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06/07 Semester II MA4221 P.D.E. Tutorial 8

1. Let $B_o(2)$ be the open ball in \mathbb{R}^3 with center at the origin and radius equal to 2.
2. Given a harmonic function u on $B_o(2) \setminus \{0\}$, we consider the possible singularity at 0 under the condition that $u > 0$.

Introduce the polar coordinates (r, θ, φ) , where

$$x = r \sin \varphi \cos \theta, \quad y = r \sin \varphi \sin \theta, \quad z = r \cos \varphi. \quad (1)$$

Similar to Q. 3 in Tutorial 1, we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} (\Delta_{S^2} u) = 0 \quad \text{for } 1 \geq r > 0. \quad (2)$$

Here

$$\Delta_{S^2} u = \frac{1}{\sin \varphi} \frac{\partial}{\partial \varphi} \left[\sin \varphi \frac{\partial u}{\partial \varphi} \right] + \frac{1}{\sin^2 \varphi} \frac{\partial^2 u}{\partial \theta^2}$$

is the “Laplacian” on the unit sphere S^2 . We won’t go into the detail here, but we know that (compare with the Green formula)

$$\int_{S^2} \Delta_{S^2} u(r, \theta, \varphi) dS_{\theta, \varphi} = 0 \quad \text{for } 1 \geq r > 0. \quad (3)$$

(A) Integrating both sides of (2) on S^2 and using (3), show that

$$\bar{u}''(r) + \frac{2}{r} \bar{u}'(r) = 0 \quad \text{for } 1 \geq r > 0. \quad (4)$$

Here the spherical average \bar{u} is defined by

$$\bar{u}(r) := \frac{1}{4\pi} \int_{S^2} u(r, \theta, \varphi) dS_{\theta, \varphi} \quad \text{for } 1 \geq r > 0. \quad (5)$$

Recall that the area of S^2 is 4π . Using the method in Q. 2 in Tutorial 1, show that

$$\bar{u}(r) = \frac{A}{r} + B \quad \text{for } 1 \geq r > 0, \quad (6)$$

where $A \geq 0$ and B are constants.

(B) For any positive number $\rho > 0$, show that the function

$$v(x, y, z) := u(\rho x, \rho y, \rho z)$$

is harmonic for $r \neq 0$ ($\rho r \leq 1$). Using this and the Harnack inequality for positive harmonic functions, obtain the following *spherical* Harnack inequality: there is a positive number $a_o < 1$ depending on n only, such that

$$a_o u(r, \theta, \varphi) < u(r, \theta', \varphi') \quad (7)$$

whenever $1 \geq r > 0$ and $0 \leq \varphi, \varphi' \leq \pi$, $0 \leq \theta, \theta' < 2\pi$.

(C) Assume that $u \equiv 0$ on $\partial B_o(1)$. Using the maximum principle, show that

$$u > \alpha_o \bar{u}, \quad \text{i.e.,} \quad u(x, y, z) - \alpha_o \bar{u}(r) > 0 \quad \text{for } 1 > r > 0,$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Here $\alpha_o < 1$ is the positive constant in (7).

(Hint: Consider applying the maximum principle on the harmonic function $u - \alpha_o \bar{u}$ in $B_o(1) \setminus B_o(\varepsilon)$.)

Define

$$w(x, y, z) = u(x, y, z) - \alpha_o \bar{u}(r) > 0 \quad \text{for } 1 > r > 0.$$

Show that

$$w(x, y, z) - \alpha_o \bar{w}(r) > 0 \quad \text{for } 1 > r > 0.$$

That is,

$$u(x, y, z) - [\alpha_o + \alpha_o(1 - \alpha_o)] \bar{u}(r) > 0 \quad \text{for } 1 > r > 0. \quad (8)$$

In the above, \bar{w} is the spherical average of w similar to (5). By iterating the above process, explain the following. For any positive integer k ,

$$u(x, y, z) - [\alpha_o + \alpha_o(1 - \alpha_o) + \alpha_o(1 - \alpha_o)^2 + \cdots + \alpha_o(1 - \alpha_o)^k] \bar{u}(r) > 0 \quad \text{for } 1 > r > 0. \quad (9)$$

Conclude that

$$u(x, y, z) - \bar{u}(r) \geq 0 \quad \text{for } 1 > r > 0. \quad (10)$$

Explain why (10) yields

$$u(x, y, z) = \bar{u}(r) = C \left(\frac{1}{r} - 1 \right) \quad \text{for } 1 > r > 0,$$

where $C \geq 0$ is a constant.

(D) Assume that u is not identically zero on $\partial B_o(1)$. Using the Poisson formula, show that

$$u(x, y, z) = \frac{a}{r} + h(x, y, z) \quad \text{for } 1 > r > 0, \quad (11)$$

where $a \geq 0$ is a constant, and $h \in C^2(B_o(1)) \cap C^o(\overline{B_o(1)})$ a harmonic function in $B_o(1)$. This is part of the Bôcher (1867-1918) theorem.