1. A unit needle is placed on the plane. Using the Cartesian coordinates \((x, y)\), the needle is initially positioned with the end at \((0, 1)\) and the head at \((1, 1)\). Consider moving the needle on the plane from the above position to the new position with its end at \((0, 0)\) and head at \((1, 0)\). Furthermore, we require that there is no self-intersection on the loci of movement, that is, if the needle occupies the unit line segment \(\ell\) at a particular instance, then at no other instance the needle can intersect \(\ell\).

Formulate conjectures on the least area needed in the transposition described above. Justify, but need not proving, your assertions.

2. The Besicovitch’s construction is simplified and organized by Perron in 1929. Consider an equilateral triangle of unit height. The idea is to halve, and halve and halve again the base. Adjacent triangles are then slid towards each other so that they overlap a little. The process is repeated (see the figure below). The result is called a Perron tree.

In order to compute efficiently the area of a Perron tree, given a positive integer \(p\), we specify how to extend the triangle \(NAB\) between \((k-1)\)-th and \(k\)-th level to \((k+1)\)-th level \((k+1 \leq p)\). Extend the line \(BN\) upward to \((k+1)\)-th level, cutting at \(S\). Likewise, extend \(AN\) upward to cut at \(Q\). The new shape is formed by joining \(S\) to \(A\), and \(Q\) to \(B\).

\((2a)\) Explain why area of \(\triangle ANB = \text{area of } \triangle ASN = \text{area of } \triangle NQB\); and area of \(\triangle ANM = \text{area of } \triangle SMN\).
For any integer \( p \geq 2 \), the Perron tree can be shown to be formed by extending the triangle with base \( \frac{4}{p} \) and height \( \frac{2}{p} \) to the next level, using the above construction, and so on, up to the \( p \)-th level. The figure below visualizes the construction when \( p = 5 \). Note that for any \( p \geq 2 \), the distance between consecutive levels is \( 1/p \), and the total height for the final step is always 1.

\[ \text{(2b)} \text{ Work out the Perron tree when } p = 6. \]
\[ \text{(2c) For every integer } p \geq 2, \text{ estimate the area of the Perron tree. Conclude that the area can be as small as we like when } p \text{ is taken large enough. (Learn how to do so first by computing the areas corresponding to } p = 5 \text{ or } 6. \]