up points in an open nonempty neighborhood of x_b. However, at first sight this gives no information on the behavior of $\{u_i\}$ near x_b. Observe that the examples in § 7 g has a lot of local maxima (spikes). One way to detect the *unevenness* is to check whether the spherical Harnack inequality

$$(8.1) \qquad \sup_{\partial B_{x_b}(r)} u_i \le C \inf_{\partial B_{x_b}(r)} u_i \quad \text{for } \ r > 0 \ \text{(small)} \quad \text{and} \quad i \gg 1$$

is fulfilled or not. Here C is a positive constant independent on i and r.

§ 8 a. *Spherical Harnack inequality.* One of the natural places to consider the spherical Harnack inequality (8.1) is on the log-cylindrical coordinates. Following § 7 b, let us introduce $(t,\ \theta)$ above y_i, where

$$(8.2) \qquad t = -\ln|y - y_i| \quad \text{and} \quad \theta = \frac{y - y_i}{|y - y_i|} \quad \text{for } \ y \in \Omega\backslash\{y_i\}\,.$$

Here Ω is an open and bounded domain in $\mathbb{R}^n$ containing y_i. The function

$$(8.3) \qquad w_{y_i}(t, \theta) := |y - y_i|^{\frac{2}{p_i - 1}}\, v(y) \quad \text{for } \ t \in (a,\ \infty) \ \text{ and } \ \theta \in S^{n-1}$$

satisfies a equation similar to (7.9) in $(a,\ \infty) \times S^{n-1}$. Here $a > 0$ is a fixed number so that $B_{y_i}(e^{-a}) \subset \Omega$.

Assume that w is bounded from above in $(b,\ \infty) \times S^{n-1}$, where $b > a$ is a fixed number. It follows from standard elliptic theory [31] that

$$(8.4) \qquad \sup_{t_o \times S^{n-1}} w \le C \inf_{t_o \times S^{n-1}} w \quad \text{for all} \quad t_o > b\,.$$

Translating back into S^n, (8.4) is equivalent to the spherical Harnack inequality (8.1). So the key is whether w is bounded from above in $(b,\ \infty) \times S^{n-1}$, that is, whether

$$u_i(y) \le \frac{C'}{|y - y_i|^{\frac{2}{p_i - 1}}} \quad \text{for} \ \ 0 < |y - y_i| \le r_o \ (:= e^{-b})\,. \tag{8.5}$$

§ 8 b. *Analytic definition.* By the stereographic projection, we can localize the discussion on an open and bounded domain $\Omega \subset \mathbb{R}^n$. Consider the equation

$$\Delta\, v_i + c_n\, \tilde{K}_p\, v_i^p = 0 \quad \text{in} \ \ \Omega\,. \tag{8.6}$$

As in § 7, let us introduce the conditions (here c and C are fixed positive constants)

$(\mathbf{B})_\Omega$: $\tilde{K}_{p_i} \in C^1(\Omega)$ with $0 < c^2 \le \tilde{K}_{p_i} \le C^2$;

$(\mathbf{S})_\Omega$: $\{v_{p_i}\} \subset C_+^{2,\,\alpha}(S^n)\,,\ p_i \uparrow \dfrac{n+2}{n-2}$; each v_{p_i} is a solution of (8.6) with $p = p_i$ and $\tilde{K}_p = \tilde{K}_{p_i}$, where $\{\tilde{K}_{p_i}\}$ satisfies $(\mathbf{B})_\Omega$.

Definition 8.7. *Let the sequence $\{v_{p_i}\}$ satisfy $(\mathbf{S})_\Omega$. A point $\bar{y} \in \Omega$ is called an isolated blow-up point of $\{v_{p_i}\}$ if there exist*

(i) *a sequence $y_i \to \bar{y}$, such that y_i is a local maximum of v_{p_i} , $\lim\limits_{i \to \infty} v_{p_i}(y_i) = \infty$; and*

(ii) *positive numbers* $\bar{r}$ and C so that $B_{\bar{y}}(2\,\bar{r}) \subset \Omega$, *with*

$$v_{p_i}(y) \le \frac{C}{|y - y_i|^{\frac{2}{p_i - 1}}} \quad \textit{for all} \ \ y \in B_{y_i}(\bar{r}) \setminus \{y_i\} \ \ \textit{and} \ \ i = 1,\ 2, \cdots . \tag{8.8}$$

Condition (8.8) implies that there is no other blow-up point in $B_{\bar{y}}(\bar{r}/2)$. For if there is another blow-up point $\bar{y}' \in B_{\bar{y}}(\bar{r}/2) \setminus \{\bar{y}\}$, with $y_i' \to \bar{y}'$ so that $\lim\limits_{i \to \infty} v_{p_i}(y_i') = \infty$, then for $i \gg 1$ so that

$$|y_i' - y_i| \ge \frac{1}{2}\, |\bar{y}' - \bar{y}|\,,$$

we have

$$v_{p_i}(y_i') \le \frac{C}{|y_i' - y_i|^{\frac{2}{p_i - 1}}} \le \frac{\left(2^{\frac{2}{p_i - 1}}\right) \cdot C}{|\bar{y}' - \bar{y}|^{\frac{2}{p_i - 1}}} \le C' \quad \text{for} \ \ i \gg 1\,.$$

But this is a contradiction. As noted above, a major property of isolated blow-up is the spherical Harnack inequality (cf. [48]).

Theorem 8.9. *Let the sequence* $\{v_{p_i}\}$ *satisfy* $(\mathbf{S})_\Omega$, *and the sequence* $y_i \to \bar y$ *fulfill the conditions in Definition 8.7, where* $\bar y \in \Omega$ *is an isolated blow-up point. Then for any* $r \in (0, \bar r/3)$*,*

$$\sup_{\partial B_{y_i}(r)} v_{p_i} \le C \inf_{\partial B_{y_i}(r)} v_{p_i}\,. \tag{8.10}$$

Here C *is a positive constant independent on* i *and* $r \in (0, \bar r/3)$.

It is asserted that

$$v_{p_i}(y) \le C\, v_{p_i}(y_i) \quad \text{for} \ \ y \in B_{y_i}(\bar r/3) \quad \text{and} \quad i \gg 1\,, \tag{8.11}$$

where C is the same constant in (8.10). Suppose that there is a point $y_\ell \in B_{y_i}(\bar r/3)$ with $v_{p_i}(y_\ell) > C\, v_{p_i}(y_i)$. Let $r_\ell = |y_i - y_\ell|$. It follows from (8.10) that

$$v_{p_i}(y_i) < \frac{1}{C} \sup_{\partial B_{y_i}(r_\ell)} v_{p_i} \le \inf_{\partial B_{y_i}(r_\ell)} v_{p_i}\,.$$

Hence there is a local minimum in $B_{y_i}(r_\ell) \subset \Omega$, contradicting that $\Delta\, v_{p_i} < 0$ in Ω. Hence (8.11) holds.

Combining (8.11) with a blow-up procedure similar to that in § 2 a, we obtain the following (see [48]).

Proposition 8.12. *Assume that the conditions in Theorem 8.9 are satisfied. In addition, assume that* $\{\tilde K_{p_i}\}$ *is bounded in* $C^1_{\text{loc}}(\Omega)$. *Then for any given sequences* $R_i \to \infty$ *and* $\varepsilon_i \to 0^+$*, after passing to a subsequence, which is still denoted by* $\{v_{p_i}\}$ *and* $\{y_i\}$, *we have*

$$\left\| \frac{v_{p_i}(y_i + \delta_i\, y)}{v_{p_i}(y_i)} - \left(\frac{1}{1 + k|y|^2} \right)^{\frac{n-2}{2}} \right\|_{C^2(B_o(R_i))} \le \varepsilon_i\,. \tag{8.13}$$

Here $\delta_i = [v_{p_i}(y_i)]^{-\frac{p_i - 1}{2}}$, $k = c_n[n(n-2)]^{-1} \lim_{i\to\infty} \tilde K_{p_i}(y_i)$, *and* $\lim_{i\to\infty} R_i\, \delta_i = 0$.

§ 8 c. *Distinct bubbling sequences.* Typical examples of isolated blow-up are similar to the towering blow-up described in § 7 f, where the bubbles are stacked up. On the other hand, if, say, 0 is a blow-up point which is not isolated, then bubbles can 'crowd' next to each other. Before we go to that, let us introduce the following.

Definition 8.14. *Let the sequence $\{v_{p_i}\}$ satisfy $(\mathbf{S})_\Omega$. $\{y_i^b\} \subset \Omega$ is called a bubbling sequence for $\{v_{p_i}\}$ at y^b if*

(i) $\lim_{i\to\infty} y_i^b = y^b \in \Omega$; and

(ii) *there exist positive numbers $\varepsilon_i \to 0$, $R_i \to \infty$ and $N_i \to \infty$, such that*

$$\left\| \frac{v_{p_i}\left(y_i^b + \delta_i\, y\right)}{v_{p_i}(y_i^b)} - \left(\frac{1}{1+k|y|^2}\right)^{\frac{n-2}{2}} \right\|_{C^2(B_o(R_i))} \le \varepsilon_i \quad \text{for } i \ge N_i\,, \tag{8.15}$$

where $\delta_i = [v_{p_i}(y_i^b)]^{-\frac{p_i-1}{2}}$, $k = c_n[n(n-2)]^{-1} \lim_{i\to\infty} \tilde K_{p_i}(y_i^b)$, *and* $\lim_{i\to\infty} R_i\,\delta_i = 0$.

Definition 8.16. *Let the sequence $\{v_{p_i}\}$ satisfy $(\mathbf{S})_\Omega$ and $\{y_i^b\}$ and $\{Y_i^b\}$ be bubbling sequences for $\{v_{p_i}\}$ at y^b. $\{y_i^b\}$ and $\{Y_i^b\}$ are called* <u>*distinct bubbling sequences*</u> *if there exist sequences $\varepsilon_i \to 0$, $R_{1_i} \to \infty$ and $R_{2_i} \to \infty$ such that* (8.15) *holds for the triples $(y_i^b,\ \varepsilon_i,\ R_{1_i})$ and $(Y_i^b,\ \varepsilon_i,\ R_{2_i})$, with*

$$B_{y_i^b}\left(\frac{R_{1_i}}{[v_{p_i}(y_i^b)]^{\frac{p_i-1}{2}}}\right) \cap B_{Y_i^b}\left(\frac{R_{2_i}}{[v_{p_i}(Y_i^b)]^{\frac{p_i-1}{2}}}\right) = \emptyset \quad \text{for } i \gg 1\,.$$

Proposition 8.17. *Let the sequence $\{v_{p_i}\}$ satisfy $(\mathbf{S})_\Omega$. In addition, assume that $\{\tilde K_{p_i}\}$ is bounded in $C^1_{\rm loc}(\Omega)$. If 0 be a blow-up point of the sequence $\{v_{p_i}\}$ which is not an isolated blow-up point, then either*

(i) *in any given open nonempty neighborhood of* 0, *there is another blow-up point besides* 0 *for a subsequence of $\{v_{p_i}\}$; or*

(ii) *after passing to a subsequence, which is still denoted by $\{v_{p_i}\}$, there exist distinct bubbling sequences $\{y_i^b\}$ and $\{Y_i^b\}$ for $\{v_{p_i}\}$ at* 0.

Proof. Suppose that 0 is not an isolated blow-up point, and, for some $\rho > 0$, there is no other blow-up point (except 0) in $B_o(\rho) \subset \Omega$ for any subsequence of $\{v_{p_i}\}$. We assert that scenario (ii) must take place. Let y_i be selected so that $v_{p_i}(y_i) = \sup\limits_{B_o(\rho)} v_{p_i}$. As 0 is the only blow-up point, we conclude $v_{p_i}(y_i) \to \infty$ and 0 is the only limit point of the sequence $\{y_i\}$. After passing to a subsequence, we may assume without loss of generality that $y_i \to 0$. Hence y_i is a local maximum for v_{p_i} $(i \gg 1)$.

For any natural number N, the inequality

$$V_i(y) := |y - y_i|^{\frac{2}{p_i - 1}}\, v_{p_i}(y) \le N \qquad \text{for } \; y \in B_o(\rho) \tag{8.18}$$

does *not* hold for all $i \gg 1$ (otherwise, 0 would have been an isolated blow-up point). Let $Y_i \in \overline{B_o(\rho)}$ be chosen so that

$$M_i := V_i(Y_i) = \sup_{B_o(\rho)} V_i\,. \tag{8.19}$$

Because (8.18) does not hold, after passing to a subsequence, which is still denoted by $\{v_{p_i}\}$, we may assume (sacrificing no generality) that

$$M_i \to \infty \quad \text{ as } \; i \to \infty\,. \tag{8.20}$$

Furthermore, $y_i \ne Y_i$, as $V_i(y_i) = 0$. By (8.18) and (8.20), we have $v_{p_i}(Y_i) \to \infty$. Since there is no other blow-up point in $B_o(\rho)$ besides 0 for any subsequence of $\{v_{p_i}\}$, we also have $Y_i \to 0$ as $i \to \infty$. Hence

$$r_i := \frac{|Y_i - y_i|}{2} \to 0 \quad \text{ as } \; i \to \infty\,. \tag{8.21}$$

We apply the blow-up procedure similar to the one used in the proof of Proposition 4 in [37]. Set

$$\mu_i := \frac{1}{[v_{p_i}(Y_i)]^{\frac{p_i - 1}{2}}} \qquad (\to 0 \;\text{ as }\; i \to \infty\,) \tag{8.22}$$

and

$$w_i\,(y) := \frac{v_{p_i}(Y_i + \mu_i\, y)}{v_{p_i}(Y_i)} \quad \text{ for } \; y \in B_o(R_i)\,, \quad i = 1,\; 2,\; \cdots. \tag{8.23}$$

Here

$$R_i := \frac{r_i}{\mu_i} = \frac{1}{2}\left[|Y_i - y_i|^{\frac{2}{p_i-1}}\, v_{p_i}(Y_i)\right]^{\frac{p_i-1}{2}} = \frac{1}{2} M_i^{\frac{p_i-1}{2}} \to \infty \text{ as } i \to \infty \quad [\text{by (8.20)}]\,.$$

For $i \gg 1$ and $y \in B_{Y_i}(r_i) \subset B_o(\rho)$, we have

$$\begin{aligned} & V_i(y) \le V_i(Y_i) \quad \text{and} \quad |y - y_i| \ge r_i = \frac{|Y_i - y_i|}{2}\,, \\ \Longrightarrow\ & |y - y_i|^{\frac{2}{p_i-1}}\, v_{p_i}(y) \le |Y_i - y_i|^{\frac{2}{p_i-1}}\, v_{p_i}(Y_i) \qquad [\text{by the definition in (8.18)}] \\ \Longrightarrow\ & v_{p_i}(y) \le 2^{\frac{2}{p_i-1}}\, v_{p_i}(Y_i)\,. \end{aligned}$$

That is,

$$w_i(y) \le 2^{\frac{2}{p_i-1}}\, w_i(0) = 2^{\frac{2}{p_i-1}} \quad \text{for } y \in B_o(R_i)\,, \quad i \gg 1\,. \tag{8.24}$$

Furthermore, w_i satisfies the equation

$$\Delta_o\, w_i(y) + c_n\, K_{p_i}(Y_i + \mu_i\, y)\, w_i^{p_i}(y) = 0 \quad \text{in } B_o(R_i)\,, \quad i \gg 1\,. \tag{8.25}$$

Standard elliptic theory implies that there is a subsequence, which is still denoted by $\{w_i\}$, converges in C^2 norm on compact subsets to a C^2 function w satisfying the equation

$$\Delta w + c_n K_\infty(0)\, w^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n. \tag{8.26}$$

The classification theorem of Gidas, Ni and Nirenberg (loc. cit. in § 2 a) implies that

$$w(y) = \left(\frac{1}{1 + k|y - y_o|^2}\right)^{\frac{n-2}{2}} \quad \text{for } y \in \mathbb{R}^n. \tag{8.27}$$

where $k = c_n[n(n-2)^{-1}]\, K_\infty(0)$. It follows from (8.21) and the definition of R_i that

$$\lim_{i\to\infty} R_i\, \mu_i = \lim_{i\to\infty} r_i = 0\,. \tag{8.28}$$

The maximum of w occurs at y_o. Thus for $i \gg 1$, w_i has a local maximum at a point y_i^s whose distant to 0 is less than C. Here C is a positive constant that depends on n and y_o only. Hence v_{p_i} has a local maximum at the point $Y_i^b := Y_i + \mu_i\, y_i^s$. Observe that

$$(8.29) \qquad |Y_i^b - Y_i| \le |y_i^s|\mu_i \le C\,\mu_i \;\le\; r_i \qquad [\,\text{by}\;\; (8.28)\,]\,.$$

By shifting the center to Y_i^b in the blow-up process in (8.23), using (8.28) and (8.29), (after possibly taking a subsequence, which is still denoted by the original one), we obtain a bubbling sequence sequence as specified in Definition 8.14. (Cf. also the argument proceeding equation (16) in [37].)

Apply the usual blow-up procedure for $\{y_i\}$ (because $v_{p_i}(y_i) = \sup\limits_{B_o(\rho)} v_{p_i}$, the convergence of a subsequence is guaranteed). After taking a subsequence, which is still denoted by $\{v_{p_i}\}$, we may take $\{y_i^b\} = \{y_i\}$. To show that there are enough spaces for the bubbling sequences $\{y_i^b\}$ and $\{Y_i^b\}$, that is, to show that (8.17) holds, we need only to expound

$$(8.30) \qquad R_{1_i} := [v_{p_i}(y_i)]^{\frac{p_i-1}{2}}\, r_i = [v_{p_i}(y_i)]^{\frac{p_i-1}{2}} \cdot \frac{|y_i - Y_i|}{2} \to \infty \qquad \text{as} \;\; i \to \infty\,.$$

Suppose that on the contrary,

$$[v_{p_i}(y_i)]^{\frac{p_i-1}{2}}\, |y_i - Y_i| \le C \quad \text{for} \;\; i \gg 1 \quad \Longrightarrow \quad v_{p_i}(y_i) \le \frac{C}{|y_i - Y_i|^{\frac{2}{p_i-1}}} \quad \text{for} \;\; i \gg 1\,.$$

On the other hand

$$(8.31) \qquad V_i(Y_i) = |y_i - Y_i|^{\frac{2}{p_i-1}}\, v_{p_i}(Y_i) > C \quad \text{for} \;\; i \gg 1 \quad \Longrightarrow \quad v_{p_i}(Y_i) > v_{p_i}(y_i)\,.$$

As soon as $i \gg 1$, $Y_i \in B_o(\rho)$. Thus (8.31) contradicts $v_{p_i}(y_i) = \sup\limits_{B_o(\rho)} v_{p_i}$. Hence we arrive at (8.30). $\square$

In dimension two, X.-X. Chen constructs examples where situation similar to scenario (ii) in Proposition 8.17 occurs [21]. The spiky blow-up described in § 7 g also carries this property.

Proposition 8.32. *Let the sequence* $\{v_{p_i}\}$ *satisfy* $(\mathbf{S})_\Omega$ *. On top of that, assume that* $\{\tilde K_{p_i}\}$ *is bounded in* $C^1_{\rm loc}(\Omega)$ *. If there exist two distinct bubbling sequences* $\{y_i\}$ *and* $\{y_i^b\}$ *for* $\{v_{p_i}\}$ *at* 0 *, then* 0 *is not an isolated blow-up point for* $\{v_{p_i}\}$ *(nor for any of its subsequences).*

Proof. Suppose that, on the contrary, 0 is an isolated blow-up point for $\{v_{p_i}\}$. We may assume without loss of generality that the sequence $\{y_i\}$ satisfies the condition in Definition 8.7. We then have the spherical Harnack inequality as described in Theorem 8.9. Consider the sphere with center at y_i and radius $r_i := |y_i - y_i^b|$. For the sequence $\{y_i^b\}$, we have (8.15) in Definition 8.14. By Definition 8.16, there is a point $y_\ell \in \partial B_{y_i}(r_i) \cap \partial B_{y_i^b}(\delta_i\, R_i)$. It follows from (8.15) that

$$(8.33)\quad v_{p_i}(y_\ell) \le \left[\left(\frac{1}{1 + k\, R_i^2} \right)^{\frac{n-2}{2}} + \varepsilon_i \right] v_{p_i}(y_i^b) \quad \Longrightarrow \quad v_{p_i}(y_\ell) \le o(1) \cdot [v_{p_i}(y_i^b)] \, .$$

As y_ℓ and y_i^b are in $\partial B_{y_i}(r_i)$, (8.33) contradicts spherical Harnack inequality (8.10) when $i \gg 1$. Hence 0 is not an isolated blow-up point for $\{v_{p_i}\}$. □

Propositions 8.16 and 8.32 together with the discussion proceeding Definition 8.7 imply the following.

Corollary 8.34. *Let the sequence* $\{v_{p_i}\}$ *satisfy* $(\mathbf{S})_\Omega$ *. On top of that, assume that* $\{\tilde K_{p_i}\}$ *is bounded in* $C^1_{\rm loc}(\Omega)$ *. Then the origin* 0 *is an isolated blow-up point for the sequence* $\{v_{p_i}\}$ *if and only if there is a nonempty open neighborhood of* 0 *in which* 0 *is the only blow-up point for any subsequence of* $\{v_{p_i}\}$ *, and for any given subsequence of* $\{v_{p_i}\}$ *, there is only one distinct bubbling sequence at* 0 *(i.e., there is no other bubbling sequence for the subsequence of* $\{v_{p_i}\}$ *at* 0 *which is distinct to the one already chosen).*

Proposition 8.35. *Let* 0 *be an isolated blow-up point for* $\{v_{p_i}\}$ *which satisfies* $(\mathbf{S})_\Omega$ *with* $\tilde K_{p_i}$ *being bounded in* $C^1_{\rm loc}(\Omega)$ *. Assume that*

(i) $v_{p_i} \to v_\infty > 0$ *in* $C^{2,\,\alpha}_{\rm loc}\,(B_o(\rho) \setminus \{0\})$ *, and*

(ii) $K_{p_i} \to K$ *in* $C^1(B_o(\rho))$ *[with* $K \in C^1(B_o(\rho))$*] .*

Here $\rho > 0$ and $B_o(\rho) \subset \Omega$. Then we have

$$v_\infty(y) \le C\,|y|^{-\frac{n-2}{2}} \quad \textit{for} \;\; y \in B_o(\rho) \setminus \{0\}\,. \tag{8.36}$$

Proof. Under the conditions, v_∞ satisfies the equation

$$\Delta_o\, v_\infty + c_n K v_\infty^{\frac{n+2}{n-2}} = 0 \quad \text{in} \;\; B_o(\rho) \setminus \{0\}\,. \tag{8.37}$$

Suppose that (8.36) fails to hold. Then the procedure described in [37] can be used to show that there exists a sequence $\{y_i\} \subset B_o(\rho)$ of local maxima of v with the following properties

(i) $\lim_{i\to\infty} y_i = 0$ and $\lim_{i\to\infty} v_\infty(y_i) = \infty\,,$ and

(ii) there exist positive numbers $\varepsilon_i \to 0$ and $R_i \to \infty$ such that

$$\left\| \frac{v_\infty\,(y_i + \mu_i\, y)}{v_\infty(y_i)} - \left(\frac{1}{1 + k|y|^2} \right)^{\frac{n-2}{2}} \right\|_{C^2(B_o(R_i))} \le \varepsilon_i\,, \tag{8.38}$$

where $\mu_i = [v_\infty(y_i)]^{\frac{2}{2-n}}\,,\; k = c_n[n(n-2)]^{-1}K(0)\,,$ and $\lim_{i\to\infty} R_i\,\mu_i = 0\,.$

(Cf. also the proof of Proposition 8.14 and [44].) Furthermore, by choosing a subsequence, which we still denote by $\{y_i\}$, we have $B_{y_i}(\mu_i\, R_i) \cap B_{y_j}(\mu_j\, R_j) = \emptyset$ for $i \ne j$. Thus $\{y_{2i}\}$ and $\{y_{2i+1}\}$ are two distinct bubbling sequences for $\{v_{p_i}\}$ at 0. It follows from Proposition 8.32 that 0 is not an isolated blow up point for $\{v_{p_i}\}$. But this is a contradiction. □

§ 8 d *Turns.* Let us consider the limit v_∞ in the log-cylindrical coordinates:

$$w_\infty(t,\,\theta) = |y|^{\frac{n-2}{2}} v_\infty(y)\,, \quad \text{where} \;\; t = -\ln|y|\,,\;\; \theta = y/|y| \quad \text{for} \;\; 0 < |y| < \rho\,.$$

Because of (8.36), which leads to the spherical Harnack inequality [15] (cf. [22]), one can analyze w effectively by using O.D.E. methods on

$$\bar{w}_\infty(t) := \int_{S^{n-1}} w_\infty(t,\,\theta)\, d\theta = r^{\frac{n-2}{2}} \left[\frac{\operatorname{Vol} S^{n-1}}{\operatorname{Vol} \partial B_o(r)} \int_{\partial B_o(r)} v_\infty\, dS \right],$$

where $r = e^{-t}$. In particular, the singularity at 0 is studied in great details for many forms of K (cf. [13], [14], [15], [22], [41], [42], [43], [53], [69] and the references inside). It is understood that, in general, there are three scenarios.

(i) v_∞ has a removable singularity at 0.

(ii) w_∞ is asymptotic to a Delaunay-Fowler type solution.

(iii) w_∞ 'degenerates' so that it has no positive lower bounds.

In the second and third cases, $\bar{w}_\infty(t)$ has infinite number of critical points (or turns) in $(-\ln \rho\,,\ \infty)$. This fine insight leads to an analytic definition of *simple* (isolated) blow-up, first propounded by Schoen [59].

9. Simple blow-up.

Definition 9.1. *Let $y_i \to \bar{y} \in \Omega$ be an isolated blow-up point for the sequence $\{v_{p_i}\}$, which satisfies $(\mathbf{S})_\Omega$. Set*

$$\bar{w}_{p_i}(r) = r^{\frac{2}{p_i - 1}} \cdot \left[\frac{\mathrm{Vol}\, S^{n-1}}{\mathrm{Vol}\, \partial B_{y_i}(r)} \int_{\partial B_{y_i}(r)} u_{p_i}\, dS \right] \qquad for \ \ r > 0 \quad with \ \ \partial B_{y_i}(r) \subset \Omega\,.$$

Suppose that there exists a positive number ρ, independent of i, such that for $i \gg 1$, $\bar{w}_{p_i}$ has precisely one critical point in $(0,\ \rho)$, then we call $\bar{y}$ a simple blow-up point. (In literature, one often uses the terms '*simple isolated blow-up point*' instead.)

Let us consider (again) the example in § 7 e:

$$v_i(y) = \left(\frac{\lambda_i}{\lambda_i^2 + |y|^2} \right)^{\frac{n-2}{2}}, \qquad \lambda_i \to 0^+ \quad \text{as} \ \ i \to \infty\,.$$

Note that the maximum value for v_i is $v_i(0) = \lambda_i^{-\frac{n-2}{2}}$. We have

$$(9.2) \qquad v_i(0) \cdot v_i(y) = \left(\frac{1}{\lambda_i^2 + |y|^2} \right)^{\frac{n-2}{2}} \longrightarrow \frac{1}{|y|^{n-2}} \quad \text{as} \ \ \lambda_i \to 0^+ \quad (y \neq 0).$$

Similarly, for any fixed point $y_o \neq 0$,

$$(9.3) \qquad \frac{v_i(y)}{v_i(y_o)} = \left(\frac{\lambda_i^2 + |y_o|^2}{\lambda_i^2 + |y|^2} \right)^{\frac{n-2}{2}} \longrightarrow \frac{|y_o|^{n-2}}{|y|^{n-2}} \quad \text{as} \ \ \lambda_i \to 0^+ \quad (y \neq 0).$$

We recognize that the last terms in (9.2) and (9.3) are harmonic functions with singularity at 0. This special property is reflected in simple isolated blow-up [48].

In what follows we consider the equation in an open domain in $\mathbb{R}^n$ by using the stereographic projection (cf. § 7 a).

Proposition 9.4 (limit harmonicity). *Let $y_i \to y^b \in \Omega$ be an simple blow-up point for the sequence $\{v_{p_i}\}$ which satisfies $(\mathbf{S})_\Omega$. Assume that $|\bigtriangledown \tilde{K}_{p_i}| \le B$ in Ω for $i = 1,\ 2,\ 3, \cdots$. Then there exist positive constants ρ and C such that*

$$v_{p_i}(y_i) \cdot v_{p_i}(y) \le \frac{C}{|y - y_i|^{n-2}} \qquad \textit{for all} \quad |y - y_i| \le \rho\,. \tag{9.5}$$

Furthermore, there is a harmonic function b in $B_{\bar{y}}(\rho)$ such that (after passing to a subsequence, which is still denoted by $\{v_{p_i}\}$)

$$v_{p_i}(y_i) \cdot v_{p_i}(y) \;\longrightarrow\; h(y) = \frac{A}{|y|^{n-2}} + b(y) \quad \textit{in} \quad C^2_{\text{loc}}(B_{\bar{y}}(\rho) \setminus \{y^b\})\,, \tag{9.6}$$

where

$$A = \left[\frac{n(n-2)}{c(n) \lim\limits_{i \to \infty} K_i(y_i)} \right]^{\frac{n-2}{2}} .$$

Key steps in the proof. We refer to [48] for a complete proof. The key point is that, for simple blow-up, $v_{p_i}(y_i + \epsilon) \sim [v_{p_i}(y_i)]^{-1}$ as $i \gg 1$, where $\vec{\epsilon} = (0, \cdots, 0, \rho/2)$ is fixed. This enable us to perform the *lift-up* argument (we may assume that $y^b = 0$):

$$\underline{V_i}(y) = \frac{v_{p_i}(y)}{v_{p_i}(y_i + \vec{\epsilon})} \qquad \text{for} \quad y \in B_o(\rho)\,.$$

v_i satisfies the equation

$$\Delta_o \underline{V_i} + c_n \left[\frac{K_{p_i}(y)}{[\, v_{p_i}(y_i + \vec{\epsilon})]^{p_i - 1}} \right] v_i^{p_i} = 0 \quad \text{in} \quad B_o(\rho)\,. \tag{9.7}$$

By the Harnack inequality, $\underline{V_i}$ is bounded from above on $B_o(\rho) \setminus B_o(k^{-1})$, where $k \gg 1$ is an integer. Hence by standard elliptic theory a subsequence of $\{\underline{V_i}\}$ converges in $C^2_{\text{loc}}(B_o(\rho) \setminus \{0\})$ to a function V. It follows from (9.7) that V is a harmonic function in $B_o(\rho) \setminus \{0\}$. It can be shown that 0 is a nonremovable singularity for V. Hence we have (9.6). □

§ 9 a. *Pohozaev identity.* In order to take advantage of the information available in (9.6), we consider the Pohozaev's identity [48]. In equation (7.4), we let

$$H(y) := \left(\frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}} = 2^{\frac{n-2}{2}} U_1(y) \quad \text{for} \;\; y \in {\rm I\!R}^n .$$

Proposition 9.8. *Let $p > 1$ and v be a positive C^2 solution of the equation*

$$\Delta_o \, v + c_n \, (H^\Upsilon \tilde K) \, v^p = 0 \qquad in \;\; B_o(R) \, .$$

We have

$$\begin{aligned} \int_{B_o(\rho)} [y \cdot \bigtriangledown (H^\Upsilon \tilde K)] \, v^{p+1} \, dy + \frac{n-2}{2} \cdot \Upsilon \int_{B_o(\rho)} (H^\Upsilon \tilde K) \, v^{p+1} \, dy \\ - \int_{\partial B_o(\rho)} \rho \, (H^\Upsilon \tilde K) \, v^{p+1} \, dS \\ = \frac{p+1}{c_n} \int_{\partial B_o(\rho)} \left[\rho \left(\frac{\partial v}{\partial \nu} \right)^2 - \frac{|\bigtriangledown v|^2}{2} \, \rho + \frac{n-2}{2} \, v \frac{\partial v}{\partial \nu} \right] dS \, , \end{aligned} \tag{9.9}$$

where $\Upsilon = (n + 2)/(n - 2) - p$, ν is the outward unit normal on $\partial B_o(\rho)$, and $R > \rho > 0$.

Let us group together the relevant terms. Define

$$\begin{aligned} \mathbf{I} \;\; &:= \;\; \int_{B_o(\rho)} [y \cdot \bigtriangledown (H^\Upsilon \tilde K)] \, v^{p+1} \, dy \, , \\ \mathbf{II} \;\; &:= \;\; \Upsilon \int_{B_o(\rho)} (H^\Upsilon \tilde K) \, v^{p+1} \, dy \ge 0 \quad \text{for} \;\; 1 < p \le \frac{n+2}{n-2} \;\; \text{and} \;\; \tilde K \ge 0 \, , \\ \mathbf{III} \;\; &:= \;\; \int_{\partial B_o(\rho)} \rho \, (H^\Upsilon \tilde K) \, v^{p+1} \, dS \, , \;\; \text{and} \\ \mathbf{IV} \;\; &:= \;\; \int_{\partial B_o(\rho)} \left[\rho \left(\frac{\partial v}{\partial \nu} \right)^2 - \frac{|\bigtriangledown v|^2}{2} \, \rho + \frac{n-2}{2} v \frac{\partial v}{\partial \nu} \right] dS \, . \end{aligned}$$

A key property of **IV** is revealed by the following elegant result (see [48]).

Lemma 9.10. *Let $w(y) = A \, |y|^{-(n-2)} + B + \alpha(y)$ for $y \ne 0$, where A and B are positive numbers, and α a C^1 function with $\alpha(0) = 0$. Let $c(n) = \dfrac{(n-2)^2}{2} \, {\rm Vol} \, (S^{n-1})$.*

We have

$$\lim_{\rho\to 0^+}\int_{\partial B_o(\rho)}\left[\rho\left(\frac{\partial w}{\partial \nu}\right)^2-\frac{|\bigtriangledown w|^2}{2}\,\rho+\frac{n-2}{2}w\frac{\partial w}{\partial \nu}\right]\,dS\;=-c(n)\,A\cdot B\,.$$

Proof. We have

$$\frac{\partial w}{\partial \nu}=\frac{\partial w}{\partial r}=-(n-2)\,\frac{A}{r^{n-1}}+\frac{\partial \alpha}{\partial r}\,,\qquad \text{where}\;\; r=|y|>0\,.$$

Furthermore,

$$\begin{aligned}
r\left(\frac{\partial w}{\partial r}\right)^2 &= (n-2)^2\frac{A^2}{r^{2n-3}}-2\,(n-2)\,\frac{A}{r^{n-2}}\frac{\partial \alpha}{\partial r}+r\left(\frac{\partial \alpha}{\partial r}\right)^2\,,\\
-\frac{r}{2}\,|\bigtriangledown w|^2 &= -\frac{(n-2)^2}{2}\,\frac{A^2}{r^{2n-3}}+(n-2)\,\frac{A}{r^{n-2}}\frac{\partial \alpha}{\partial r}-\frac{r}{2}\,|\bigtriangledown \alpha|^2\,,\\
\frac{n-2}{2}\,w\frac{\partial w}{\partial \nu} &= -\frac{(n-2)^2}{2}\,\frac{A^2}{r^{2n-3}}+\frac{n-2}{2}\,\frac{A}{r^{n-2}}\frac{\partial \alpha}{\partial r}-\frac{(n-2)^2}{2}\,\frac{AB}{r^{n-1}}+\frac{n-2}{2}B\,\frac{\partial \alpha}{\partial r}\\
&\qquad -\frac{(n-2)^2}{2}\,\frac{A}{r^{n-1}}\cdot\alpha(y)+\frac{n-2}{2}\,\frac{\partial \alpha}{\partial r}\cdot\alpha(y)\,.
\end{aligned}$$

The leading terms in the above are canceled when the three formulas are added together. It follows that

$$\begin{aligned}
&\lim_{\rho\to 0^+}\int_{\partial B_o(\rho)}\left[\rho\left(\frac{\partial w}{\partial \nu}\right)^2-\frac{|\bigtriangledown w|^2}{2}\,\rho+\frac{n-2}{2}w\frac{\partial w}{\partial \nu}\right]\,dS\\
=&\;-\frac{(n-2)^2}{2}\lim_{\rho\to 0^+}\int_{\partial B_o(\rho)}\frac{A\,[B+\alpha(y)]}{\rho^{n-1}}\,dS=-\left[\frac{(n-2)^2}{2}\,\mathrm{Vol}\,(S^{n-1})\right]\,A\cdot B\,.
\end{aligned}$$

Hence the limit is verified. □

The boundary integral in **IV** can be viewed as a link between the bubble and the other part. Also, when the harmonic function $A\,|y|^{2-n}+B$ is expressed in the log-cylindrical coordinates, the quantity $A\cdot B$ is equal to the square of the neck-size (cf. § 7 c).

Applying Pohozaev's identity (9.9) to the sequence $\{v_{p_i}\}$ which satisfies the conditions in Proposition 9.4, using y_i as the center, and multiplying both sides by $M_i^2 = v_{p_i}^2(y_i)\,,$ we obtain

$$(9.11) \qquad M_i^2 \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown (H^{\Upsilon_i} K_{p_i})]\, v_{p_i}^{p_i+1}\, dy + \frac{n-2}{2}\cdot \Upsilon_i\, M_i^2 \int_{B_{y_i}(\rho)} (H^{\Upsilon_i} K_{p_i})\, v_{p_i}^{p+1}\, dy - \frac{1}{M_i^{p_i-1}} \int_{\partial B_{y_i}(\rho)} \rho\, (H^{\Upsilon_i} K_{p_i})\, (M_i\, v_{p_i})^{p_i+1}\, dS$$

$$= \frac{p_i+1}{c_n} \int_{\partial B_{y_i}(\rho)} \left[\rho \left(\frac{\partial (M_i\, v_{p_i})}{\partial \nu} \right)^2 - \frac{|\bigtriangledown (M_i\, v_{p_i})|^2}{2}\,\rho + \frac{n-2}{2}\, (M_i\, v_{p_i})\, \frac{\partial (M_i\, v_{p_i})}{\partial \nu} \right] dS\,.$$

Proposition 9.4 implies that

$$(9.12) \qquad \left| \frac{1}{M_i^{p_i-1}} \int_{\partial B_{y_i}(\rho)} \rho\, (H^{\Upsilon_i} K_{p_i})\, (M_i\, v_{p_i})^{p_i+1}\, dS \right| \le \frac{C}{M_i^{p_i-1}}\, \rho^{(2-n)p_i} \to 0$$

as $i \to \infty$. Using Lemma 9.10, we prearrange a small number $\rho > 0$ and obtain

$$(9.13) \qquad \int_{\partial B_{y_i}(\rho)} \left[\rho \left[\frac{\partial (M_i\, v_{p_i})}{\partial \nu} \right]^2 - \frac{|\bigtriangledown (M_i\, v_{p_i})|^2}{2}\,\rho + \frac{n-2}{2}\, (M_i\, v_{p_i})\, \frac{\partial (M_i\, v_{p_i})}{\partial \nu} \right] dS$$

$$= \; - \left[\frac{(n-2)^2}{2}\, \mathrm{Vol}\,(S^{n-1}) \right] A\cdot B \, + \, o(1) \qquad\qquad \text{for} \;\; i \gg 1\,.$$

As in (9.6), here $B = b(0)$.

The second term in (9.11) is known to be nonnegative (cf. **II**). So the major challenge is to estimate the term

$$(9.14) \qquad M_i^2 \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown (H^{\Upsilon_i} K_{p_i})]\, v_{p_i}^{p_i+1}\, dy$$

$$= \; M_i^2 \int_{B_{y_i}(\rho)} H^{\Upsilon_i} [(y-y_i)\cdot \bigtriangledown K_{p_i}]\, v_{p_i}^{p_i+1}\, dy + M_i^2 \cdot \Upsilon_i \int_{B_{y_i}(\rho)} H^{\Upsilon_i - 1} K_{p_i} [(y-y_i)\cdot \bigtriangledown H]\, v_{p_i}^{p_i+1}\, dy\,.$$

§ 9 b. *Reasonable expectations.* Before we go to the method to estimate (9.14), we use a simple example to illustrate what are the likely outcomes. Let

$$\mathcal{U}_i(y) = \left(\frac{\lambda_i}{\lambda_i^2 + r^2} \right)^{q_i}, \qquad \text{where} \;\; 0 < q_i \uparrow \frac{n-2}{2}\,, \qquad r = |y - y_i|\,.$$

Here $y_i \to 0$. We have

$$(9.15)\quad \Delta_o \mathcal{U}_i \;=\; \mathcal{U}_i'' + \frac{n-1}{r}\mathcal{U}_i' = -2\,n\,q_i\left[1 + \frac{2}{n}\,\frac{r^2}{\lambda_i^2}\left(\frac{\Upsilon_i}{2(p_i-1)(n-2)}\right)\right]\mathcal{U}_i^{p_i}$$
$$\left(\text{where}\ \ p_i = \frac{q_i+2}{q_i} \uparrow \frac{n+2}{n-2}\,, \qquad \Upsilon_i = \frac{n+2}{n-2} - p_i\right)$$

$$\Longrightarrow\quad \Delta_o \mathcal{U}_i + 2\,n\,q_i\left[1 + \frac{r^2}{n(p_i-1)(n-2)}\,\frac{\Upsilon_i}{\lambda_i^2}\right]\mathcal{U}_i^{p_i} = 0$$

$$\Longrightarrow\quad \Delta_o \mathcal{U}_i + K_i\,\mathcal{U}_i^{p_i} = 0$$
$$\left(\text{where}\ \ K_i := 2\,n\,q_i\left[1 + \frac{r^2}{n(p_i-1)(n-2)}\,\frac{\Upsilon_i}{\lambda_i^2}\right]\right)\,.$$

For blow-up to occur, we take $\lambda_i \to 0^+$. In other for the term in the square brackets in (9.15) to stay bounded, we need

$$\frac{\Upsilon_i}{\lambda_i^2} \le C \quad\Longleftrightarrow\quad \Upsilon_i \le C\,\lambda_i^2\,.$$

Hence we have

$$(9.16)\qquad M_i := \max \mathcal{U}_i = \lambda_i^{-q_i} = \lambda_i^{-\frac{2}{p_i-1}}$$
$$\Longrightarrow\quad \lambda_i = M_i^{-\frac{p_i-1}{2}}$$
$$\Longrightarrow\quad \Upsilon_i \le C\,M_i^{-(p_i-1)} \qquad (\text{by}\ \ (9.15)\,,\ \ \text{recall that}\ \ p_i \uparrow (n+2)/(n-2)\,)\,.$$

In particular,

$$(9.17)\qquad \lim_{i\to\infty} M_i^{\Upsilon_i} = 1 \qquad (\,\text{cf.}\ \ (7.16)\ \&\ (9.21))\,.$$

It follows from (9.16) and (9.17) that

$$(9.18)\qquad M_i \le C\,\Upsilon_i^{-\frac{n-2}{4}} \qquad (\text{cf.}\ \ \text{Proposition}\ \ 1.1\ \ \text{in}\ \ [63])\,.$$

We intend to see what may happen when $y_i \ne 0$. If we take $\Upsilon_i = \nu\,\lambda_i^2$, where $\nu \ge 0$ is a constant, we have

$$(9.19)\qquad K_i(y) = 2\,n\,q_i\left[1 + \frac{r^2}{n(p_i-1)(n-2)}\,\nu\right] \;\longrightarrow\; n(n-2)[1 + 4\,n\,\nu\,r^2\,]$$

as $i \to \infty$. *Assume that*

$$K_i(y) = n(n-2) + O(|y|^h) \quad \text{for} \ \ |y| \approx 0\,.$$

Here h is a fixed positive number. By (9.19), we have

$$\begin{aligned} & |K_i(y_i) - n(n-2)| = O(|y_i|^h) \quad \text{for} \ \ i \gg 1 \\ \Longrightarrow \quad & |n(n-2) - 2\,q_i\,n| = O(|y_i|^h) \qquad (\text{as} \ \ r = |y - y_i| = 0 \quad \text{when} \ \ y = y_i) \\ \Longrightarrow \quad & \Upsilon_i = O(|y_i|^h)\,. \end{aligned}$$

Combining with (9.16) we obtain

$$M_i \le C\,|y_i|^{-\frac{h}{p_i - 1}}\,.$$

In particular, if $h = 2$ (cf. the situation when K_i is C^3 and 0 is a critical point), then

$$\text{(9.20)} \qquad M_i \le C|y_i|^{-\frac{2}{p_i-1}} \ \Longrightarrow \ M_i \le C'|y_i|^{-\frac{n-2}{2}} \ \Longleftrightarrow \ |y_i| \le C M_i^{-\frac{2}{n-2}}\,,$$

where we use (9.17). Cf. the decay estimate in (8.36). (9.20) indicates that when K_i is flat enough at 0, the flexibility of the bubbles (expressed as how freely y_i and M_i can change independently on each other) is restricted.

§ 9 c. *Integral estimates.* To begin with the estimate on (9.14), we first observe that

$$\text{(9.21)} \qquad \int_{B_o(\rho)} \left(\frac{\lambda}{\lambda^2 + |y|^2} \right)^n dy = C(n) + O(\lambda^n) \quad \text{for} \ \ \lambda > 0 \ \ \text{close} \ \ \text{to} \ \ 0\,.$$

Here ρ is a fixed positive number. We expect similar formula to hold for simple blow-up:

$$\text{(9.22)} \qquad \int_{B_{y_i}(\rho)} v_{p_i}^{p_i+1}\,dy = C(n) + o(1) \quad \text{for} \ \ i \gg 1\,.$$

As u_i concentrates around y_i, one would expect the integral to become smaller if we multiple the integrand by $|y - y_i|^\sigma$. Here $\sigma > 0$ is a fixed number. This is made precise by the following (see [48]).

Proposition 9.23. *Under the assumption of Proposition 9.4, after passing to a subsequence, which is still denoted by* $\{v_{p_i}\}$, *we have*

$$\int_{B_{y_i}(\rho)} |y-y_i|^\sigma \, v_{p_i}^{p_i+1} \, dy \; = \; \begin{cases} O\left(M_i^{-2\sigma/(n-2)}\right) & for \;\; 0 < \sigma < n\,, \\ O\left(M_i^{-2n/(n-2)} \cdot \ln M_i\right) & for \;\; \sigma = n\,, \\ O\left(M_i^{-2n/(n-2)}\right) & for \;\; \sigma > n\,. \end{cases} \tag{9.24}$$

Here $i \gg 1$ *and* $M_i = v_{p_i}(y_i)$.

Observe that once $\sigma > n$, the decrease near the center is balanced by the increase at the periphery so that there is no net gain in the order of decay, which is maximized at $O(M_i^{-2n/(n-2)})$.

We derive (9.22) to illustrate the method used to obtain 9.24. As in Proposition 8.12, we choose the sequences of positive numbers $\{R_i\}$ and $\{\varepsilon_i\}$ such that

$$R_i \to \infty\,, \quad \varepsilon_i \to 0\,, \quad \varepsilon_i\, R_i^n \to 0 \qquad \text{as} \;\; i \to \infty\,. \tag{9.25}$$

Let $\delta_i = M_i^{-(p_i-1)/2}$ and $r_i = R_i\,\delta_i$. By (8.13), we have (after passing to a subsequence, which is still denoted by $\{v_{p_i}\}$, and without lost of generality, we assume that $k = 1$)

$$\left| v_{p_i}(y) - \left(M_i^{\Upsilon_i}\right)^{\frac{n-2}{4}} \left(\frac{\lambda_i}{\lambda_i^2 + |y-y_i|^2} \right)^{\frac{n-2}{2}} \right| \le \varepsilon_i\, M_i \qquad \text{for} \;\; y \in B_{y_i}(r_i)\,. \tag{9.26}$$

Here, as before, $\Upsilon_i = (n+2)/(n-2) - p_i$. We use $\lambda_i = \delta_i$. Recall the simple inequality,

$$|x^{p+1} - y^{p+1}| \le \; (p+1)\,|x-y| \cdot \max\{\, x^p, \;\; y^p \,\}\,, \tag{9.27}$$

which holds for positive numbers $x,\; y$ and p .

Under the condition of Proposition 9.23, it is known that [48]

$$\lim_{i\to\infty} M_i^{\Upsilon_i} = 1\,. \tag{9.28}$$

Thus, for $y \in B_{y_i}(r_i)$ (where $r_i = R_i\,\delta_i$) , we obtain

$$
\begin{aligned}
(9.29)\qquad & \left| \int_{B_{y_i}(r_i)} v_{p_i}^{p_i+1}\, dy - (M_i^{\Upsilon_i})^{\frac{n-2}{4}(p_i+1)} \int_{B_{y_i}(r_i)} \left(\frac{\lambda_i}{\lambda_i^2 + |y-y_i|^2} \right)^{\frac{n-2}{2}(p_i+1)} dy \right| \\
\le\ & \int_{B_{y_i}(r_i)} \left| v_{p_i}^{p_i+1} - \left[(M_i^{\Upsilon_i})^{\frac{n-2}{4}} \left(\frac{\lambda_i}{\lambda_i^2 + |y-y_i|^2} \right)^{\frac{n-2}{2}} \right]^{p_i+1} \right| dy \\
\le\ & C \left[(\varepsilon_i M_i) \cdot \max \left\{ M_i^{p_i},\ \ (M_i^{\Upsilon_i})^{\frac{n-2}{4}p_i} \cdot M_i^{\frac{p_i-1}{2}\frac{n-2}{2}p_i} \right\} \right] \int_{B_{y_i}(r_i)} dy \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (7.26) \& (9.27))} \\
\le\ & C\, (\varepsilon_i M_i)\, M_i^{p_i}\, \mathrm{Vol}\,(B_{y_i}(r_i)) = C\, \mathrm{Vol}\,(B_o(1))\, \varepsilon_i\, M_i^{p_i+1}\, r_i^n \\
\le\ & C'\, \varepsilon_i \left[\frac{R_i}{M^{(p_i-1)/2}} \right]^n M_i^{p_i+1} = C'\, (\varepsilon_i R_i^n)(M^{\Upsilon_i})^{\frac{n-2}{2}} \longrightarrow 0 \quad \text{when} \ \ i \to \infty
\end{aligned}
$$

$$
\left(\text{as} \ \ \lim_{i\to\infty} \varepsilon_i R_i^n = 0\,, \ \ \lim_{i\to\infty} M_i^{\Upsilon_i} = 1 \ \ \text{and} \ \ \lim_{i\to\infty} p_i = \frac{n+2}{n-2} \right).
$$

Likewise,

$$
\begin{aligned}
(9.30)\qquad & (M_i^{\Upsilon_i})^{\frac{n-2}{4}(p_i+1)} \int_{B_{y_i}(r_i)} \left(\frac{\lambda_i}{\lambda_i^2 + |y-y_i|^2} \right)^{\frac{n-2}{2}(p_i+1)} dy \\
=\ & (M_i^{\Upsilon_i})^{\frac{n-2}{4}} \int_{B_{y_i}(r_i)} \left(\frac{\lambda_i}{\lambda_i^2 + |y-y_i|^2} \right)^{n} dy \\
\longrightarrow\ & \int_{\mathbb{R}^n} \left(\frac{1}{1+|y|^2} \right)^n dy = C(n) \quad \text{as} \ \ i \to \infty \quad \text{(cf. (1.5))}.
\end{aligned}
$$

Using (9.5), we have

$$
\begin{aligned}
(9.31)\qquad & \int_{B_{y_i}(\rho)\setminus B_{y_i}(r_i)} v_{p_i}^{p_i+1}\, dy \\
\le\ & \frac{C}{M_i^{p_i+1}} \int_{r_i}^{\rho} \frac{r^{n-1}\, dr}{r^{(n-2)(p_i+1)}} \\
=\ & \frac{C'}{M_i^{p_i+1}} \left[\frac{1}{r_i^{(n-2)(p_i+1)-n}} - \frac{1}{\rho^{(n-2)(p_i+1)-n}} \right] \\
=\ & \frac{C'}{M_i^{\frac{n-2}{2}\cdot\Upsilon_i\cdot p_i}} \frac{1}{R_i^{(n-2)(p_i+1)-n}} - o(1) \longrightarrow 0 \quad \text{as} \ \ i \to \infty\,.
\end{aligned}
$$

Recall that $r_i = R_i\, \delta_i = R_i\, M_i^{-(p_i-1)/2}$. Combining (9.29), (9.30) and (9.31), we obtain (9.22).

We end this section with the observation that the rough estimate of (9.12) by (9.24), namely,

$$M_i^2 \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown (H^{\Upsilon_i} K_{p_i})]\, v_{p_i}^{p_i+1}\, dy \Big| \le C M_i^2 \int_{B_{y_i}(\rho)} |y-y_i|\, v_{p_i}^{p_i+1}\, dy \le C M_i^{\frac{2n-6}{n-2}}\,,$$

is not good enough except when $n = 3$. Thus, in higher dimensions, we need to understand better

(i) relation between Υ_i and M_i, and

(ii) the behavior of $\bigtriangledown K_{p_i}$ near y_i.

If $y_i \equiv 0$ for $i \gg 1$, then we just need to know the infinitesimal behavior of K_{p_i} at the origin. In general, one quickly discovers that the flexibility of y_i (and hence $\bigtriangledown K_{p_i}(y_i)$) is a major issue in estimating (9.12). Cf. (9.20).

§ 9 d. *Relation between* Υ_i *and* M_i. We present some key steps in the argument. For details, refer to [48]. By the limit harmonicity (Proposition 9.4), (9.11)–(9.14) and (9.22), we have

$$\begin{aligned}
(9.32)\qquad & M_i^2 \int_{B_{y_i}(\rho)} H^{\Upsilon_i}[(y-y_i)\cdot \bigtriangledown K_{p_i}]\, v_{p_i}^{p_i+1}\, dy \\
&\qquad + M_i^2\, \Upsilon_i \int_{B_{y_i}(\rho)} H^{\Upsilon_i - 1} K_{p_i} [(y-y_i)\cdot \bigtriangledown H]\, v_{p_i}^{p_i+1}\, dy \\
&= O(\Upsilon_i\, M_i^2) + O(1) \\
\Longrightarrow\quad & \Upsilon_i \le C \Bigg[M_i^{-2} + \Upsilon_i \int_{B_{y_i}(\rho)} |y-y_i|\, v_{p_i}^{p_i+1}\, dy \\
&\qquad\qquad + \int_{B_{y_i}(\rho)} |y-y_i|\cdot|\bigtriangledown K_{p_i}|\, v_{p_i}^{p_i+1}\, dy \Bigg] \\
\Longrightarrow\quad & \Upsilon_i \le C \Bigg[M_i^{-2} + \Upsilon_i\, M_i^{-\frac{2}{n-2}} + \int_{B_{y_i}(\rho)} |y-y_i|\cdot|\bigtriangledown K_{p_i}|\, v_{p_i}^{p_i+1}\, dy \Bigg] \\
&\qquad\qquad\qquad\qquad \text{(by Proposition 9.23)} \\
\Longrightarrow\quad & \Upsilon_i \le C' \Bigg[M_i^{-2} + \int_{B_{y_i}(\rho)} |y-y_i|\cdot|\bigtriangledown K_{p_i}|\, v_{p_i}^{p_i+1}\, dy \Bigg] \quad \text{for } i \gg 1\,.
\end{aligned}$$

In order to estimate the last integral in (9.32), we apply the strategy of first controlling the center, and then spread over to the neighborhood.

We have

$$
(9.33)\qquad \Big|\int_{B_{y_i}(\rho)} |y-y_i|\cdot|\bigtriangledown K_{p_i}|\, v_{p_i}^{p_i+1}\,dy\Big|
$$
$$
\begin{aligned}
&\le |\bigtriangledown K_{p_i}(y_i)| \int_{B_{y_i}(\rho)} |y-y_i|\, v_{p_i}^{p_i+1}\,dy \\
&\qquad + \int_{B_{y_i}(\rho)} |y-y_i|\cdot|\bigtriangledown K_{p_i}(y)-\bigtriangledown K_{p_i}(y_i)|\, v_{p_i}^{p_i+1}\,dy \\
&\le C\,\frac{|\bigtriangledown K_{p_i}(y_i)|}{M_i^{\frac{2}{n-2}}} + \int_{B_{y_i}(\rho)} |y-y_i|\cdot|\bigtriangledown K_{p_i}(y)-\bigtriangledown K_{p_i}(y_i)|\, v_{p_i}^{p_i+1}\,dy\,.
\end{aligned}
$$

Using the Taylor expansion, we obtain

$$
(9.34)\qquad \Big|\frac{\partial K_{p_i}}{\partial y_j}(y)-\frac{\partial K_{p_i}}{\partial y_j}(y_i)\Big|
$$
$$
\begin{aligned}
&\le \sum_{s=1}^{k} |\bigtriangledown^{s+1} K_{p_i}(y_i)|\cdot|y-y_i|^s \\
&\qquad + \max_{0\le t\le 1} |\bigtriangledown^{k+2} K_{p_i}(y_i+t(y-y_i))\,|\cdot|y-y_i|^{k+1}.
\end{aligned}
$$

Here $k\ge 2$ is an integer.

For $n\ge 5$, consider functions $K_{p_i}\in C^{n-2}(\Omega)$ satisfying

$$
(9.35)\quad |\bigtriangledown^{s+1}K_{p_i}(y)|\le C|\bigtriangledown K_{p_i}(y)| \quad \text{for } s=1,\ \cdots,\ n-4\,, \quad \text{and} \quad \text{for } y\in B_o(\rho)\,.
$$

Here C is a positive constant independent on i . Take $k=n-4$ in (9.34), (9.35) and Proposition 9.23, we have

$$
(9.36)\qquad \int_{B_{y_i}(\rho)} |y-y_i|\,|\bigtriangledown K_{p_i}(y)-\bigtriangledown K_{p_i}(y_i)|\, v_{p_i}^{p_i+1}\,dy
$$
$$
\begin{aligned}
&\le C|\bigtriangledown K_{p_i}(y_i)| \sum_{s=1}^{n-4}\int_{B_{y_i}(\rho)} |y-y_i|^{s+1}\, v_{p_i}^{p_i+1}\,dy + C'\int_{B_{y_i}(\rho)} |y-y_i|^{n-2}\, v_{p_i}^{p_i+1}\,dy \\
&\le C\cdot\frac{|\bigtriangledown K_{p_i}(y_i)|}{M_i^{\frac{4}{n-2}}} + \frac{C'}{M_i^2}\,.
\end{aligned}
$$

It follows from (9.33) that

$$
(9.37)\qquad \int_{B_{y_i}(\rho)} |y-y_i|\cdot|\bigtriangledown K_{p_i}|\, v_{p_i}^{p_i+1}\,dy \le C\cdot\frac{|\bigtriangledown K_{p_i}(y_i)|}{M_i^{\frac{2}{n-2}}} + \frac{C'}{M_i^2}\,.
$$

Combining with (9.32) we obtain

$$\Upsilon_i \le \frac{C_1}{M_i^2} + C_2\,\frac{|\bigtriangledown K_{p_i}(y_i)\,|}{M_i^{\frac{2}{n-2}}} \qquad \text{for} \quad i \gg 1\,. \tag{9.38}$$

§ 9 e. *Condition on* $\bigtriangledown K_{p_i}$. Unfortunately the condition in (9.35) is too restrictive. Consider the following

$$\begin{aligned} f(t) = t^\ell \quad \Longrightarrow \quad |f^{(s)}(t)| \;&=\; \frac{\ell\,!}{(\ell - s)\,!}\,|t|^{\ell - s} \\ &\le\; C_{s,\,\ell}\,|f'(t)|^{\frac{\ell-s}{\ell-1}} \quad \text{for} \;\; \ell \ge s > 1 \;\; (\;t \ne 0 \text{ when } s = \ell)\,. \end{aligned}$$

Generalizing to functions of several variables, let us introduce the 'well-tempered' condition, which is a simplified version of the condition in [48] (Definition 0.4; pp. 322).

Definition 9.39. *For a real number $\beta \ge 2$, a sequence of functions $\{K_{p_i}\} \subset C^{[\beta]+1}(\Omega)$ is said to satisfy the condition $\mathbf{WT}_\beta$ if there exist positive constants C_1 and C_2 (independent on i) so that $\|\bigtriangledown K_{p_i}\|_{C^o(\Omega)} \le C_1$ for all i, and for $2 \le s \le [\beta]$,*

$$|\bigtriangledown^s K_{p_i}(y)| \le C_2\,|\bigtriangledown K_{p_i}(y)|^{\frac{\beta-s}{\beta-1}} \qquad \textit{for all} \;\; y \in \Omega \;\; \textit{with} \;\; \bigtriangledown K_{p_i}(y) \ne 0 \tag{9.40}$$

and for all i. (Here $[\beta]$ is the integer part of β.)

Example 9.41. Given a positive integer $\ell \ge 2$, suppose that K_{p_i} has the following Taylor expansion

$$K_{p_i}(y) = K_{p_i}(0) + Q_i^{(\ell)}(y) + R_i(y) \quad \text{for} \;\; y \in B_o(1)\,, \quad i = 1,\ 2, \cdots,$$

where $Q_i^{(l)}$ is a *homogeneous polynomial* of degree ℓ satisfying

$$C'|y|^{\ell-1} \le |\bigtriangledown Q_i^{(\ell)}(y)\,| \quad \text{for} \;\; y \in B_o(1)\,,$$

and the remainder R_i is uniformly bounded in $C^{\ell+1}(B_o(1))$, satisfying

$$\sum_{s=0}^{\ell} |\bigtriangledown^s R_i(y)| \cdot |y|^{-\ell+s} \longrightarrow 0 \quad \text{uniformly} \quad \text{on} \;\; i \;\; \text{as} \quad |y| \to 0\,.$$

Then $\{K_{p_i}\}$ satisfies condition $\mathbf{WT}_\beta$ in $B_o(1)$ for $\beta = \ell$ [48].

Assume that $\{K_{p_i}\}$ satisfies $\mathbf{WT}_\beta$. We have

$$(9.42)\qquad \begin{aligned} |\bigtriangledown^s K_{p_i}(y)| \cdot |y - y_i|^{s-1} &\le C\,|\bigtriangledown K_{p_i}(y)|^{\frac{\beta-s}{\beta-1}} \cdot |y-y_i|^{s-1} \\ &\le |\bigtriangledown K_{p_i}(y)| + C'\,|y-y_i|^{\beta-1} \end{aligned}$$

for $[\beta] \ge s \ge 2$. Here we use the Cauchy-Schwartz inequality. Together with (9.33), (9.34) and Proposition 9.23, if $\beta \ge n-2$ and $n \ge 4$, then we obtain

$$(9.43)\qquad \begin{aligned} &\int_{B_{y_i}(\rho)} |y-y_i| \cdot |\bigtriangledown K_{p_i}(y) - \bigtriangledown K_{p_i}(y_i)|\, v_{p_i}^{p_i+1}\, dy \\ &\le C \cdot \frac{|\bigtriangledown K_{p_i}(y_i)|}{M_i^{\frac{2}{n-2}}} + \frac{C'}{M_i^{\frac{2\beta}{n-2}}} \quad \Big(\leftarrow \text{ replaced by } \frac{C'}{M_i^{\frac{2n}{n-2}}} \text{ if } \beta > n; \\ &\qquad\qquad \text{likewise for } \beta = n, \text{ see Proposition 9.23}\Big) \\ \Longrightarrow\ &\int_{B_{y_i}(\rho)} |y-y_i| \cdot |\bigtriangledown K_{p_i}|\, v_{p_i}^{p_i+1}\, dy \le C \cdot \frac{|\bigtriangledown K_{p_i}(y_i)|}{M_i^{\frac{2}{n-2}}} + \frac{C'}{M_i^2} \\ &\qquad\qquad (\text{as } \beta \ge n-2)\,. \end{aligned}$$

It follows from (9.32) and (9.43) that

$$(9.44)\qquad \Upsilon_i \le \frac{C}{M_i^2} + C'\,\frac{|\bigtriangledown K_{p_i}(y_i)|}{M_i^{\frac{2}{n-2}}} \qquad \text{for} \quad i \gg 1\,.$$

§ 9 f. *Relation between* $|y_i|$ *and* M_i. For a fixed i, let η be a smooth cut-off function so that $\eta(y) = 1$ for $|y-y_i| \le \rho/4$ and $\eta(y) = 0$ for $|y-y_i| \ge \rho/2$. Multiplying both sides of equation (7.4) by $\eta\,(\partial v_{p_i}/\partial y_j)$ and integrating by parts in $B_{y_i}(\rho)$, we obtain

$$(9.45)\qquad \begin{aligned} &\frac{c_n}{p_i+1} \int_{B_{y_i}(\rho)} \frac{\partial (K_{p_i} H^{\Upsilon_i})}{\partial y_j}\, v_{p_i}^{p_i+1} \cdot \eta\, dy \\ = \ &\frac{1}{2} \int_{D_i} |\bigtriangledown v_{p_i}|^2 \frac{\partial \eta}{\partial y_j}\, dy - \int_{D_i} \frac{\partial v_{p_i}}{\partial y_j}\, (\bigtriangledown v_{p_i} \cdot \bigtriangledown \eta)\, dy \\ &\qquad - \frac{c_n}{p_i+1} \int_{D_i} K_{p_i} H^{\Upsilon_i} v_{p_i}^{p_i+1}\, \frac{\partial \eta}{\partial y_j}\, dy\,, \end{aligned}$$

where $D_i := B_{y_i}(\rho/2) \setminus B_{y_i}(\rho/4)$. For $i \gg 1$, v_{p_i} is uniformly bounded from above in D_i. Standard elliptic theory implies the gradient estimate $|\bigtriangledown v_{p_i}(y)| \le C v_{p_i}(y)$

for $y \in D_i$ and $i \gg 1$. Using the product formula, Propositions 9.4 and 9.23, we have

$$\Big| \int_{B_{y_i}(\rho)} \frac{\partial K_{p_i}}{\partial y_j} v_{p_i}^{p_i+1} \cdot \eta \; dy \Big| \le C \int_{B_{y_i}(\rho/2) \setminus B_{y_i}(\rho/4)} | \bigtriangledown v_{p_i}|^2 \, dy + \frac{C}{M_i^{p_i+1}} + C \Upsilon_i$$
$$\le \frac{C}{M_i^2} + \frac{C}{M_i^{p_i+1}} + C\, \Upsilon_i$$

$$\Longrightarrow \; \Big| \frac{\partial K_{p_i}}{\partial y_j}(y_i) \Big| \le \frac{1}{c} \Big| \frac{\partial K_{p_i}}{\partial y_j}(y_i) \int_{B_{y_i}(\rho)} v_{p_i}^{p_i+1} \cdot \eta \; dy \Big|$$
$$\le \Big| \int_{B_{y_i}(\rho)} \left\{ \frac{\partial K_{p_i}}{\partial y_j}(y_i) - \frac{\partial K_{p_i}}{\partial y_j}(y) \right\} v_{p_i}^{p_i+1} \cdot \eta \; dy \Big| + \Big| \int_{B_{y_i}(\rho)} \frac{\partial K_{p_i}}{\partial y_j} v_{p_i}^{p_i+1} \, \eta \, dy \Big|$$
$$\le \int_{B_{y_i}(\rho)} \Big| \frac{\partial K_{p_i}}{\partial y_j}(y_i) - \frac{\partial K_{p_i}}{\partial y_j}(y) \Big| v_{p_i}^{p_i+1} \cdot \eta \; dy + \frac{C}{M_i^2} + \frac{C}{M_i^{p_i+1}} + C\, \Upsilon_i \, .$$

If we assume that $\{K_{p_i}\}$ satisfies $\mathbf{WT}_\beta$ with $\beta \ge 2$, then the last integral can be estimated in a similar fashion as in (9.34) and (9.42), leading to

$$\int_{B_{y_i}(\rho)} \Big| \frac{\partial K_{p_i}}{\partial y_j}(y_i) - \frac{\partial K_{p_i}}{\partial y_j}(y) \Big| v_{p_i}^{p_i+1} \cdot \eta \; dy \le \frac{1}{2n} | \bigtriangledown K_{p_i}(y_i)| + \frac{C'}{M_i^{\frac{2(\beta-1)}{n-2}}} \, .$$

Here we use Proposition 9.23 with $n > \beta - 1$. Thus

$$\Big| \frac{\partial K_{p_i}}{\partial y_j}(y_i) \Big| \le \frac{1}{2n} \cdot | \bigtriangledown K_{p_i}(y_i)| + \frac{C'}{M_i^{\frac{2(\beta-1)}{n-2}}} + \cdot \frac{C}{M_i^2} + C\, \Upsilon_i$$
$$\Longrightarrow \; | \bigtriangledown K_{p_i}(y_i)| \le \frac{C}{M_i^2} + \frac{C'}{M_i^{\frac{2(\beta-1)}{n-2}}} + C\, \Upsilon_i \, .$$

Combining with (9.37) we obtain

$$\text{(9.46)} \qquad \Upsilon_i \le \frac{C}{M_i^2} \quad \text{and} \quad | \bigtriangledown K_{p_i}(y_i)| \le \frac{C'}{M_i^2} + \frac{C'}{M_i^{\frac{2(\beta-1)}{n-2}}} \quad \text{for } \; i \gg 1 \, .$$

In the above, by Proposition 9.23, the term $M_i^{\frac{2(\beta-1)}{n-2}}$ should be replaced by $M_i^{\frac{2n}{n-2}}$ if $\beta - 1 > n$. (Similar replacement for $\beta - 1 = n$, cf. Proposition 9.23.) We are less concerned with this, as we find out eventually what we want is $O\left(M_i^{-(2+\varepsilon)}\right)$ for a small positive number ε.

In case K_{p_i} satisfies the conditions in Example 9.41 for $\ell = \beta$, we have

$$| \bigtriangledown K_{p_i}(y_i)| \ge C|y_i|^{\beta-1}.$$

Hence for $n-2 \ge \beta - 1$, it follows from (9.46) that

$$|y_i| \le C M_i^{-\frac{2}{n-2}}.$$

Cf. (9.20).

§ 9 g. *Estimating the leading term in Pohozaev's identity.* We start with

$$\begin{aligned}
&\left| \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown (K_{p_i} H^{\Upsilon_i})]\, v_{p_i}^{p_i+1}\, dy \right| \\
\le\ & \left| \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown K_{p_i}]\, H^{\Upsilon_i}\, v_{p_i}^{p_i+1}\, dy \right| \\
&\qquad + \Upsilon_i \left| \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown H]\, H^{\Upsilon_i - 1} K_{p_i}\, v_{p_i}^{p_i+1}\, dy \right| \\
\le\ & C \left| \bigtriangledown K_{p_i}(y_i) \int_{B_{y_i}(\rho)} |y-y_i|\, v_{p_i}^{p_i+1}\, dy \right| \\
&\qquad + C \left| \int_{B_{y_i}(\rho)} |y-y_i|\, | \bigtriangledown K_{p_i}(y) - \bigtriangledown K_{p_i}(y_i)|\, v_{p_i}^{p_i+1}\, dy \right| \\
&\qquad + \Upsilon_i \int_{B_{y_i}(\rho)} |y-y_i|\, v_{p_i}^{p_i+1}\, dy \\
\le\ & C\, M_i^{-(2+\frac{2}{n-2})} + C' | \bigtriangledown K_{p_i}(y_i)|\, M_i^{-\frac{2}{n-2}} \\
&\qquad + \int_{B_{y_i}(\rho)} |y-y_i|\, | \bigtriangledown K_i(y) - \bigtriangledown K_i(y_i)|\, v_{p_i}^{p_i+1}\, dy\, .
\end{aligned}$$

The last integral can be estimated as in (9.43). If $\beta > n-2$, then there is a positive number ε such that

(9.47) $$\left| \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown (K_{p_i} H^{\Upsilon_i})]\, v_{p_i}^{p_i+1}\, dy \right| \le C\, M_i^{-(2+\frac{2}{n-2})} + C' M_i^{-(2+\varepsilon)}$$

for $i \gg 1$. In particular, we have

(9.48) $$\lim_{i\to\infty} M_i^2 \int_{B_{y_i}(\rho)} [(y-y_i)\cdot \bigtriangledown (K_{p_i} H^{\Upsilon_i})]\, v_{p_i}^{p_i+1}\, dy = 0\, .$$

§ 9 h. *Sharpness.* In case $y_i \equiv 0$ for $i \gg 1$ and

(9.49) $$| \bigtriangledown K_{p_i}(y)| \le C|y|^{\ell-1} \quad \text{for } \ y \in B_o(\sigma) \qquad (\ell < n)\, ,$$

we have

$$\Big| \int_{B_o(\rho)} (y \,\cdot\, \bigtriangledown K_{p_i})\; v_{p_i}^{p_i+1}\, dy \Big| \le C \int_{B_o(\rho)} |y|^\ell v_{p_i}^{p_i+1}\, dy\,.$$

By Proposition 9.23, the last integral is of order $O(M^{-(2+\varepsilon)})$ when $\ell > n-2$. This shows that in (9.47) the effect of off-set poles (i.e., $y_i \ne 0$) is ignorable. Without taking 'internal cancelation' of $(y-y_i)\cdot\bigtriangledown K_{p_i}$ into account (instead of $|y-y_i|\cdot|K_{p_i}|$), one has limited room to improve the estimates (cf. [49]).

Note that when K_{p_i} is radially symmetric above 0, basically no information is lost by considering $|y| \cdot | \bigtriangledown K_{p_i}|$ instead of $y \cdot \bigtriangledown K_{p_i}$. Cf. § 4 a.

§ 9 i. *One bubble.* (9.11)–(9.13) & (9.48) can be used to eliminate the possibility of two or more simple blow-ups, as $B = b(0) > 0$ in case there is more than one simple blow-up, see Proposition 9.4 and § 6 b. When there is just one simple blow-up, the Taylor expansion of K above the blow-up point follows some special arrangement. We refer the readers to [48], [49] and [46].

10. Reflections.

The moving planes/spheres method is introduced by Gidas, Ni and Nirenberg in [29] and [30], developing an idea of Serrin on a version of Alexandrov's reflection principle (see [34]). The method is found to an integrated part of the problem and deep results are obtained by Caffarelli, Gidas and Spruck [10], W.-X. Chen and C.-M. Li [19], C.-C. Chen and C.-S. Lin [13] [14] [17] , and others.

We demonstrate that, in a broad sense, *supported* blow-up does not occur when $K \equiv$ const. While this is not an original result, it serves the purpose of illustrating the basic principles of the reflection method. In the following discourse, we incorporate quite a few points from the work of Taliaferro and Zhang [69].

Let $\{v_i\} \subset C^2_+(B_o(r))$ be a sequence of solutions of the equation

$$\Delta\, v + n(n-2)\, v^{\frac{n+2}{n-2}} = 0 \quad \text{ in } \; B_o(r) \subset \mathbb{R}^n, \tag{10.1}$$

such that $M_i := v_i(y_i) = \max\limits_{B_o(r/2)} v_i \to \infty$. We say that the blow-up is supported if there is a positive number c such that

$$v_i \ge c^2 > 0 \quad \text{in} \;\; B_o(r) \;\; \text{for} \quad \text{all} \;\; i \gg 1\,. \tag{10.2}$$

(Cf. the examples in § 7 f and § 7 g .) Without loss of generality, we take $r = 4$ in (10.1) & (10.2). For simplicity sake, *we assume that there is no blow-up points on* $\partial B_o(2)$.

As there is no blow-up point on $\partial B_o(2)$, a subsequence of $\{y_i\}$, which we still denote by $\{y_i\}$, converges to a point inside $B_o(2)$. Without loss of generality, we may assume that $\lim\limits_{i\to\infty} y_i = 0$. Define

$$w_i\,(y) := \frac{v_i\,(y_i + M_i^{-\frac{2}{n-2}}\,y)}{M_i} \qquad \text{for} \;\; |y| \le M_i^{\frac{2}{n-2}}\,, \quad i \gg 1\,. \tag{10.3}$$

By the discussion in § 2 a , we may assume that, for every $R > 0$, w_i converges uniformly in $C^2(B_o(R))$ norm to $U_1(y) = \left(\dfrac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}$. In what follows, we make use of the following property of superharmonic functions.

Lemma 10.4. *Let W be a positive smooth superharmonic function (i.e., $\Delta W \le 0$) on $\mathbb{R}^n$. Given $R > r_o > 0$, we have*

$$W(y) \ge \frac{C_{R\,,\,r_o}}{|y|^{n-2}} \qquad \text{for} \;\; R > |y| > r_o\,,$$

where $C_{R\,,\,r_o} := \min\left\{ \inf\limits_{|y|=r_o} [\, r_o^{n-2}\, W(y)]\,, \;\; \inf\limits_{|y|=R} [R^{n-2}\, W(y)] \right\}$.

Proof. Let

$$\phi(y) := \frac{C_{R\,,\,r_o}}{|y|^{n-2}} \quad \text{and} \;\; D(y) := W(y) - \phi(y) \quad \text{for} \;\; y \in \mathbb{R}^n \setminus \{0\}\,.$$

We have

$$\Delta\, D \le 0 \quad \text{in} \;\; \mathbb{R}^n \setminus \{0\} \quad \text{and} \;\; D(y) \ge 0 \quad \text{for} \;\; |y| = r_o \;\; \text{or} \;\; |y| = R\,.$$

Thus if there is a point y_- with $r_o < |y_-| < R$ such that $D(y_-) < 0$, we may assume that y_- is local minimum. As $\Delta\, D \le 0$, we are led to a contradiction via the maximum principle. $\square$

For a number $\lambda > 0$, the reflection of a point $y \neq 0$ upon the sphere with center at 0 and radius λ is given by

$$y^\lambda := \frac{\lambda^2 y}{|y|^2} . \tag{10.5}$$

The Kelvin transform of w_i about the sphere with center at 0 and radius λ is given by

$$\begin{aligned} w_i^\lambda(y) &:= \left(\frac{\lambda}{|y|}\right)^{n-2} w_i(y^\lambda) \\ &= \left(\frac{\lambda}{|y|}\right)^{n-2} w_i\left(\frac{\lambda^2 y}{|y|^2}\right) \quad \text{for} \quad \lambda^2 M_i^{-\frac{2}{n-2}} \leq |y| < \infty . \end{aligned} \tag{10.6}$$

Observe that for any fixed $\lambda > 0$, $\lambda > \lambda^2 M_i^{-\frac{2}{n-2}}$ for i large enough.

The behavior of w_i in $B_o(\lambda)$ is well-controlled because of the convergence to U_1 (the rigidity part). It follows that we have good control on w_i^λ outside $B_o(\lambda)$ via the reflection. On the other hand, we do not know much about $w_i(y)$ when $|y| > \lambda \gg 1$ (the flexibility part). The simple but profound idea is the comparison

$$\begin{aligned} \Phi_i^\lambda(y) &:= w_i(y) - w_i^\lambda(y) \\ &= r^{-\frac{n-2}{2}} \left[r^{\frac{n-2}{2}} w_i(y) - \left(\frac{\lambda^2}{r}\right)^{\frac{n-2}{2}} w_i\left(\frac{\lambda^2 y}{r^2}\right)\right] \end{aligned} \tag{10.7}$$

for $\lambda < |y| = r < M_i^{\frac{2}{n-2}}$.

For $\Phi_i^\lambda(y) \neq 0$, Φ_i^λ satisfies the equation

$$\Delta \Phi_i^\lambda + n(n-2) \left[\frac{w_i^{\frac{n+2}{n-2}} - (w_i^\lambda)^{\frac{n+2}{n-2}}}{w_i - w_i^\lambda}\right] \Phi_i^\lambda = 0 \quad \text{for} \ \ \lambda < |y| < M_i^{\frac{2}{n-2}}. \tag{10.8}$$

When $\Phi_i^\lambda(y) = 0$, $\Delta \Phi_i^\lambda(y) = 0$. The term in the square brackets is positive once $\Phi_i^\lambda \neq 0$.

Lemma 10.9. *Let Φ_i^λ be defined by* (10.7) *and* $\Omega \subset B_o(M_i^{\frac{2}{n-2}}) \backslash \overline{B_o(\lambda)}$ *be an open connected set. Assume that $\Phi_i^\lambda \ge 0$ in Ω. Then we have the following conclusions.*

(a) *Either $\Phi_i^\lambda > 0$ or $\Phi_i^\lambda \equiv 0$ in Ω.*

(b) *Suppose that there is a ball B in Ω and a point $p \in \partial B \cap \partial \Omega$ with $\Phi_i^\lambda(p) = 0$. Then if $\Phi_i^\lambda \not\equiv 0$ in B, we have $\dfrac{\partial \Phi_i^\lambda}{\partial \mathbf{n}_p} < 0$, where $\mathbf{n}_p$ is the unit outward normal on ∂B at p.*

The proof of the above lemma follows from the maximum principle and Hopf's lemma (cf. [37]).

§ 10 a. *Key property*: $\Phi_i^\lambda > 0$ *on* $\partial B_o(M_i^{\frac{2}{n-2}})$ *for* $i \gg 1$. It follows from the definition of w_i^λ that

(10.10) $$w_i^\lambda(y) = \left(\frac{\lambda}{|y|}\right)^{n-2} w_i(y^\lambda) \le \left(\sup_{B_o(R_o)} w_i(y)\right) \cdot \left(\frac{\lambda}{|y|}\right)^{n-2} \quad \text{for } |y| > R_o\,.$$

In particular,

(10.11) $$w_i^\lambda(y) \le \left(\sup_{B_o(R_o)} w_i(y)\right) \cdot \lambda^{n-2} M_i^{-2} \quad \text{for } |y| = M_i^{\frac{2}{n-2}}.$$

As the blow-up is supported, we have

(10.12)
$$v_i\,(y_i + M_i^{-\frac{2}{n-2}}\, y) \ge c^2 \implies w_i(y) \ge c^2 M_i^{-1} \quad \text{for } |y| = M_i^{\frac{2}{n-2}} \text{ and } i \gg 1\,.$$

In particular, for any $\lambda > 0$, we have

(10.13) $$w_i(y) > w_i^\lambda(y) \implies \Phi_i^\lambda(y) > 0\,, \quad \text{where } |y| = M_i^{\frac{2}{n-2}} \text{ and } i \gg 1\,.$$

(10.13) encourages us to focus on the assertion:

(**) *for any $\lambda > 0$, there is a positive integer i_λ, such that for all $i \ge i_\lambda$,*

$$\Phi_i^{\lambda_o}(y) > 0 \quad \text{for } \lambda_o < |y| < M_i^{\frac{2}{n-2}} \quad \textit{and for all} \quad \lambda_o \in (0\,,\ \lambda\,]\,.$$

We first explain why (**) leads to a contradiction. Letting $i \to \infty$ in (**) and using a diagonal argument, we obtain

(10.14) $$U_1(y) \ge U_1^\lambda(y) \quad \text{for } |y| > \lambda \quad \text{for all } \lambda > 0\,.$$

One can check this is impossible. In general, we have the following [44].

Lemma 10.15. *Let U be a spherical solution given by*

$$U(y) = \left(\frac{a}{a^2 + |y - \xi|^2} \right)^{\frac{n-2}{2}} \qquad for \ \ y \in \mathbb{R}^n,$$

and U^λ the Kelvin transform of U about the sphere with center at the origin and radius $\lambda > 0$. Then we have

$$U^\lambda(y) = \left(\frac{\tilde{a}}{\tilde{a}^2 + |y - \tilde{\xi}|^2} \right)^{\frac{n-2}{2}} \qquad for \ \ y \in \mathbb{R}^n,$$

where

$$\tilde{a} = \frac{\lambda^2 a}{a^2 + |\xi|^2} \qquad and \qquad \tilde{\xi} = \frac{\lambda^2\, \xi}{a^2 + |\xi|^2}\,.$$

In particular, if $\xi = 0$, then $U(y) < U^\lambda(y)$ for $\lambda > a$ and $|y| > \lambda$. For fixed a and $\xi \neq 0$, if $\lambda \gg 1$, then

$$|\tilde{\xi}| > \lambda \gg 1 \implies U(\tilde{\xi}) = O(\lambda^{-2(n-2)}) < U^\lambda(\tilde{\xi}) = O(\lambda^{-(n-2)})\,.$$

§ 10 b. (**) *holds for small* $\lambda > 0$. Observe that, by Lemma 10.15,

$$U_1(y) > U_1^\lambda(y) = \left(\frac{\lambda^2}{\lambda^4 + |y|^4} \right)^{\frac{n-2}{2}} \qquad \text{for} \ \ \lambda < |y| \ \ \text{and} \ \ \lambda \in (0\,,\ 1)\,.$$

Let $f_i(y) := r^{\frac{n-2}{2}}\, w_i(y)$, where $r = |y| > 0$. It follows from (10.7) that

$$\Phi_i^\lambda(y) = r^{-\frac{n-2}{2}} \left[f_i(y) - f_i(y^\lambda) \right] \qquad \text{for} \ \ \lambda < |y| < M_i^{\frac{2}{n-2}}\,. \tag{10.16}$$

By using the C^2 convergence of w_i to the standard bubble U_1 in $B_o(R)$, one can check that there is a positive number R_o such that

$$\frac{\partial f_i}{\partial r}(r\,, \theta) > 0 \qquad \text{for} \ \ 0 < r \le R_o < R\,, \quad \theta \in S^{n-1} \ \ \text{and} \ \ i \gg 1.$$

As the reflection decreases the radial distance for $r > \lambda$ (i.e., $|y^\lambda| = \lambda^2/r < |y|$), we have

$$\Phi_i^\lambda(y) > 0 \qquad \text{for} \ \ 0 < \lambda < r \le R_o \ \ \text{and} \ \ i \gg 1\,. \tag{10.17}$$

By Lemma 10.4, we have

$$\begin{aligned} w_i(y) &\geq \min\left\{ \inf_{|y|=R_o} (R_o^{n-2}\, w_i)\,, \quad \inf_{|y|=M_i^{\frac{2}{n-2}}} (M_i^2\, w_i) \right\} |y|^{-(n-2)} \\ &\geq \min\left\{ \inf_{|y|=R_o} (R_o^{n-2}\, w_i)\,, \;\; c^2 M_i \right\} |y|^{-(n-2)} \qquad \text{(by } \; 10.12) \\ &\geq \left[\inf_{|y|=R_o} (R_o^{n-2}\, w_i) \right] |y|^{-(n-2)} \qquad \text{for} \quad R_o < |y| < M_i^{\frac{2}{n-2}} \;\; \text{and} \;\; i \gg 1\,, \end{aligned}$$

as $c^2 M_i \gg 1$ when $i \gg 1$ and $w_i(y) \leq 1$ for $|y| < M_i^{\frac{2}{n-2}}$. Using the convergence of w_i, there is a positive number λ_o such that

$$\left(\sup_{B_o(R_o)} w_i(y) \right) \lambda^{n-2} < \inf_{|y|=R_o} (R_o^{n-2}\, w_i) \quad \text{for} \;\; \lambda \in (0, \;\lambda_o] \;\; \text{and} \;\; i \gg 1\,.$$

Together with (10.10), we obtain

$$\begin{aligned} w_i(y) \quad &> \quad \left(\sup_{B_o(R_o)} w_i(y) \right) \lambda^{n-2}\, |y|^{-(n-2)} \geq w_i^\lambda(y) \\ &\Longrightarrow \quad \Phi_i^\lambda(y) > 0 \quad \text{for} \;\; R_o < |y| < M_i^{\frac{2}{n-2}}\,, \quad \lambda \in (0, \;\lambda_o] \;\; \text{and} \;\; i \gg 1\,. \end{aligned}$$

Combining with (10.17) we have

$$\text{(10.18)} \qquad \Phi_i^\lambda(y) > 0 \quad \text{for} \;\; \lambda < |y| < M_i^{\frac{2}{n-2}}\,, \quad \lambda \in (0, \;\lambda_o] \;\; \text{and} \;\; i \gg 1\,.$$

§ 10 c. (**) *holds for all* $\lambda > 0$. For large λ, we make use of (10.13) to 'slide' w_i^λ under w_i. Precisely, let

$$\text{(10.19)} \qquad \bar\lambda := \sup\{\lambda \in \mathbb{R}^+ \;\mid\; (**) \;\; \text{holds} \;\; \text{for}\; \lambda\}\,.$$

Suppose that $\bar\lambda < \infty$. For $\lambda \in (\bar\lambda - \varepsilon, \;\bar\lambda + \varepsilon) \in \mathbb{R}^+$, from (10.7), together with the triangle inequality, we have

$$\begin{aligned} \text{(10.20)} \qquad |\Phi_i^\lambda(y) - \Phi_i^{\bar\lambda}(y)| \;\; &\leq \;\; \frac{|\lambda^{n-2} - (\bar\lambda)^{n-2}|}{|y|^{n-2}}\, w_i\!\left(\frac{\lambda^2 y}{|y|^2}\right) \\ &\qquad + \frac{(\bar\lambda)^{n-2}}{|y|^{n-2}} \left| w_i\!\left(\frac{\lambda^2 y}{|y|^2}\right) - w_i\!\left(\frac{\bar\lambda^2 y}{|y|^2}\right) \right|. \end{aligned}$$

Using the uniform convergence of w_i to U_1 in $B_o((\bar\lambda+\varepsilon)^2/(\bar\lambda-\varepsilon))$, and taking $\lambda \in (\bar\lambda-\varepsilon,\ \bar\lambda)$ close to $\bar\lambda$, we see from (10.20) and the positivity of Φ_i^λ that

$$\Phi_i^{\bar\lambda}(y) \ge 0 \quad \text{for } \ \bar\lambda < |y| < M_i^{\frac{2}{n-2}} \ \text{ and } \ i \gg 1\,. \tag{10.21}$$

By (10.17) and (a) in Lemma 10.9, we have

$$\Phi_i^{\bar\lambda}(y) > 0 \quad \text{for } \ \bar\lambda < |y| < M_i^{\frac{2}{n-2}} \ \text{ and } \ i \gg 1\,. \tag{10.22}$$

By (b) in Lemma 10.9,

$$\frac{\partial \Phi_i^{\bar\lambda}}{\partial r}(y) > 0 \quad \text{for } \ |y| = \bar\lambda \ \text{ and } \ i \gg 1\,.$$

Because of the continuity in r and λ, the compactness of $\partial B_o(\bar\lambda)$, and the convergence of w_i to U_1, there is a positive number ε_1 such that

$$\frac{\partial \Phi_i^{\lambda}}{\partial r}(y) > 0 \quad \text{for } \ \bar\lambda \le |y| \le \bar\lambda+\varepsilon_1\,, \qquad \bar\lambda \le \lambda \le \bar\lambda+\varepsilon_1 \ \text{ and } \ i \gg 1\,. \tag{10.23}$$

Take a positive number $\varepsilon_2 \le \varepsilon_1$. By lemma 10.4, (10.23) and (10.12), we have

$$\begin{aligned}
\Phi_i^{\bar\lambda}(y) \ &\ge\ \min\left\{ \inf_{|y|=\bar\lambda+\varepsilon_2} (\bar\lambda+\varepsilon_2)^{n-2}\,\Phi_i^{\bar\lambda}\,,\ \ \frac{1}{2}c^2 M_i \right\} |y|^{2-n} \\
&\ge\ \left[\inf_{|y|=\bar\lambda+\varepsilon_2} (\bar\lambda+\varepsilon_2)^{n-2}\,\Phi_i^{\bar\lambda} \right] |y|^{2-n} > 0 \\
&\qquad\qquad\qquad \text{for } \quad \bar\lambda+\varepsilon_2 < |y| < M_i^{\frac{2}{n-2}} \ \text{ and } \ i \gg 1\,.
\end{aligned}$$

It follows from (10.20) there is a positive number $\varepsilon_3 < \varepsilon_2$ such that

$$\Phi_i^{\bar\lambda+\varepsilon}(y) > 0 \quad \text{for } \ \bar\lambda+\varepsilon_2 < |y| < M_i^{\frac{2}{n-2}}, \ \ \varepsilon \in (0,\ \varepsilon_3] \ \text{ and } \ i \gg 1\,. \tag{10.24}$$

As $\Phi_i^{\bar\lambda+\varepsilon}(y) = 0$ for $|y| = \bar\lambda+\varepsilon$. By (10.22), we have

$$\Phi_i^{\bar\lambda+\varepsilon}(y) > 0 \quad \text{for } \ \bar\lambda+\varepsilon < |y| \le \bar\lambda+\varepsilon_2\,, \ \ \varepsilon \in (0,\ \varepsilon_3] \ \text{ and } \ i \gg 1\,. \tag{10.25}$$

Combining (10.24) and (10.25), we find that $\bar\lambda+\varepsilon_3$ also satisfies (**). This contradicts the definition of $\bar\lambda$. Hence $\bar\lambda$ must be infinite.

§ 10 d. *Final remarks.* In this discussion, a point of interest is that we have to go far ($|y| = M_i^{2/(n-2)}$) and use some property of w_i there to draw the conclusion. The method relies heavily on (10.8). For general K, the term which should be in the square brackets in (10.8) becomes harder to control. Indeed, after a rearrangement, we obtain

$$\begin{aligned} \Delta\, \Phi_i^\lambda + K_i \left[\frac{w_i^{\frac{n+2}{n-2}} - (w_i^\lambda)^{\frac{n+2}{n-2}}}{w_i - w_i^\lambda} \right] \Phi_i^\lambda &= (K_i^\lambda - K_i) \cdot (w_i^\lambda)^{\frac{n+2}{n-2}} \quad \text{for} \ \ \lambda < |y| < M_i^{\frac{2}{n-2}}\,, \\ \Delta\, \Phi_i^\lambda(y) &= (K_i^\lambda - K_i) \cdot (w_i^\lambda)^{\frac{n+2}{n-2}} \quad \text{for} \ \ \Phi_i^\lambda(y) = 0\,. \end{aligned}$$

When $K_i > 0\,,$ the term

$$b_i^\lambda := K_i \left[\frac{w_i^{\frac{n+2}{n-2}} - (w_i^\lambda)^{\frac{n+2}{n-2}}}{w_i - w_i^\lambda} \right] > 0 \quad \text{when} \ \ w_i \neq w_i^\lambda\,.$$

The extra term (the 'shifted' term)

$$J_i^\lambda := (K_i^\lambda - K_i) \cdot (w_i^\lambda)^{\frac{n+2}{n-2}}$$

has undetermined signs and is the source of difficulty.

The key idea here, by Chen and Lin, is to produce an auxiliary function $h_i^\lambda \le 0$ so that

$$\Delta\, h_i^\lambda(y) + J_i^\lambda(y) \le 0 \quad \text{for} \ \ \lambda < |y| < M_i^{\frac{2}{n-2}}\,.$$

Granting this, then the equation for $\Phi_i^\lambda + h_i^\lambda$ is given by

$$\Delta\,(\Phi_i^\lambda + h_i^\lambda) + b_i^\lambda(\Phi_i^\lambda + h_i^\lambda) = J_i^\lambda + \Delta h_i^\lambda + b_i^\lambda\, h_i^\lambda \le 0\,.$$

Thus we consider $\Phi_i^\lambda + h_i^\lambda$ instead of Φ_i^λ The better we can control h_i^λ, the finer the information we can gain for Φ_i^λ. We refer the interested readers to the wonderful works of Chen and Lin on the development and applications of this PDE insight. (Similar idea is used by Taliaferro and Zhang in smoothing out the scalar curvature function when the bubbles are superimposed, see [67] & [68].)

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