Bounded Positive Solutions of Rotationally Symmetric Harmonic Map Equations

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Abstract

We consider bounded positive solutions α of rotationally symmetric harmonic map equations. We study the continuity of the maps $\alpha'(0) \mapsto \alpha(\infty)$ and $\alpha(1) \mapsto \alpha(\infty)$ in connection with the Dirichlet problem at infinity. Regularity at zero, local properties and conditions for positive solutions to be blowing up, unbounded, or bounded are discussed.

KEY WORDS: positive solutions, harmonic maps, rotationally symmetric.

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1. Introduction

In this paper we study positive C^2-solutions α of the rotationally symmetric harmonic map equation

$$\alpha''(r) + (n-1)\frac{f'(r)}{f(r)}\alpha'(r) - (n-1)\frac{g(\alpha(r))g'(\alpha(r))}{f^2(r)} = 0 \tag{1.1}$$

for $r > 0$, with prescribed limit

$$\lim_{r \to 0^+} \alpha(r) = 0\,. \tag{1.2}$$

We assume that

$$f,\, g \in C^4([0,\infty))\,, \qquad f(0) = g(0) = 0\,, \tag{1.3}$$
$$f'(0) = g'(0) = 1 \qquad \text{and} \qquad f(r) > 0\,,\ g(y) > 0$$

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for $r > 0,\ y > 0$. Equation (1.1) arises as the Euler-Lagrange equation of the energy functional for rotationally symmetric maps between model Riemannian manifolds [18]

$$M(f) = ([0, \infty) \times S^{n-1}\,,\ dr^2 + f^2(r)\, d\vartheta^2\ )$$

and

$$N(g) = ([0, \infty) \times S^{n-1}\,,\ dr^2 + g^2(r)\, d\vartheta^2\ )\,,$$

where $(S^{n-1},\, d\vartheta^2)$ is the unit sphere in $\mathbb{R}^n$ with the standard Riemannian metric $d\vartheta^2$, f and g satisfy the conditions in (1.3). $M(f)$ and $N(g)$ are complete non-compact Riemannian manifolds [11]. Examples are Euclidean space and hyperbolic space, which correspond to $f(r) = r$ and $f(r) = \sinh r$ for $r > 0$, respectively. In the context, $\alpha(r)$ represents the radial distance and it is natural to require that $\alpha \ge 0$ and α satisfies prescribed limit (1.2). Equation (1.1) is studied in [18] by Ratto and Rigoli for Liouville type theorems and existence of bounded positive solutions. Local existence of positive solutions of equation (1.1) with prescribed limit (1.2) is studied in [3] and [6]. The solution is said to be entire if it is defined and satisfies equation (1.1) in $\mathbb{R}^+$. Consider the collection of all positive C^2-solutions of equation (1.1) with prescribed limit (1.2). There are four possible cases.

(I) Every local positive C^2-solution of equation (1.1) with prescribed limit (1.2) blows up in a bounded interval when it is extended.

(II) All positive entire C^2-solutions of equation (1.1) with prescribed limit (1.2) are unbounded from above in $\mathbb{R}^+$ (a Liouville type theorem holds).

(III) All positive entire C^2-solutions of equation (1.1) with prescribed limit (1.2) are bounded from above in $\mathbb{R}^+$.

(IV) There is an unbounded and a bounded positive entire C^2-solution of equation (1.1) with prescribed limit (1.2).

A main purpose is to identify the conditions on f and g for the possible cases to occur. We show that, for $n \ge 2$ and under some conditions on f and g (see section 5), if

$$(1.4) \qquad \int_1^\infty \frac{dr}{f^{n-1}(r)} = \infty \quad \text{and} \quad \int_1^\infty \frac{dy}{g(y)} < \infty\,,$$

then we have case (I). If we assume that

$$(1.5) \qquad \int_1^\infty \frac{dr}{f^{n-1}(r)} = \infty \quad \text{and} \quad \int_1^\infty \frac{dy}{g(y)} = \infty\,,$$

then we have case (II), that is, every positive entire C^2-solution with prescribed limit (1.2) is unbounded. If we assume that

$$(1.6) \qquad \int_1^\infty \frac{dr}{f(r)} < \infty \quad \text{and} \quad \int_1^\infty \frac{dy}{g(y)} = \infty\,,$$

then we have case (III).

The most interesting case is when

$$\int_1^\infty \frac{dr}{f(r)} < \infty \quad \text{and} \quad \int_1^\infty \frac{dy}{g(y)} < \infty, \tag{1.7}$$

which is related to case (IV). It is shown in [7] that when $n = 2$, there is a unique unbounded positive entire C^2-solution α_τ of equation (1.1) with prescribed limit (1.2). Furthermore, any positive C^2-solution with prescribed limit (1.2) that is smaller than $\alpha_\tau(t)$ for a positive number t is bounded from above in $\mathbb{R}^+$, and any positive C^2-solution with prescribed limit (1.2) that is larger than $\alpha_\tau(t)$ for a positive number t blows up in a bounded interval when it is extended. For $n > 2$, under the condition of (1.7), we show that if a non-negative C^2-solution α of equation (1.1) with prescribed limit (1.2) and with $\alpha(1)$ small enough, then α can be extended to a bounded non-negative entire C^2-solution in $\mathbb{R}^+$. On the other hand, if α grows up fast, then α blows up in a bounded interval when it is extended .

We study case (III) and case (IV) in more detail, in association with the Dirichlet problems at infinity. Consider the map

$$\phi_1 : \ \alpha(1) \mapsto \alpha(\infty)\,, \tag{1.8}$$

where $\alpha(\infty) = \lim_{r\to\infty} \alpha(r)$. We define the domain of the map to be

$$I \ = \ \Big\{\alpha(1) \mid \alpha \in C^2((0,\infty) \cap C^1([0,\infty)) \ \text{ is a bounded non-negative solution of equation (1.1) in } \mathbb{R}^+ \text{ with } \alpha(0) = 0\Big\}\,.$$

If $g'(y) > 0$ for $y > 0$, then a positive C^2-solution of equation (1.1) with prescribed limit (1.2) is an increasing function. If α is bounded from above in $\mathbb{R}^+$, then $\alpha(\infty)$ exists and hence the map ϕ_1 is well-defined. We show in section 6 that I is path connected. Therefore $I = [0, M)$ or $I = [0, M]$, where in the first case M is a positive number or infinity, and in the second case M is a positive number or 0. It is proved that the map $\alpha(1) \mapsto \alpha(\infty)$ is a continuous map from I to $\mathbb{R}^+$ (continuous from the right for 0, and from the left for M, when $I = [0, M]$ and $0 < M < \infty$). If there exists an unbounded positive entire C^2-solution of equation (1.1) with prescribed limit (1.2), then $M < \infty$. We obtain the following result.

Theorem A. *In equation (1.1), assume that* $n \ge 2$, $f''(0) = g''(0) = 0$, $f''(r) \ge 0$ *for* $r > 0$, *and* $g''(y) \ge 0$ *for* $y > 0$. *Assume also that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty\,. \tag{1.9}$$

If there exists an unbounded positive solution $\alpha_\tau \in C^2((0,\infty))$ *of equation (1.1) in* $\mathbb{R}^+$ *with prescribed limit (1.2), then* $I = [0, M)$ *for some* $M > 0$ *and the* *map ϕ_1 is a homeomorphism from the I to $[0, \infty)$. In particular, for every non-negative number a, there exists a unique bounded non-negative entire solution $\alpha \in C^2((0,\infty)) \cap C^1([0,\infty))$ of equation (1.1) with $\alpha(0) = 0$ and*

$$\alpha(\infty) = \lim_{r \to \infty} \alpha(r) = a\,. \tag{1.10}$$

We note that if $f = g$ in $\mathbb{R}^+$, then we may take the α_τ in theorem A to be the identity map, that is, $\alpha_\tau(r) = r$ for $r > 0$. Moreover, functions of the type $f(r) \sim r^k$ for $k > 1$ satisfy condition (1.9). For $n \geq 2$, the radial Ricci curvature of the Riemannian manifold $M(f)$ is given by $-(n-1)f''(r)/f(r)$ for $r > 0$. The assumption in theorem A that $f'' \geq 0\,,\ g'' \geq 0$ in $\mathbb{R}^+$ is equivalent to the assumption that the radial Ricci curvatures of $M(f)$ and $N(g)$ are non-positive. Existence results on bounded harmonic maps often require the sectional curvature to be non-positive (cf. [1] [2] [10] [12] [13]). As indicated by the proof of theorem A, the condition $f''(r) \geq 0$ for $r > 0$ can be replaced by weaker assumption on the growth of f. In section 5, it is shown that the conditions in theorem A are close to being necessary by obtaining the Liouville type theorems 5.57 and 5.58.

The geometric meaning of the condition $f''(0) = 0$ and $g''(0) = 0$ can be observed from the fact that if the Riemannian metrics $dr^2 + f^2(r)\,d\vartheta^2$ and $dr^2 + g^2(r)\,d\vartheta^2$ are smooth at the origin, then all the odd order derivatives of the functions f^2 and g^2 are equal to zero at $r = 0$. This is because

$$\frac{\partial^k f^2}{\partial\, x_1^k}(0) = (f^2)^{(k)}(0) = (-1)^k \frac{\partial^k f^2}{\partial\,(-x_1)^k}(0)\,,$$

where k is a positive integer and $(x_1, ..., x_n)$ are Cartesian coordinates on $\mathbb{R}^n$. In particular, we have

$$0 = \left(f^2\right)'''(0) = 6f''(0)f'(0) + 2f'''(0)f(0) = 6f''(0)\,,$$

as $f(0) = 0$ and $f'(0) = 1$. Therefore if the Riemannian metrics are smooth at the origin, then $f''(0) = g''(0) = 0$ (with all this, compare also [15]).

For $n = 2$, equation (1.1) becomes

$$\alpha'(r) = \frac{g(\alpha(r))}{f(r)} \quad \text{for} \quad r > 0 \tag{1.11}$$

(cf. [6]). It is a first order semi-linear differential equation in its simple form. We treat this special case separately in another paper [7], where a characterization of cases (I) to (IV) is obtained in terms of whether $1/f$ and $1/g$ are in $L^1((1,\infty))$ or not. It is also shown that the conclusions in theorem A remain valid under the assumption that $1/f \in L^1((1,\infty))$.

Dirichlet problem at infinity for equation (1.1) is considered by Ratto and Rigoli in [18], where they show that if there exist constants $\lambda \in (0,1)$, $k > 1$ and $R_o > 0$ such that

$$(1.12)\quad (n-1)\lambda f'(r)/f(r) > k/r \quad \text{and} \quad (1-\lambda) f'(r) f(r) > r^k \quad \text{for} \quad \text{all} \;\; r \ge R_o\,,$$

then there exists a positive number L such that the Dirichlet problem at infinity admits a non-negative C^2-solution α for any limit value $\alpha(\infty) \in (0, L]$. The proof is by constructing a sub-solution of equation (1.1) in $\mathbb{R}^+$. They also note that if there is an unbounded positive entire C^2-solution, then, under the conditions in (1.12), the Dirichlet problem at infinity is actually solvable for any limit value $\alpha(\infty) \in [0, \infty)$. Condition (1.9) is weaker than condition (1.12), as condition (1.12) implies that $f(r) \ge C r^{1+(k-1)/2}$ for large r and for a positive constant C. Hence $1/f \in L^1((1,\infty))$. While functions of the form

$$f(r) = (r+1)\,[\ln{(r+1)}]^m \qquad \text{for} \;\; r \ge 1\,,$$

where $m > 1$ is a constant, satisfy $1/f \in L^1((1,\infty))$ but do not satisfy the second condition in (1.12).

We note that, by a result of Milnor [17], the Riemannian metric $dr^2 + f^2(r)\,d\vartheta^2$ on $\mathbb{R}^2$ is hyperbolic (that is, conformally equivalent to the open unit disc) if and only if $1/f \in L^1((1,\infty))$.

The proof of continuity of the map ϕ_1 consists of a local and global argument. A main step in the proof of theorem A is a stability result which says that I is open on the right, that is, $I = [0, M)$ for some $M > 0$. To this end the regularity of α at zero is examined. Then we show that the Dirichlet problem at infinity has a non-negative C^2-solution for any limit value in $[0,\infty)$.

For local properties, we show that if $f''(0) = g''(0) = 0$ and $g''(y) \ge 0$ for $y > 0$, then the map $\alpha'(0) \mapsto \alpha(1)$ is continuous. This can be used to show the existence of non-negative solutions with prescribed values $\alpha(1)$. The regularity at zero for non-negative solutions of equation (1.1) with prescribed limit (1.2) is considered. While the Frobenius method indicates that in a formal power series solution the term $-r\ln r$ may be present, we show that this is the worst possible situation. Under additional conditions on f and g, non-negative C^2-solutions of equation (1.1) with prescribed limit (1.2) are actually C^1 up to zero.

We describe briefly the content of each section. In section 2 we discuss existence, uniqueness and basic properties of non-negative C^2-solutions of equation (1.1). Continuity of the map $\alpha'(0) \mapsto \alpha(1)$ is studied in section 3. Section 4 is devoted to obtaining estimates and regularity of positive solutions at zero. In section 5 we discuss conditions for cases (I) to (IV) mentioned above, together with Liouville type theorems. Theorem A is proved in section 6, where we show that the map ϕ_1 is continuous, $I = [0, M)$ and the Dirichlet problem at infinity is solvable for any non-negative number.

2. Preliminaries

Local existence of positive solutions of equation (1.1) is studied in [6].

Local Existence Theorem 2.1 [6]. *For $n \geq 1$, given any $a \geq 0$, there exist a positive number ϵ and a unique positive function $\alpha_a \in C^2([0, \epsilon))$ which satisfies equation (1.1) in $(0, \epsilon)$ with $\alpha_a(0) = 0$ and $\alpha'_a(0) = a$.*

In fact, only the case $a > 0$ is studied in [6]. By modifying the argument there, one can also settle the case $a = 0$. We note that when $n = 1$, equation (1.1) reduces to $\alpha''(r) = 0$. Therefore if α is a non-negative solutions of equation (1.1) with prescribed limit (1.2), then we have $\alpha(r) = ar$ for $r > 0$ and for a non-negative number a. We have the following uniqueness theorem [16].

Uniqueness Theorem 2.2 [16]. *For $n \geq 1$, assume that $g''(y) \geq 0$ for $y > 0$. Let $\alpha, \beta \in C^2((0, R))$ be positive solutions of equation (1.1) in $(0, R)$ for some $R > 0$, with prescribed limit (1.2). If there is a number $R_o \in (0, R)$ such that $\alpha(R_o) = \beta(R_o)$, then $\alpha \equiv \beta$ in $(0, R)$.*

Theorem 2.2 shows that, under the conditions of the theorem, if $\alpha(r_o) > \beta(r_o)$ for a number $r_o \in (0, R)$, then $\alpha(r) > \beta(r)$ for all $r \in (0, R)$. For $n > 1$, the assumption in theorem 2.2 is equivalent to the assumption that the radial Ricci curvature of $N(g)$ is non-positive. We note that for harmonic maps, uniqueness results often require the sectional curvature of $N(g)$ to be non-positive (cf. [12] [14]), while existence results on bounded harmonic maps often require the sectional curvature of $M(f)$ to be non-positive (cf. [1] [2] [12]). We consider the case when the solution is non-negative. The proof of the following lemma is essentially the same as the proof of lemma 2.2 in [18].

Lemma 2.3. *For $n \geq 1$, let $\beta \in C^2((0, R))$ be a non-negative C^2-solution of equation (1.1) in $(0, R)$ for some $R > 0$, with prescribed limit (1.2). Assume that $g'(\beta(r)) > 0$ for $r \in (0, R)$. Then either $\beta \equiv 0$ in $(0, R)$ or $\beta'(r) > 0$ for $r \in (0, R)$.*

Lemma 2.3 implies that, under the conditions of the lemma, if $\beta(R_o) = 0$ for some $R_o \in (0, R)$, then $\beta \equiv 0$ on $(0, R)$. It follows from uniqueness theorem 2.2 that if $\alpha, \beta \in C^2((0, R)) \cap C^1([0, R))$ are non-negative solutions of equation

(1.1) in $(0, R)$ for some $R > 0$, with $\alpha(0) = \beta(0) = 0$ and $\alpha'(0) > \beta'(0)$, then we have $\alpha > \beta$ in $(0, R)$. Furthermore, local existence theorem 2.1 shows that if $\alpha'(0) = \beta'(0) \geq 0$, then $\alpha = \beta$ in $(0, R)$. Thus we have the following lemma.

Lemma 2.4. *For $n \geq 1$, assume that $g''(y) \geq 0$ for $y > 0$. Let $\alpha, \beta \in C^2((0,R)) \cap C^1([0,R))$ be non-negative solutions of equation (1.1) in $(0,R)$ for some $R > 0$, with $\alpha(0) = \beta(0) = 0$. If $\alpha'(0) \geq \beta'(0)$, then $\alpha(r) \geq \beta(r)$ for $r \in (0, R)$.*

Lemma 2.5. *For $n \geq 2$, let $\alpha \in C^2((0,R))$ be a non-negative C^2-solution of equation (1.1) in $(0,R)$ for some $R > 0$, with prescribed limit (1.2). Then*

$$\lim_{r \to 0^+} f^{n-1}(r)\alpha'(r) = 0\,. \tag{2.6}$$

Proof. As $g'(0) = 1$, we have $g'(y) > 0$ for $y > 0$ close to zero. From equation (1.1) we obtain

$$\left(f^{n-1}(r)\alpha'(r)\right)' = (n-1)f^{n-3}(r)\, g(\alpha(r))\, g'(\alpha(r)) \geq 0 \tag{2.7}$$

for $r > 0$ close to zero. We conclude that $f^{n-1}(r)\alpha'(r)$ is non-increasing as r decreases to 0, for $r > 0$ close to zero. By lemma 2.3, we need only to consider the case $\alpha'(r) > 0$ for $r > 0$ close to zero. Thus

$$\lim_{r \to 0^+} f^{n-1}(r)\alpha'(r) = a \geq 0\,. \tag{2.8}$$

Suppose that $a > 0$. Then for all $r > 0$ sufficiently small we have

$$f^{n-1}(r)\alpha'(r) \geq \frac{a}{2}\,. \tag{2.9}$$

We have $0 < f(r) \leq Cr$ for $r > 0$ close to zero, where C is a positive constant. From (2.9) we obtain

$$\alpha'(r) \geq \frac{C_o}{r^{n-1}} \tag{2.10}$$

for $r > 0$ close to zero. Here $C_o = a/(2C^{n-1})$ is a positive constant. Integrating both sides (2.10) from ε to r, we have

$$\alpha(r) - \alpha(\varepsilon) \geq \frac{C_o}{n-2}\left(\frac{1}{\varepsilon^{n-2}} - \frac{1}{r^{n-2}}\right) \quad \text{if } \ n > 2\,, \tag{2.11}$$

where $r > \varepsilon > 0$ and r is close to zero, and similar formula for $n = 2$. As $\lim_{\epsilon \to 0^+} \alpha(\epsilon) = 0$, (2.11) provides a contradiction as $\varepsilon \to 0^+$. Hence we have $a = 0$ in (2.8). □

Lemma 2.5 provides a crude but useful result on the regularity of α' at zero. The result is generalized in lemma 4.22. We note that lemma 2.5 does not hold when $n = 1$, as $\alpha'(0)$ may be positive. In [5] (see also [7]), it is shown that if $\alpha \in C^2((0,R))$ is a non-negative function with $\lim_{r\to 0^+} \alpha(r) = 0$, then α satisfies equation (1.1) with $n = 2$ in $(0,R)$ if and only if

$$\alpha'(r) = \frac{g(\alpha(r))}{f(r)} \quad \text{for} \quad r \in (0,R)\,. \tag{2.12}$$

3. Local Properties

We have the following local version of uniqueness theorem 2.2.

Lemma 3.1. *For $n \geq 1$, given positive numbers C and ε, there exists a positive number δ which depends on C, ε, and g only, such that for any non-negative solutions $\alpha\,,\ \beta \in C^2((0,\varepsilon))$ of equation (1.1) in $(0,\varepsilon)$ with prescribed limit (1.2), if*

$$\alpha(r) \leq Cr\,, \quad \beta(r) \leq Cr \quad \textit{for} \quad r \in (0,\varepsilon)\,,$$

and there is a number $r_1 \in (0,\delta)$ such that $\alpha(r_1) = \beta(r_1)$, then $\alpha \equiv \beta$ in $(0,\varepsilon)$. Moreover, if $\beta(r) \leq \alpha(r) \leq Cr$ for $r \in (0,\delta)$, then $F'(r) \geq 0$ for $r \in (0,\delta)$, where $F = \alpha - \beta$.

Proof. The result holds when $n = 1$. Assume that $n \geq 2$. From equation (1.1) we have

$$F''(r) + (n-1)\frac{f'(r)}{f(r)}F'(r) = \frac{(n-1)}{f^2(r)}\left[g(\alpha(r))g'(\alpha(r)) - g(\beta(r))g'(\beta(r))\right] \tag{3.2}$$

for $r \in (0,\varepsilon)$. Hence we obtain

$$F''(r) + (n-1)\frac{f'(r)}{f(r)}F'(r) = \frac{(n-1)}{2\,f^2(r)}\left\{\left[g^2(y)\right]'\Big|_{y=\alpha(r)} - \left[g^2(y)\right]'\Big|_{y=\beta(r)}\right\} \tag{3.3}$$

for $r \in (0,\varepsilon)$ We have

$$\left[g^2(y)\right]' = 2g(y)\,g'(y) \quad \text{and} \quad \left[g^2(y)\right]'' = 2\,[g'(y)]^2 + 2g(y)\,g''(y) \quad \text{for} \quad y > 0\,.$$

As $g(0) = 0$ and $g'(0) = 1$, there exists a positive number δ such that $\delta \leq \varepsilon$ and

$$\left[g^2(y)\right]'' > 0 \quad \text{for} \quad y \in (0, C\delta)\,.$$

Hence $(g^2)'$ is an increasing function on $(0,C\delta)$. Assume that there is a number $r_1 \in (0,\delta)$ such that $\alpha(r_1) = \beta(r_1)$. Suppose that $\alpha \not\equiv \beta$ in $(0,\delta)$. As

$$\lim_{r\to 0^+} \alpha(r) = \lim_{r\to 0^+} \beta(r) = 0\,,$$

without loss of generality, we may assume that there is a number $\bar{r} \in (0, r_1)$ such that

$$\alpha(\bar{r}) > \beta(\bar{r}) , \qquad \alpha'(\bar{r}) = \beta'(\bar{r}) \quad \text{and} \quad \alpha''(\bar{r}) \le \beta''(\bar{r}) .$$

As $\beta(\bar{r}) < \alpha(\bar{r}) < C\delta$, it follows from (3.3) that

$$\alpha''(\bar{r}) - \beta''(\bar{r}) = F''(\bar{r}) = \frac{(n-1)}{2f^2(\bar{r})} \left\{ \left[g^2(y)\right]'\Big|_{y=\alpha(\bar{r})} - \left[g^2(y)\right]'\Big|_{y=\beta(\bar{r})} \right\} > 0 ,$$

which is a contradiction. Hence $\alpha \equiv \beta$ in $(0, \delta)$. A uniqueness result in [8], pp. 259, shows that $\alpha \equiv \beta$ in $(0, \varepsilon)$. Assume that $\beta(r) \le \alpha(r)$ for $r \in (0, \delta)$. From (3.3) we have

$$\left[f^{n-1}(r)F'(r)\right]' = \frac{n-1}{2} f^{n-3}(r) \left\{ \left[g^2(y)\right]'\Big|_{y=\alpha(r)} - \left[g^2(y)\right]'\Big|_{y=\beta(r)} \right\} \ge 0 \tag{3.4}$$

for $r \in (0, \delta)$. Integrating both sides of the above formula from r to r_o, letting $r_o \to 0^+$ and using lemma 2.5, we obtain $F'(r) \ge 0$ for $r \in (0, \delta)$. □

From the proof of lemma 3.1, we have the following result.

Lemma 3.5. *For $n \ge 1$, let α, $\beta \in C^2((0, R))$ be non-negative solutions of equation (1.1) in $(0, R)$ for some $R > 0$, with prescribed limit (1.2). Let*

$$F(r) = \alpha(r) - \beta(r) \quad \textit{for} \;\; r \in (0, R) . \tag{3.6}$$

Assume that $g''(\alpha(r)) \ge 0$ for $r \in (0, R)$. If $\alpha(r) \ge \beta(r)$ for $r \in (0, R)$, then $F'(r) \ge 0$ for $r \in (0, R)$.

Lemma 3.7. *For $n \ge 1$, assume that $f''(0) = g''(0) = 0$. Let α, $\beta \in C^2((0, R)) \cap C^1([0, R))$ be non-negative solutions of equation (1.1) in $(0, R)$ for some $R > 0$, with $\alpha(0) = \beta(0) = 0$. Assume that $\alpha'(0) > \beta'(0) \ge 0$ and $\alpha(r) \le C_o r$ for a positive number C_o and for $r \in (0, R)$. There exist positive numbers ϵ, C_1 and C_2, where ϵ depends on C_o , f , g , n and R only, and C_1 and C_2 depend on ϵ but not on α and β, such that $\epsilon < R$ and*

$$C_1 \left[\alpha'(0) - \beta'(0)\right] \le \alpha(\epsilon) - \beta(\epsilon) \le C_2 \left[\alpha'(0) - \beta'(0)\right] . \tag{3.8}$$

Proof. For the case $n = 1$ we have $\alpha(r) = ar$ and $\beta(r) = br$, where $a = \alpha'(0)$ and $b = \beta'(0)$. The conclusion holds. For $n > 1$, using Taylor's expansion and the assumption that $f''(0) = g''(0) = 0$, we have

$$\begin{aligned} f(r) &= r + O(r^3) , \qquad g(r) = r + O(r^3) ; \\ f'(r) &= 1 + O(r^2) , \qquad g'(r) = 1 + O(r^2) ; \\ g''(r) &= r + O(r^2) \quad \text{for} \;\; r \in (0, R) . \end{aligned} \tag{3.9}$$

By lemma 3.1, there exists a positive number δ, which depends on C_o, R and g only, such that $\delta < R$ and $F(r) \geq 0$ for $r \in (0, \delta)$. Lemma 3.1 also shows that $F'(r) \geq 0$ for $r \in (0, \delta)$ if we choose δ to be small enough. Consider equation (3.2). We observe that

$$
\begin{aligned}
(3.10) \qquad & g(\alpha(r))g'(\alpha(r)) - g(\beta(r))g'(\beta(r)) \\
= \; & g'(\alpha(r))\,[g(\alpha(r)) - g(\beta(r))] + g(\beta(r))\,[g'(\alpha(r)) - g'(\beta(r))]
\end{aligned}
$$

for $r \in (0, \delta)$. We have

$$
\begin{aligned}
(3.11) \qquad g(\alpha(r)) - g(\beta(r)) &= \int_{\beta(r)}^{\alpha(r)} g'(s)\,ds \leq \int_{\beta(r)}^{\alpha(r)} (1 + C_1 s^2)\,ds \\
&= \left(s + C_1 \frac{s^3}{3} \right) \Bigg|_{\beta(r)}^{\alpha(r)} \\
&= F(r) + \frac{C_1}{3} F(r) \left[\alpha^2(r) + \alpha(r)\beta(r) + \beta^2(r) \right]
\end{aligned}
$$

for $r \in (0, \delta)$, where C_1 is a positive constant that depends on C_o, δ and g only. Therefore

$$
(3.12) \qquad g(\alpha(r)) - g(\beta(r)) = F(r)[1 + O(r^2)]\,,
$$

as $0 \leq \beta(r) < \alpha(r) \leq C_o r$ for $r \in (0, \delta)$. Similarly,
(3.13)

$$
g'(\alpha(r)) - g'(\beta(r)) = \frac{F(r)}{2}\,[\alpha(r) + \beta(r)] + \frac{C_2}{3} F(r) \left[\alpha^2(r) + \alpha(r)\beta(r) + \beta^2(r) \right]
$$

for $r \in (0, \delta)$, where C_2 is a positive constant that depends on C_o, δ and g only. Therefore

$$
(3.14) \qquad g'(\alpha(r)) - g'(\beta(r)) = F(r) O(r)\,,
$$

as $0 \leq \beta(r) < \alpha(r) \leq C_o r$ for $r \in (0, \delta)$. Hence

$$
\begin{aligned}
(3.15) \qquad & g(\alpha(r))g'(\alpha(r)) - g(\beta(r))g'(\beta(r)) \\
\leq \; & \left(1 + O(r^2)\right) F(r) \left(1 + O(r^2)\right) + \left(C_o r + O(r^3)\right) F(r) O(r) \\
= \; & F(r) + F(r) O(r^2) \quad \text{for} \;\; r \in (0, \delta)\,.
\end{aligned}
$$

Furthermore we have

$$
(3.16) \qquad \frac{f'(r)}{f(r)} = \frac{1}{r} + O(r) \quad \text{and} \quad \frac{1}{f^2(r)} = \frac{1}{r^2} + O(1) \quad \text{for} \;\; r \in (0, \delta)\,.
$$

From equation (3.2), we obtain

$$
\begin{aligned}
F''(r) &\leq -(n-1) \left[\frac{1}{r} + O(r) \right] F'(r) + (n-1) \left[\frac{1}{r^2} + O(1) \right] \left[F(r) + F(r) O(r^2) \right] \\
&= -(n-1) \left[\frac{F'(r)}{r} - \frac{F(r)}{r^2} \right] + F'(r) O(r) + F(r) O(1)
\end{aligned}
$$

for $r \in (0, \delta)$. Therefore

$$F''(r) \le -(n-1)\left(\frac{F(r)}{r}\right)' + C_3\, rF'(r) + C_4 F(r) \quad \text{for } \ r \in (0, \delta)\,, \tag{3.17}$$

where C_3 and C_4 are positive constants that depend on C_o , δ , f and g only. We have $F'(r) \ge 0$ for $r \in (0, \delta)$ and hence F is a non-decreasing function on $(0, \delta)$. Integrating both sides of (3.17) we obtain

$$\begin{aligned} & F'(r) - F'(r_o) \\ \le\ & -(n-1)\left[\frac{F(r)}{r} - \frac{F(r_o)}{r_o}\right] + C_3 \int_{r_o}^{r} sF'(s)\, ds + C_4 \int_{r_o}^{r} F(s)\, ds \\ \le\ & -(n-1)\left[\frac{F(r)}{r} - \frac{F(r_o)}{r_o}\right] + C_3\, r[F(r) - F(r_o)] + C_4 F(r)(r - r_o)\,, \end{aligned} \tag{3.18}$$

where $\delta > r > r_o > 0$. Let $r_o \to 0^+$ we have

$$\lim_{r_o \to 0^+} \frac{F(r_o)}{r_o} = F'(0)\,. \tag{3.19}$$

It follows that

$$F'(r) - F'(0) \le -(n-1)\frac{F(r)}{r} + (n-1)F'(0) + (C_3 + C_4)\, rF(r) \tag{3.20}$$

for $r \in (0, \delta)$. As $F'(r) > 0$ for $r \in (0, \delta)$, we obtain

$$(n-1)\frac{F(r)}{r} - (C_3 + C_4)\, rF(r) \le nF'(0) \quad \text{for } \ r \in (0, \delta)\,. \tag{3.21}$$

That is,

$$\left[\frac{n-1}{nr} - \frac{(C_3 + C_4)r}{n}\right] F(r) \le F'(0) \quad \text{for } \ r \in (0, \delta)\,. \tag{3.22}$$

By taking $r = \epsilon > 0$ to be small (ϵ depends on C_o , δ , f and g only) so that

$$\frac{n-1}{n\epsilon} - \frac{(C_3 + C_4)\epsilon}{n} > 0\,. \tag{3.23}$$

then we obtain the right hand side inequality in (3.8). Similarly, from (3.2) we obtain

$$\begin{aligned} F''(r) \ \ge\ & -(n-1)\left[\frac{1}{r} + O(r)\right] F'(r) + \left[\frac{1}{r^2} + O(1)\right]\left[F(r) - F(r)O(r^2)\right] \\ \ge\ & -(n-1)\left(\frac{F(r)}{r}\right)' - C_5\, rF'(r) - C_6 F(r) \quad \text{for } \ r \in (0, \delta)\,, \end{aligned} \tag{3.24}$$

where C_5 and C_6 are positive constants that depend on C_o, δ and f only. Integrating both sides of (3.24) from $r > 0$ to r_o and letting $r_o \to 0^+$ we have

$$F'(r) - F'(0) \geq -(n-1)\left[\frac{F(r)}{r} - F'(0)\right] - C_5\, rF(r) - C_6 F(r) r \tag{3.25}$$

for $r \in (0, \delta)$, where we make use of the fact that $F' \geq 0$ and F is non-decreasing on $(0, \delta)$. Therefore we have

$$F'(r) + (n-1)\frac{F(r)}{r} + C_7\, rF(r) \geq nF'(0) \quad \text{for } \; r \in (0, \delta)\,, \tag{3.26}$$

where $C_7 = C_5 + C_6$ is a positive constant. That is,

$$\left(r^{n-1}F(r)\right)' + C_7\, r^n F(r) \geq nr^{n-1}F'(0) \quad \text{for } \; r \in (0, \delta)\,. \tag{3.27}$$

An integration from $r > 0$ to 0 we obtain

$$r^{n-1}F(r) + \frac{C_7}{n+1} r^{n+1} F(r) \geq r^n F'(0) \quad \text{for } \; r \in (0, \delta)\,, \tag{3.28}$$

again making use of the fact that F in non-decreasing on $(0, \delta)$. Therefore

$$\left[\frac{1}{\epsilon} + \frac{C_7}{n+1}\epsilon\right] F(\epsilon) \geq F'(0)\,. \tag{3.29}$$

This completes the proof of the lemma. □

Lemma 3.30. *For $n \geq 2$, let α, $\beta \in C^2((0, R)) \cap C^1([0, R))$ be non-negative solutions of equation (1.1) in $(0, R)$ for some $R > 1$, with $\alpha(0) = \beta(0) = 0$. Assume that $\alpha(r) > \beta(r) \geq 0$ and $\alpha(r) < C'_o$ for a positive constant C'_o and for $r \in (0, R)$. Assume also that $g''(y) \geq 0$ for $y \in (0, C'_o)$. For every $\epsilon \in (0, 1)$, there exists a positive number C' that depends on R, C'_o, ϵ, f, n and g only, such that*

$$\alpha(1) - \beta(1) \leq C'\,[\alpha(\epsilon) - \beta(\epsilon)]\,. \tag{3.31}$$

Proof. Let $F(r) = \alpha(r) - \beta(r)$ for $r \in (0, R)$. We claim that there exists a positive constant C_1 depending on R, C'_o, f, n and g only, such that

$$F'(r) \leq \frac{C_1}{f(r)} F(r) \quad \text{for } \; r \in [\epsilon, R)\,. \tag{3.32}$$

Assuming the claim for the moment, integrating both sides of (3.32) from 1 to $\epsilon \in (0, 1)$ we have

$$\ln\left[\frac{F(1)}{F(\epsilon)}\right] \leq \int_\epsilon^1 \frac{C_1}{f(r)}\, dr\,. \tag{3.33}$$

Let

$$\int_{\epsilon}^{1} \frac{dr}{f(r)} = C_2 \,. \tag{3.34}$$

We have

$$F(1) \le e^{C_1 C_2} F(\epsilon)\,. \tag{3.35}$$

Let $C' = e^{C_1 C_2}$, then we have (3.31). To proof the claim, consider equation (3.2) and (3.10). As $g \in C^2([0,\infty))$ and $\alpha(r) < C'_o$ in $(0,R)$, there exist positive constants C'_1 and C'_2 depending on C'_o and g only, such that

$$|g(x) - g(y)| \le C'_1 |x-y| \quad \text{and} \quad |g'(x) - g'(y)| \le C'_2 |x-y| \tag{3.36}$$

for $x,\ y \in [0\,,\,C'_o]$. When $n=2$, using equation (2.12) and (3.36) we obtain

$$F'(r) = \frac{g(\alpha(r)) - g(\beta(r))}{f(r)} \le \frac{C'_1}{f(r)} F(r) \quad \text{for} \ \ r \in (0,R)\,. \tag{3.37}$$

That is the claim (3.32). For $n \ge 3$, we have

$$g(\alpha(r))g'(\alpha(r)) - g(\beta(r))g'(\beta(r)) \le C_3\,[\alpha(r) - \beta(r)]\ , \tag{3.38}$$

by using (3.36) and the fact that $\beta(r) < \alpha(r) < C'_o$ for $r \in (0,R)$, where C_3 is a positive constant that depends on R, C'_o and g only. Putting the information into equation (3.2) we obtain

$$F''(r) + (n-1)\frac{f'(r)}{f(r)} F'(r) \le \frac{C_4}{f^2(r)} F(r) \quad \text{for} \ \ r \in (0,R)\,, \tag{3.39}$$

where C_4 is a positive constant that depends on R, C'_o, n and g only. That is,

$$\left(f^{n-1}(r) F'(r) \right)' \le C_4 f^{n-3}(r) F(r) \quad \text{for} \ \ r \in (0,R)\,. \tag{3.40}$$

As $\alpha,\ \beta \in C^2((0,R)) \cap C^1([0,R))$ and $n \ge 3$, we obtain

$$\lim_{r_o \to 0^+} f^{n-1}(r_o) F'(r_o) = 0\,. \tag{3.41}$$

By lemma 3.1 we have $F'(r) \ge 0$ for $r \in (0,R)$. Integrating both sides of (3.40) from r_o to r we have

$$f^{n-1}(r)F'(r) - f^{n-1}(r_o)F'(r_o) \le C_4 \int_{r_o}^{r} f^{n-3}(s)F(s)\,dr \le C_4 F(r) \int_{r_o}^{r} f^{n-3}(s)\,ds$$

for $R > r > r_o > 0$. Here we make use of the fact that F is non-decreasing on (r_o, R_o). Letting $r_o \to 0^+$ we have

$$\begin{aligned} F'(r) \ &\le\ \frac{C_4}{f^{n-1}(r)} F(r) \lim_{r_o \to 0^+} \int_{r_o}^{r} f^{n-3}(s)\,ds \ \le\ \frac{C_5}{f(r)} F(r) \frac{1}{r^{n-2}} \int_0^r s^{n-3}\,ds \\ &\le\ \frac{C_6}{f(r)} F(r) \frac{1}{r^{n-2}} r^{n-2} \ \le\ \frac{C_6}{f(r)} F(r) \quad \text{for} \ \ r \in (0,R)\,, \end{aligned} \tag{3.42}$$

where we make use of the fact that there exist positive constants C_8 and C_9 such that $C_8\, r \le f(r) \le C_9\, r$ for $r \in (0, R)$. Here C_5 and C_6 and are positive constants that depends on R, C_o, n and g only. The proof of the claim is completed. $\square$

Remark 3.43. We note that, under the assumption of lemma 3.1, $F'(r) \ge 0$ for $r \in (0, R)$. Hence

$$\alpha(\epsilon) - \beta(\epsilon) \le \alpha(1) - \beta(1) \tag{3.44}$$

for $\epsilon \in (0, 1)$.

Theorem 3.45. *In equation (1.1), assume that $n \ge 2$, $f''(0) = g''(0) = 0$ and $g''(y) \ge 0$ for $y > 0$. Let $\alpha \in C^2((0, R)) \cap C^1([0, R))$ be a positive solution of equation (1.1) in $(0, R)$ for some $R > 1$, with $\alpha(0) = 0$. Then for every $b_1 \in [0,\, \alpha(1)]$, there is a unique non-negative C^2-solution $\beta \in C^2((0, R)) \cap C^1([0, R))$ of equation (1.1) in $(0, R)$ with $\beta(0) = 0$ and $\beta(1) = b_1$.*

Proof. By lemma 2.4, as α is positive, we have $a_o = \alpha'(0) > 0$. Given any $c_o \in [0, a_o]$, by theorem 2.1, there exist a positive number $\epsilon > 0$ and a non-negative solution $\gamma \in C^2((0, \epsilon)) \cap C^1([0, \epsilon))$ of equation (1.1) in $(0, \epsilon)$, such that $\gamma(0) = 0$ and $\gamma'(0) = c_o$. By lemma 2.4 we have $\alpha(r) \ge \gamma(r)$ for $r \in (0, \epsilon)$. It follows that γ is bounded and can be extended to a non-negative C^2-solution of equation (1.1) in $(0, R)$ (cf. [9] pp.15). Take a number $R_o \in (1, R)$. There is a positive constant C_o such that $\alpha(r) \le C_o r$ for $r \in (0, R_o)$. Lemma 3.7 and lemma 3.30 show that the map

$$\phi : [0, a_o] \to [0, \alpha(1)]\,, \qquad \phi(\gamma'(0)) = \gamma(1) \tag{3.46}$$

is a continuous map from $[0, a_o]$ to $[0, \alpha(1)]$. As 0 and $\alpha(1)$ are in the image of this map, by the intermediate value theorem, for every $b_1 \in [0, \alpha(1)]$, there is a $b_o \in [0, a_o]$ such that $\phi(b_o) = b_1$. That is, there exists a non-negative solution $\beta \in C^2((0, R)) \cap C^1([0, R))$ such that $\beta(0) = 0$ and $\beta(1) = b_1$. Uniqueness follows from uniqueness theorem 2.2. $\square$

Remark 3.47. It follows from lemma 3.7, lemma 3.30 and remark 3.43 that under the condition of theorem 3.45, the map ϕ in (3.46) is a homeomorphism.

4. Regularity at Zero

We proceed to study regularity at zero for non-negative C^2-solutions of equation (1.1) with prescribed limit (1.2). Consider the following example.

Example 4.1. Let $g(y) = y$ for $y \geq 0$. Assume that f is real analytic at zero, $f(0) = 0$ and $f'(0) = 1$. Equation (1.1) becomes

$$\alpha''(r) + (n-1)\frac{f'(r)}{f(r)}\alpha'(r) - (n-1)\frac{1}{f^2(r)}\alpha(r) = 0 \quad \text{for } r > 0\,. \tag{4.2}$$

We have

$$\frac{f'(r)}{f(r)} = \frac{1}{r}\left(1 + a_1 r + \cdots\right) \quad \text{and} \quad \frac{1}{f^2(r)} = \frac{1}{r^2}\left(1 + b_1 r + \cdots\right)\,. \tag{4.3}$$

The indicial equation of (4.2) is given by

$$y''(r) + \frac{n-1}{r}y'(r) - \frac{n-1}{r^2}y(r) = 0\,. \tag{4.4}$$

The indicial roots are $\lambda_1 = 1$ and $\lambda_2 = -(n-1)$. If we assume that n is an integer larger than one, then $\lambda_1 - \lambda_2 = n$ is a positive integer. The Frobenius method (cf. [4]) shows that the general solution of equation (4.2) near zero is given by

$$\alpha(r) = (A_1 r - A_2\, r \ln r)\left(\sum_{k=0}^{\infty} c_k\, r^k\right) + A_3\left(r^{-(n-1)}\sum_{k=0}^{\infty} d_k\, r^k\right)\,, \tag{4.5}$$

where A_1 , A_2 and A_3 are constants.

If the term $-r\ln r$ is present for small r, then α may not have higher order regularity at zero. Nevertheless, we show that this is the worst situation in general.

Theorem 4.6. *For $n \geq 1$, let $\alpha \in C^2((0,R))$ be a non-negative solution of equation (1.1) in $(0,R)$ for some $R > 0$, with prescribed limit (1.2). There exist positive constants C , C' and $\epsilon < \min\{e^{-1}, R\}$ such that*

$$\alpha(r) \leq -Cr\ln r \quad \textit{and} \quad \alpha'(r) \leq -C'\ln r \quad \textit{for } r \in (0,\epsilon)\,. \tag{4.7}$$

Proof. The conclusion holds when $n = 1$, as $\alpha(r) = cr$ in $(0,R)$ for some $c \geq 0$. We may assume that $n > 1$. For a positive constant C, let

$$\beta_C(r) = -Cr\ln r \quad \text{for } r \in (0,1)\,. \tag{4.8}$$

Note that $\ln r < 0$ for $r \in (0,1)$. Let y_o be a positive number such that $(g^2)' = 2gg'$ is an increasing function on $(0, y_o)$ (cf. the proof of lemma 3.1). There exist positive numbers c_1 and c_2 such that

$$g'(y) \geq 1 - c_1 y \quad \text{and} \quad g(y) \geq y - c_2\, y^2 \quad \text{for } y \in (0, y_o)\,. \tag{4.9}$$

For every number $C > e\, y_o$, let r_c be the positive number in $(0, e^{-1})$ such that

$$y_o = \beta_C\,(r) = -C\, r_c \ln r_c\,. \tag{4.10}$$

We have $0 < \beta_C\,(r) < y_o$ for $r \in (0, r_c)$. We note that as $C \to \infty$, $r_c \to 0$. Using (4.8), (4.9) and (4.10) we obtain

$$\begin{aligned} &\beta''_C(r) + (n-1)\frac{f'(r)}{f(r)}\beta'_C(r) - (n-1)\frac{g(\beta_C\,(r))g'(\beta_C\,(r))}{f^2(r)} \\ \le\ & -\frac{C}{r} - C(n-1)\left[\frac{1}{r} + O(1)\right](\ln r + 1) \\ & + C(n-1)\left[\frac{1}{r^2} + O\left(r^{-1}\right)\right]\left[r\ln r + (c_1+c_2)(r\ln r)^2 + c_1c_2(r\ln r)^3\right] \\ =\ & -C\left[\frac{n}{r} + O\left((\ln r)^2\right) + O(1)\right] \end{aligned} \tag{4.11}$$

for $r \in (0, r_c)$. Hence there exists a positive constant $C_o \ge e^{-1}$ such that for all $C \ge C_o$ we have

$$\beta''_C(r) + (n-1)\frac{f'(r)}{f(r)}\beta'_C(r) - (n-1)\frac{g(\beta_C\,(r))g'(\beta_C\,(r))}{f^2(r)} \le 0 \tag{4.12}$$

for $r \in (0, r_c)$, as r_c is small when C is large. The condition $\lim_{r\to 0^+}\alpha(r) = 0$ implies that there exists a positive constant $C > C_o$ such that

$$0 \le \alpha(r) < \frac{y_o}{2} \quad \text{for} \;\; r \in (0, r_c)\ . \tag{4.13}$$

We may also assume that C is large enough so that $r_c < R$. By (4.10) and (4.13) we have

$$\alpha(r_c) < y_o = \beta_C\,(r_c)\,. \tag{4.14}$$

Suppose that there is a point $r' \in (0, r_c)$ such that

$$\alpha(r') > \beta_C\,(r')\,. \tag{4.15}$$

As $\lim_{r\to 0^+}\alpha(r) = \lim_{r\to 0^+}\beta_C\,(r) = 0$ and $\beta_C\,(r_c) > \alpha(r_c)\,$, there is a point $\bar r \in (0, r_c)$ such that

$$\alpha(\bar r) > \beta_C\,(\bar r)\,, \quad \alpha'(\bar r) = \beta'_C(\bar r) \quad \text{and} \quad \alpha''\,(\bar r) \le \beta''_C(\bar r)\,. \tag{4.16}$$

As $g\,g'$ is an increasing function on $(0, y_o)$ and $\alpha(\bar r) < y_o$, equation (1.1) and inequality (4.12) together with (4.16) give

$$\begin{aligned} \alpha''(\bar r) &= (n-1)\left\{\frac{g(\alpha(\bar r))g'(\alpha(\bar r))}{f^2(r)} - \frac{f'(\bar r)}{f(\bar r)}\alpha'(\bar r)\right\} \\ &> (n-1)\left\{\frac{g(\beta_C\,(\bar r))g'(\beta_C\,(\bar r))}{f^2(\bar r)} - \frac{f'(\bar r)}{f(\bar r)}\beta'_C(\bar r)\right\} \ \ge\ \beta''_C\,(\bar r)\,, \end{aligned}$$

which is a contradiction. Therefore

$$\alpha(r) \le -Cr\ln r \quad \text{for } \ r \in (0, r_c)\,. \tag{4.17}$$

To show the derivative bound, we observe that there exist positive constant C_1, C_2, C_3 and C_4 such that

$$\begin{aligned} &C_1\, r \le f(r) \le C_2\, r \quad \text{for } \ r \in (0, r_c)\,, \\ &g(y) \le C_3\, y \quad \text{and} \quad g'(y) \le C_4 \quad \text{for } \ y \in (0, y_o)\,. \end{aligned} \tag{4.18}$$

Using equation (1.1), (4.17) and (4.18) we have

$$\begin{aligned} \left(f^{n-1}(r)\alpha'(r)\right)' &= (n-1)f^{n-3}(r)g(\alpha(r))g'(\alpha(r)) \\ &\le C_5\, f^{n-3}(r)\,(-r\ln r) \\ &\le -C_6\, r^{n-2}\ln r \quad \text{for } \ r \in (0, r_c)\,, \end{aligned} \tag{4.19}$$

where C_5 and C_6 are positive constants. As $n > 1$, by lemma 2.5 we have

$$0 \le f^{n-1}(r_o)\alpha'(r_o) \to 0 \quad \text{as } \ r_o \to 0^+\,. \tag{4.20}$$

Integrating both sides of (4.19) from $r \in (0, r_c)$ to r_o and letting $r_o \to 0^+$, we have

$$f^{n-1}(r)\alpha'(r) \le -C_7\, r^{n-1}\ln r + C_8 r^{n-1} \quad \text{for } \ r \in (0, r_c)\,, \tag{4.21}$$

where C_7 and C_8 are positive constants. Using (4.18), there exists a positive constant C' such that $\alpha'(r) \le -C'\ln r$ for $r > 0$ close to zero. □

It follows directly from the above theorem that we have a generalization of lemma 2.5.

Lemma 4.22. *For $n \ge 1$, let α, $\beta \in C^2((0,R))$ be non-negative solutions of equation (1.1) in $(0,R)$ for some $R > 0$, with prescribed limit (1.2). Assume that $\alpha(r) \ge \beta(r)$ for $r \in (0,R)$. Let $F(r) = \alpha(r) - \beta(r)$ for $r \in (0,R)$. Then for every $\lambda > 0$, there exists a sequence $\{r_i\} \subset (0,R)$ such that*

$$\lim_{i\to\infty} r_i = 0 \quad \textit{and} \quad \lim_{i\to\infty} r_i^{\lambda}\, F'(r_i) = 0\,.$$

We recall the following regularity result, which is proved in [6].

Theorem 4.23 [6]. *For $n \ge 1$, assume that $f''(0) = g''(0) = 0$. Let $\alpha \in C^2((0,R))$ be a non-negative solution of equation (1.1) in $(0,R)$ for some $R > 0$, with prescribed limit (1.2). Then $\lim_{r\to 0^+}\alpha'(r)$ exists. Moreover, if we defined $\alpha(0) = 0$, then $\alpha \in C^2((0,R)) \cap C^1([0,R))$.*

5. Blow-up, Bounded and Unbounded Solutions

Lemma 5.1. *For $n \ge 2$, assume that $f'(r) \ge 0$ for $r > 0$ and $g'(y) > 0$ for $y > 0$. Let $\alpha \in C^2((0,R))$ be a positive solution of equation (1.1) for some $R > 0$, with prescribed limit (1.2). We have*

$$\int_{\alpha(r_o)}^{\alpha(r)} \frac{dy}{g(y)} \le \sqrt{n-1}\int_{r_o}^{r} \frac{ds}{f(s)} \quad \text{for} \ \ R > r > r_o > 0\,. \tag{5.2}$$

Proof. By lemma 2.3 we have $\alpha(r) > 0$ and $\alpha'(r) > 0$ for $r \in (0,R)$. As $f'(r) \ge 0$ for $r > 0$, it follows from equation (1.1) that

$$\alpha''(r) + \frac{f'(r)}{f(r)}\alpha'(r) \le (n-1)\frac{g(\alpha(r))g'(\alpha(r))}{f^2(r)} \quad \text{for} \ \ r \in (0,R)\,. \tag{5.3}$$

Hence we obtain

$$f(r)\left(f(r)\alpha'(r)\right)'\alpha'(r) \le (n-1)g(\alpha(r))g'(\alpha(r))\alpha'(r) \quad \text{for} \ \ r \in (0,R)\,. \tag{5.4}$$

That is,

$$\left[(f(r)\alpha'(r))^2\right]' \le \left\{\left[\sqrt{n-1}\,g(\alpha(r))\right]^2\right\}' \quad \text{for} \ \ r \in (0,R)\,. \tag{5.5}$$

We have $\lim_{r\to 0^+} g(\alpha(r)) = 0$. By lemma 4.22, there exists a sequence $\{r_i\} \subset (0,R)$ such that

$$\lim_{i\to\infty} r_i = 0 \quad \text{and} \quad \lim_{i\to\infty} f(r_i)\alpha'(r_i) = 0\,. \tag{5.6}$$

Integrating both sides of equation (5.5) from r_i to r and letting $i \to \infty$, we obtain

$$f(r)\alpha'(r) \le \sqrt{n-1}\,g(\alpha(r)) \quad \text{for} \ \ r \in (0,R)\,. \tag{5.7}$$

That is,

$$\frac{\alpha'(r)}{g(\alpha(r))} \le \frac{\sqrt{n-1}}{f(r)} \quad \text{for} \ \ r \in (0,R)\,. \tag{5.8}$$

Integrating both sides of (5.8) and using the change of variables $y = \alpha(r)$, we have

$$\int_{\alpha(r_o)}^{\alpha(r)} \frac{dy}{g(y)} \le \sqrt{n-1}\int_{r_o}^{r} \frac{ds}{f(s)}\,, \tag{5.9}$$

where $R > r > r_o > 0$. □

Theorem 5.10. *For $n \geq 2$, assume that $f'(r) \geq 0$ for $r > 0$ and $g'(y) > 0$ for $y > 0$. Let $\alpha \in C^2((0,R))$ be a non-negative solution of equation (1.1) for some $R > 0$, with prescribed limit (1.2). Assume that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty \quad \text{and} \quad \int_1^\infty \frac{dy}{g(y)} = \infty\,. \tag{5.11}$$

Then α can be extended to a bounded entire non-negative C^2-solution of equation (1.1) with prescribed limit (1.2). Furthermore, any non-negative entire C^2-solution of equation (1.1) with prescribed limit (1.2) is bounded from above in $\mathbb{R}^+$.

Proof. The conclusion holds when $\alpha \equiv 0$ in $(0,R)$. We may assume that $\alpha \not\equiv 0$ in $(0,R)$. By lemma 2.3 we have $\alpha(r) > 0$ for $r \in (0,R)$. If we fix $r_o \in (0,R)$, then using the conditions in (5.11) and inequality (5.2) in lemma 5.1 we conclude that there exists a positive constant C depending on $\alpha(r_o)$ such that

$$\alpha(r) \leq C \quad \text{for} \ \ r \in (0,R)\,.$$

Hence α can be extended to a positive entire C^2-solution to equation (1.1) with prescribed limit (1.2) [9]. Furthermore, using (5.2) and the conditions in (5.11), we see that any non-negative entire C^2-solution of equation (1.1) with prescribed limit (1.2) is bounded from above in $\mathbb{R}^+$. □

Remark 5.12. It follows also from lemma 5.1 that, under the conditions of the lemma, if $1/f \in L^1((1,\infty))$ and there exists an unbounded positive entire C^2-solution of equation (1.1) in $\mathbb{R}^+$, with prescribed limit (1.2), then $1/g \in L^1((1,\infty))$. Thus the conditions in theorem A in the introduction are related to the case $1/f, 1/g \in L^1((1,\infty))$.

Lemma 5.13. *For $n \geq 2$, assume that $g'(y) > 0$ for $y > 0$ and there exist positive constants c and r_o such that*

$$f(r) \geq c^2 \quad for \ \ r \geq r_o\,. \tag{5.14}$$

For every positive solution $\alpha \in C^2((0,\infty))$ of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2), there exists a positive constant ϵ depending on α, r_o and c such that

$$\int_{\alpha(r_o)}^{\alpha(r)} \frac{dy}{g(y)} \geq \epsilon \int_{r_o}^{r} \frac{ds}{f^{n-1}(s)} \quad for \ \ r > r_o\,. \tag{5.15}$$

Proof. Lemma 2.3 shows that $\alpha(r) > 0$ and $\alpha'(r) > 0$ for $r > 0$. Using equation (1.1) we have

$$f^{n-1}(r)\left(f^{n-1}(r)\alpha'(r)\right)'\alpha'(r) = (n-1)f^{2n-4}(r)g(\alpha(r))g'(\alpha(r))\alpha'(r) \tag{5.16}$$

for $r > 0$. Therefore we obtain

$$\left[\left(f^{n-1}(r)\alpha'(r)\right)^2\right]' \geq (n-1)c^{4n-8}\left\{[g(\alpha(r))]^2\right\}' \quad \text{for } r \geq r_o. \tag{5.17}$$

Choose a positive number ϵ such that

$$(n-1)c^{4n-8} \geq \epsilon^2 \quad \text{and} \quad \left(f^{n-1}(r_o)\alpha'(r_o)\right)^2 \geq \epsilon^2\left[g(\alpha(r_o))\right]^2. \tag{5.18}$$

Hence we have

$$\left[\left(f^{n-1}(r)\alpha'(r)\right)^2\right]' \geq \left\{[\epsilon g(\alpha(r))]^2\right\}' \quad \text{for } r \geq r_o. \tag{5.19}$$

Integrating both sides of (5.19) from r_o to r and using (5.18) we have

$$\left(f^{n-1}(r)\alpha'(r)\right)^2 \geq \epsilon^2\left[g(\alpha(r))\right]^2 \quad \text{for } r \geq r_o. \tag{5.20}$$

We obtain

$$\frac{\alpha'(r)}{g(\alpha(r))} \geq \frac{\epsilon}{f^{n-1}(r)} \quad \text{for } r \geq r_o. \tag{5.21}$$

Integrating both sides of (5.21) and using the change of variables $y = \alpha(r)$, we have (5.15). □

Theorem 5.22. *For $n \geq 2$, assume that $g'(y) > 0$ for $y > 0$ and there exist positive constants c and r_o such that $f(r) \geq c^2$ for $r \geq r_o$. If*

$$\int_1^\infty \frac{dr}{f^{n-1}(r)} = \infty \quad \textit{and} \quad \int_1^\infty \frac{dy}{g(y)} < \infty, \tag{5.23}$$

then every positive C^2-solution of equation (1.1) with prescribed limit (1.2) blows up in a bounded interval when it is extended.

Proof. Let $\alpha \in C^2((0,R))$ be a positive solution of equation (1.1) for some $R > 0$, with prescribed limit (1.2). Assume that α does not blow up in a bounded interval when it is extended. By using a result in [9] pp.15 and lemma 2.3, α can be extended to a positive C^2-solution of equation (1.1) in $\mathbb{R}^+$. Letting $r \to \infty$ in (5.15) and using (5.23), we have a contradiction. Hence α has to blow up in a bounded interval when it is extended. □

Theorem 5.24. *For $n \geq 2$, assume that $g'(y) > 0$ for $y > 0$. Assume also that*

$$\int_1^\infty \frac{dy}{g(y)} = \infty. \tag{5.25}$$

Then every non-negative local C^2-solution of equation (1.1) with prescribed limit (1.2) can be extended to an entire non-negative C^2-solution of equation (1.1) in

$I\!R^+$. In addition, if we also assume that there exist positive constants c and r_o such that

$$f(r) \geq c^2 \quad for \;\; r \geq r_o\,, \tag{5.26}$$

and

$$\int_1^\infty \frac{dr}{f^{n-1}(r)} = \infty\,, \tag{5.27}$$

then every positive entire C^2-solution of equation (1.1) with prescribed limit (1.2) is unbounded from above in $I\!R^+$.

Proof. Let $\alpha \in C^2((0,R))$ be a non-negative solution of equation (1.1) for some $R > 0$, with prescribed limit (1.2). Assume that $\alpha \not\equiv 0$ in $\mathrm{I\!R}^+$. Lemma 2.3 implies that $\alpha(r) > 0$ and $\alpha'(r) > 0$ for $r \in (0,R)$. Suppose that α blows up in $(0,R)$. That is, $\lim_{r\to R^-} \alpha(r) = \infty$. There exists a positive constant C such that

$$(n-1) f^{2n-4}(r) \leq C^2 \quad \text{for} \;\; r \in (0,R)\,. \tag{5.28}$$

Using (5.16) we have

$$\left[\left(f^{n-1}(r)\alpha'(r)\right)^2\right]' \leq C^2 \left\{[g(\alpha(r))]^2\right\}' \quad \text{for} \;\; r \in (0,R)\,. \tag{5.29}$$

For a number $r_o < R$, we may choose the constant C large enough so that

$$\left(f^{n-1}(r_o)\alpha'(r_o)\right)^2 \leq C^2 \left[g(\alpha(r_o))\right]^2 . \tag{5.30}$$

Integrating (5.29) from r to r_o and using (5.30) we have

$$f^{n-1}(r)\alpha'(r) \leq C\, g(\alpha(r)) \quad \text{for} \;\; R > r > r_o\,. \tag{5.31}$$

Integrating both sides of (5.31) and using the change of variables $y = \alpha(r)$, we obtain

$$\int_{\alpha(r_o)}^{\alpha(r)} \frac{dy}{g(y)} \leq C \int_{r_o}^{r} \frac{ds}{f^{n-1}(s)} \quad \text{for} \;\; R > r > r_o\,. \tag{5.32}$$

Letting $r \to R^-$ in (5.32) and using (5.25) together with the assumption that α blows up at R, we have a contradiction. We conclude that α cannot blow up in a bounded interval. That is, α remains bounded in the domain of definition. Hence α can be extended to an entire non-negative C^2-solution of equation (1.1) in $\mathrm{I\!R}^+$ with prescribed limit (1.2) [9]. Assume also that there exist positive constants c and r_o such that

$$f(r) \geq c^2 \quad \text{for} \;\; r \geq r_o\,. \tag{5.33}$$

Using lemma 5.13 we have

$$\int_{\alpha(r_o)}^{\alpha(r)} \frac{dy}{g(y)} \geq \epsilon \int_{r_o}^{r} \frac{ds}{f^{n-1}(s)} \quad \text{for } r \geq r_o > 0\,, \tag{5.34}$$

where ϵ is a positive constant. If (5.27) holds, then (5.34) shows that α cannot be bounded from above in $\mathbb{R}^+$. We conclude that $\lim_{r\to\infty} \alpha(r) = \infty$. □

Lemma 5.35. *In equation (1.1), assume that $n \geq 2$, $f'(r) \geq 0$ for $r > 0$, and $g'(y) > 0$ for $y > 0$. Assume also that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty \quad \textit{and} \quad \int_1^\infty \frac{dy}{g(y)} < \infty\,. \tag{5.36}$$

There exists a positive constant δ_o such that for any non-negative solution $\alpha \in C^2((0,\infty))$ of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2), if $\alpha(1) \leq \delta_o$, then α is bounded from above in $\mathbb{R}^+$.

Proof. There exists a positive constant C such that $g(y) \leq Cy$ for $y \in (0,1)$. Hence

$$\lim_{y_o\to 0^+} \int_{y_o}^1 \frac{dy}{g(y)} = \infty\,. \tag{5.37}$$

Therefore there exists a positive number δ_o such that

$$\int_{\delta_o}^\infty \frac{dy}{g(y)} > \sqrt{n-1} \int_1^\infty \frac{dr}{f(r)}\,. \tag{5.38}$$

If $\alpha(1) \leq \delta_o$, then lemma 5.1 shows that

$$\int_{\delta_o}^{\alpha(r)} \frac{dy}{g(y)} \leq \sqrt{n-1} \int_1^r \frac{ds}{f(s)} \quad \text{for } r > 1\,. \tag{5.39}$$

(5.38) and (5.39) show that α is bounded from above in $\mathbb{R}^+$. □

Theorem 5.40. *In equation (1.1), assume that $n \geq 2$, $f''(0) = g''(0) = 0$, $f'(r) \geq 0$ for $r > 0$, and $g'(y) > 0$ for $y > 0$. Assume also that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty \quad \textit{and} \quad \int_1^\infty \frac{dy}{g(y)} < \infty\,. \tag{5.41}$$

There exists a positive number c_o such that for any non-negative solution $\alpha \in C^2((0,R)) \cap C^1([0,R))$ of equation (1.1) in $(0,R)$ for some $R > 0$, with $\alpha(0) = 0$, if $\alpha'(0) \leq c_o^2$, then α can be extended to an entire non-negative C^2-solution of equation (1.1) in $\mathbb{R}^+$ which is bounded from above in $\mathbb{R}^+$.

Proof. Let $\alpha_1 \in C^2((0,R_1)) \cap C^1([0,R_1))$ be a non-negative solution of equation

(1.1) in $(0, R_1)$ for some $R_1 > 0$, with $\alpha_1(0) = 0$ and $\alpha_1'(0) = 1$. By choosing a smaller R_1 if necessary, we may assume that there exists a positive constant C_o such that $\alpha_1(r) \le C_o r$ for $r \in (0, R_1)$. Using lemma 2.4, if $\alpha'(0) \le 1$, then

$$\alpha(r) \le \alpha_1(r) \le C_o r \quad \text{for } r \in (0, R_1) \cap (0, R) . \tag{5.42}$$

By using an extension if necessary, we may assume that α is a C^2-solution of equation (1.1) in $(0, R_1)$ with prescribed limit (1.2). Let $\epsilon \in (0, R_1)$ be the constant in lemma 3.7. As in the proof of lemma 5.35, there exists a positive constant δ such that

$$\int_\delta^\infty \frac{dy}{g(y)} > \sqrt{n-1} \int_\epsilon^\infty \frac{dr}{f(r)} . \tag{5.43}$$

Lemma 3.7 implies that there exists a positive number c_o such that if $\alpha'(0) \le c_o^2$, then $\alpha(\epsilon) \le \delta$. As $\alpha(\epsilon) \le \delta$, lemma 5.1 implies that

$$\int_\delta^{\alpha(r)} \frac{dy}{g(y)} \le \sqrt{n-1} \int_\epsilon^r \frac{ds}{f(s)} \tag{5.44}$$

for $r > \epsilon$ in the domain of α. (5.43) and (5.44) show that α is bounded from above and can be extended to a bounded non-negative solution of equation (1.1) in $\mathbb{R}^+$, which is bounded from above in $\mathbb{R}^+$.. □

By using theorem 2.1, theorem 5.10 and theorem 5.40 we have the following result.

Corollary 5.45. *In equation (1.1), assume that $n \ge 2$, $f''(0) = g''(0) = 0$, $f'(r) \ge 0$ for $r > 0$, and $g'(y) > 0$ for $y > 0$. Assume also that $1/f \in L^1((1, \infty))$. Then there is a bounded positive entire solution $\alpha \in C^2((0, \infty)) \cap C^1([0, \infty))$ of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2).*

Corollary 5.46. *In equation (1.1), assume that $n \ge 2$, $f''(0) = g''(0) = 0$, $f'(r) \ge 0$ for $r > 0$, and $g''(y) \ge 0$ for $y > 0$. Assume also that there exists a positive constant C such that*

$$C f(r) \ge g(r) \quad \textit{for } r \ge 1 . \tag{5.47}$$

Then there is a positive entire C^2-solution of equation (1.1) with prescribed limit (1.2).

Proof. We have

$$\frac{1}{f(r)} \le \frac{C}{g(r)} \quad \text{for } r \ge 1 . \tag{5.48}$$

If $1/g \in L^1((1,\infty))$, then theorem 5.40 implies that there is a bounded positive entire C^2-solution of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2). If $1/g \notin L^1((1,\infty))$, then theorem 5.24 implies that there is a positive entire C^2-solution of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2). □

Theorem 5.49. *In equation (1.1), assume that $n \geq 2$, $f'(r) \geq 0$ for $r > 0$, and $g'(y) > 0$ for $y > 0$. Assume also that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty \quad \text{and} \quad \int_1^\infty \frac{dy}{g(y)} < \infty\,. \tag{5.50}$$

For a number $r_o \in (0,R)$, where R is a positive number, there exists a positive constant R_1 such that for any non-negative solution $\alpha \in C^2((0,R))$ of equation (1.1) in $(0,R)$, with prescribed limit (1.2), if

$$\alpha(r_o) \geq R_1 \quad \text{and} \quad \alpha'(r_o) \geq \sqrt{n-1}\,\frac{g(\alpha(r_o))}{f(r_o)}\,, \tag{5.51}$$

then α blows up in a bounded interval when it is extended.

Proof. As f is non-decreasing and $n \geq 2$,

$$\int_1^\infty \frac{dr}{f(r)} < \infty \Rightarrow \int_1^\infty \frac{dr}{f^{n-1}(r)} < \infty\,.$$

It follows from equation (5.16) that

$$\left[\left(f^{n-1}(r)\alpha'(r)\right)^2\right]' \geq (n-1)f^{2n-4}(r_o)\left\{[g(\alpha(r))]^2\right\}' \quad \text{for } R > r \geq r_o\,. \tag{5.52}$$

From (5.51) we have

$$\left[f^{n-1}(r_o)\alpha'(r_o)\right]^2 \geq (n-1)f^{2n-4}(r_o)\left[g(\alpha(r_o))\right]^2\,. \tag{5.53}$$

Using (5.52) and (5.53), as in the proof of lemma 5.13, we have

$$\int_{\alpha(r_o)}^{\alpha(r)} \frac{dy}{g(y)} \geq \sqrt{n-1}\,f^{n-2}(r_o)\int_{r_o}^{r} \frac{ds}{f^{n-1}(s)} \quad \text{for } R > r \geq r_o\,. \tag{5.54}$$

There exists a positive constant R_1 such that

$$\int_{R_1}^\infty \frac{dy}{g(y)} < \sqrt{n-1}\,f^{n-2}(r_o)\int_{r_o}^\infty \frac{dr}{f^{n-1}(r)}\,. \tag{5.55}$$

If $\alpha(r_o) \geq R_1$, then by (5.54) we have

$$\int_{R_1}^{\alpha(r)} \frac{dy}{g(y)} \geq \int_{\alpha(r_o)}^{\alpha(r)} \frac{dy}{g(y)} \geq \sqrt{n-1}\,f^{n-2}(r_o)\int_{r_o}^{r} \frac{ds}{f^{n-1}(s)} \quad \text{for } R > r \geq r_o\,. \tag{5.56}$$

Using (5.55), we see that (5.56) cannot hold for large r. Hence α blows up in a bounded interval when it is extended. $\square$

We conclude this section by obtaining some Liouville type theorems. We note that, under the assumptions of theorem 5.22 and theorem 5.24, there is no bounded positive entire C^2-solution of equation (1.1) with prescribed limit (1.2). Hence we have the following result.

Theorem 5.57. *For $n \ge 2$, assume that $g'(y) > 0$ for $y > 0$ and there exist positive constants c and r_o such that $f(r) \ge c^2$ for $r \ge r_o$. If $1/f^{n-1} \notin L^1((1,\infty))$, then there do not exist bounded positive entire C^2-solutions of equation (1.1) with prescribed limit (1.2).*

Theorem 5.58. *For $n \ge 3$, assume that $g''(y) \ge 0$ for $y > 0$. Assume also that either f is bounded from above, or*

$$\lim_{r\to\infty} f(r) = \infty \quad \textit{and} \quad \int_1^\infty \frac{1}{f(r)}\, dr = \infty \tag{5.59}$$

and there exist positive constants C and R_o such that

$$\int_0^R f^{n-3}(r)\, dr \ge C f^{n-2}(R) \quad \textit{for} \;\; R \ge R_o\,. \tag{5.60}$$

Let $\alpha \in C^2((0,\infty))$ be a non-negative solution of equation (1.1) in $I\!R^+$ with prescribed limit (1.2). If α is bounded from above in $I\!R^+$, then $\alpha \equiv 0$ in $I\!R^+$.

Proof. Assume that $\alpha \not\equiv 0$ in $\mathbb{R}^+$. Lemma 2.3 shows that $\alpha(r) > 0$ for $r > 0$. Suppose that f is bounded from above in $\mathbb{R}^+$. From equation (1.1) and an integration we obtain

$$\begin{aligned} f^{n-1}(R)\alpha'(R) &= (n-1)\int_0^R f^{n-3}(r) g(\alpha(r)) g'(\alpha(r))\, dr \\ &\ge \int_0^1 f^{n-3}(r) g(\alpha(r)) g'(\alpha(r))\, dr > 0\,, \end{aligned} \tag{5.61}$$

where $R \ge 1$ and lemma 2.5 is used. As f is bounded from above, (5.61) implies that there exists a positive constant c such that $\alpha'(R) \ge c^2 \;$ for $\; R \ge 1$. An integration shows that α is unbounded from above in $\mathbb{R}^+$. Assume that $\lim_{r\to\infty} f(r) = \infty$. As α, g and g' are non-decreasing functions, it follows from equation (1.1) that there exists a positive constant c_o such that

$$\left(f^{n-1}(r)\alpha'(r)\right)' \ge c_o f^{n-3}(r) \quad \text{for} \;\; r \ge R_o\,. \tag{5.62}$$

Integrating both sides of (5.64) from R to R_o with $R \ge R_o$, and using condition (5.60), we have

$$(5.63)\quad f^{n-1}(R)\alpha'(R) - f^{n-1}(R_o)\alpha'(R_o) \;\geq\; c_o\int_0^R f^{n-3}(r)\,dr - c_o\int_0^{R_o} f^{n-3}(r)\,dr$$
$$\geq\; C\,c_o\,f^{n-2}(R) - c_o\int_0^{R_o} f^{n-3}(r)\,dr\,.$$

Hence there exists a positive constant C_1 such that

$$(5.64)\qquad f^{n-1}(R)\alpha'(R) \geq C\,c_o\,f^{n-2}(R) - C_1 \quad \text{for}\;\; R \geq R_o\,.$$

We obtain

$$(5.65)\qquad \alpha'(R) \geq \frac{1}{f(R)}\left(C\,c_o - \frac{C_1}{f^{n-2}(R)}\right) \quad \text{for}\;\; R \geq R_o\,.$$

As $\lim_{r\to\infty} f(r) = \infty$, there exists a positive number $R_1 \geq R_o$ such that

$$(5.66)\qquad \alpha'(R) \geq \frac{C\,c_o}{2f(R)} \quad \text{for}\;\; R \geq R_1\,.$$

Integrating both sides of (5.66) and using the second condition in (5.59), we conclude that α is unbounded.

We note that functions of the type $f(r) \sim r^\delta$ near infinity with $\delta \in [0,1]$ satisfy the conditions in theorem 5.58. Furthermore, if f' is bounded from above in $\mathbb{R}^+$, then the second condition in (5.59) is satisfied.

6. Bounded Positive Entire Solutions

We begin with a proof of continuity of the map ϕ_1.

Lemma 6.1. *For $n \geq 3$, assume that $g''(y) \geq 0$ for $y > 0$ and there exist positive constants C_1 and C_2 such that*

$$(6.2)\qquad \int_1^\infty \frac{dr}{f(r)} \leq C_1 \quad \textit{and} \quad \int_0^R f^{n-3}(r)\,dr \leq C_2 f^{n-2}(R) \quad \textit{for}\;\; R \geq 1\,.$$

Let α , $\beta \in C^2((0,\infty))$ be non-negative solutions of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2). For every positive constant C_o, there exists a positive constant C depending on C_o , C_1 , C_2 , n and g only, such that if

$$(6.3)\qquad \beta(r) < \alpha(r) \leq C_o \quad \textit{for}\;\; r > 0\,,$$

then

$$(6.4)\qquad \alpha(1) - \beta(1) \;\leq\; \lim_{r\to\infty}\,[\alpha(r) - \beta(r)] \;\leq\; C[\alpha(1) - \beta(1)]\,.$$

Proof. As α is positive, lemma 2.3 implies that $\alpha'(r) > 0$ for $r > 0$. Likewise, either $\beta \equiv 0$ or $\beta'(r) > 0$ for $r > 0$. They are bounded from above, therefore

$$\alpha(\infty) \stackrel{\text{def}}{=} \lim_{r\to\infty} \alpha(r) \quad \text{and} \quad \beta(\infty) \stackrel{\text{def}}{=} \lim_{r\to\infty} \beta(r)$$

exist. Let $F(r) = \alpha(r) - \beta(r)$ for $r > 0$. We claim that there exists a positive constant C_3 depending on C_o, C_2, n and g only, such that

$$F'(r) \le \frac{C_3}{f(r)} F(r) \quad \text{for} \ \ r \ge 1\,. \tag{6.5}$$

Assuming the claim for the moment, integrating both sides of (6.5) we have

$$\ln\left(\frac{F(R)}{F(1)}\right) \le C_3 \int_1^R \frac{dr}{f(r)} \quad \text{for} \ \ R > 1\,. \tag{6.6}$$

By the first assumption in (6.2) we obtain

$$\ln\left(\frac{F(R)}{F(1)}\right) \le C_1\, C_3 \quad \text{for} \ \ R > 1\,. \tag{6.7}$$

Thus

$$F(R) \le e^{C_1 C_3} F(1) \quad \text{for} \ \ R > 1\,. \tag{6.8}$$

Letting $R \to \infty$ and $C = e^{C_1 C_3}$ we have $\alpha(\infty) - \beta(\infty) \le C\,[\alpha(1) - \beta(1)]$, which is the right hand side of (6.4). Furthermore, by lemma 3.5 we have $F'(r) \ge 0$ for $r > 0$. It follows that $\alpha(\infty) - \beta(\infty) \ge \alpha(1) - \beta(1)$. To proof the claim, we note that, by lemma 2.5,

$$\lim_{r\to 0^+} f^{n-1}(r) F'(r) = 0\,. \tag{6.9}$$

Consider (3.2). As $g \in C^2([0,\infty))$, there exist positive constants C_4 and C_5 depending on C_o and g only, such that

$$|g(x) - g(y)| \le C_4\,|x - y| \quad \text{and} \quad |g'(x) - g'(y)| \le C_5\,|x - y| \tag{6.10}$$

for $x,\ y \in [0, C_o]$. As in (3.10), we have

$$g(\alpha(r))g'(\alpha(r)) - g(\beta(r))g'(\beta(r)) \le C_6\,[\alpha(r) - \beta(r)] \quad \text{for} \ \ r > 0\,. \tag{6.11}$$

where we use (6.10) and the fact that $\beta < \alpha \le C_o$ in $\mathbb{R}^+$. Here C_6 is a positive constant that depends on C_o and g only. Putting the information into (3.2) we obtain

$$F''(r) + (n-1)\frac{f'(r)}{f(r)} F'(r) \le \frac{C_7}{f^2(r)} F(r) \quad \text{for} \ \ r > 0\,, \tag{6.12}$$

where C_7 is a positive constant that depends on C_o, n and g only. From (6.12) we have,

$$\left(f^{n-1}(r)F'(r)\right)' \le C_7 f^{n-3}(r)F(r) \quad \text{for } r > 0\,. \tag{6.13}$$

For $R \ge 1$, integrating both sides of (6.13) from r_o to R, letting $r_o \to 0^+$ and using (6.9), we obtain

$$f^{n-1}(R)F'(R) \le C_7 F(R)\int_0^R f^{n-3}(r)\,dr \quad \text{for } R \ge 1\,. \tag{6.14}$$

By the second assumption in (6.2), there exists a positive constant C_3 depending on C_o, C_2, n and g only, such that

$$F'(r) \le \frac{C_3}{f(r)}F(r) \quad \text{for } r \ge 1\,. \tag{6.15}$$

This completes the proof of the claim. □

Remark 6.16. We note that functions of the type $f(r) \sim r^k$ near infinity for $k > 1$ satisfy the conditions in lemma 6.1. If there exists a positive constant c such that $f'(r) \ge c^2$ for $r \ge 0$ and $n \ge 3$, then

$$\int_0^R f^{n-3}(r)\,dr \le c^{-2}\int_0^R f^{n-3}(r)f'(r)\,dr = \frac{1}{c^2(n-2)}\,f^{n-2}(R) \quad \text{for } R > 1\,. \tag{6.17}$$

That is, the second condition in (6.2) is satisfied. In particular, if we assume that $f''(r) \ge 0$ for $r > 0$, then we have the second condition in (6.2).

Remark 6.18. In the case $n = 2$, as in the proof of lemma 3.30 (cf. inequality (3.37)), the claim (6.5) in the proof of lemma 6.1 holds. Hence if we assume that $1/f \in L^1((1,\infty))$ and $g''(y) \ge 0$ for $y > 0$, then the conclusion in lemma 6.1 is valid for $n = 2$.

We consider a collection of bounded non-negative solutions to equation (1.1). Recall that I is defined in the introduction.

Lemma 6.19. *For $n \ge 2$, if $f''(0) = g''(0) = 0$ and $g''(y) \ge 0$ for $y > 0$, then I is path connected.*

Proof. If $I = \{0\}$, then I is path connected. Let $a \in I$ be a positive number. By definition, there exists a bounded non-negative solution $\alpha \in C^2((0,\infty)\cap C^1([0,\infty))$ of equation (1.1) in $\mathbb{R}^+$, with $\alpha(0) = 0$ and $\alpha(1) = a$. For any $b \in [0,a)$, by theorem 3.45, there exists a non-negative solution $\beta \in C^2((0,R)\cap C^1([0,R))$ for some $R > 1$ such that $\beta(0) = 0$ and $\beta(1) = b$. Theorem 2.2 implies that $\alpha > \beta$ in $(0,R)$. It follows that β can be extended to a non-negative C^2-solution in $\mathbb{R}^+$ and $\alpha > \beta$ in

$\mathbb{R}^+$ ([9] pp.15). Therefore $b \in I$ for all $b \in [0, a)$. Thus I is path connected. $\Box$

It follows that $I = [0, M)$ or $I = [0, M]$, where in the first case M is a positive number or infinity, and in the second case M is a positive number or 0. Under the conditions of Lemma 6.1, it follows that the map $\alpha(1) \to \alpha(\infty)$ is a continuous map from I to $\mathbb{R}^+$ (continuous from the right for 0, and left for M, if $I = [0, M]$ and $0 < M < \infty$). Therefore the image of this map is also a connected set containing 0. If a Liouville type theorem holds, then $I = \{0\}$, as there are no bounded positive entire C^2-solutions.

Lemma 6.20. *For $n \geq 2$, assume that $g''(y) \geq 0$ for $y > 0$. For any non-negative function $A \in C^o([0, \infty))$, there exists a non-negative continuous function G on $[1, \infty)$ which can be defined in terms of f, g, g', g''', A and n, such that for any non-negative solutions $\alpha, \beta \in C^2((0, \infty))$ of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2), which satisfy*

$$0 \leq \beta(r) \leq \alpha(r) \leq A(r) \quad for \;\; r > 0,$$

we have

$$\alpha(r) - \beta(r) \leq G(r)\,[\alpha(1) - \beta(1)] \quad for \;\; r \geq 1. \tag{6.21}$$

Proof. We first note that the conclusion of the lemma holds for $n = 1$ without any condition, with $G(r) = r$ for $r \in [1, \infty)$. For $n \geq 2$, let $F(r) = \alpha(r) - \beta(r)$ for $r > 0$. Consider (3.2) and (3.10). As $g'' \geq 0$ in $\mathbb{R}^+$, g' is non-decreasing. We have

$$\begin{aligned} g(\alpha(r)) - g(\beta(r)) &= \int_{\beta(r)}^{\alpha(r)} g'(s)\,ds \leq g'(\alpha(r)) \int_{\beta(r)}^{\alpha(r)} ds \\ &\leq g'(A(r))F(r) \quad \text{for} \;\; r > 0. \end{aligned} \tag{6.22}$$

For $n = 2$, from equation (2.12) we obtain

$$F'(r) = \frac{g(\alpha(r)) - g(\beta(r))}{f(r)} \leq \frac{g'(A(r))}{f(r)} F(r) \quad \text{for} \;\; r > 0. \tag{6.23}$$

Using (6.23) and an integration from r to 1 with $r \geq 1$, we have

$$F(r) \leq F(1) \exp\left(\int_1^r \frac{g'(A(s))}{f(s)}\,ds \right) \quad \text{for} \;\; r \geq 1.$$

Hence we may take

$$G(r) = \exp\left(\int_1^r \frac{g'(A(s))}{f(s)}\,ds \right) \quad \text{for} \;\; r \geq 1. \tag{6.24}$$

For $n \geq 3$, we have

$$g''(y) = g''(0) + \int_0^y g'''(s)\, ds \leq g''(0) + \int_0^y |g'''(s)|\, ds \quad \text{for } \ y > 0\,. \tag{6.25}$$

Let

$$h(y) = g''(0) + \int_0^y |g'''(s)|\, ds \quad \text{for } \ y > 0\,. \tag{6.26}$$

We note that $g''(0) \geq 0$ as $g \in C^2([0,\infty))$ and $g''(y) \geq 0$ for $y > 0$. Hence h is a non-negative C^1-function such that

$$g''(y) \leq h(y) \quad \text{for } \ y > 0\,. \tag{6.27}$$

Furthermore h is non-decreasing on $\mathbb{R}^+$. As in (6.22) we have

$$\begin{aligned} g'(\alpha(r)) - g'(\beta(r)) &= \int_{\beta(r)}^{\alpha(r)} g''(s)\, ds \leq \int_{\beta(r)}^{\alpha(r)} h(s)\, ds \\ &\leq h(A(r))F(r) \end{aligned} \tag{6.28}$$

for $r > 0$. Therefore we have

$$g(\alpha(r))g'(\alpha(r)) - g(\beta(r))g'(\beta(r)) \leq \left[g'^2(A(r)) + g(A(r))h(A(r))\right] F(r) \tag{6.29}$$

for $r > 0$. Substituting into (3.2) and we obtain

$$F''(r) + (n-1)\frac{f'(r)}{f(r)}F'(r) \leq (n-1)\frac{g'^2(A(r)) + g(A(r))h(A(r))}{f^2(r)}\, F(r) \tag{6.30}$$

for $r > 0$. That is,

$$\left(f^{n-1}(r)F'(r)\right)' \leq (n-1)f^{n-3}(r)\left(g'^2(A(r)) + g(A(r))h(A(r))\right) F(r) \tag{6.31}$$

for $r > 0$. Integrating both sides of (6.31) from r_o to r, letting $r_o \to 0^+$ and using lemma 2.5, we have

$$f^{n-1}(r)F'(r) \leq (n-1)\int_0^r f^{n-3}(s)\left(g'^2(A(s)) + g(A(s))h(A(s))\right) F(s)\, ds \tag{6.32}$$

for $r > 0$. As $n \geq 3$, using (6.32) and the fact that F is non-decreasing on $(0,\infty)$, we have

$$f^{n-1}(r)F'(r) \leq (n-1)F(r)\int_0^r f^{n-3}(s)\left(g'^2(A(s)) + g(A(s))h(A(s))\right)\, ds \tag{6.33}$$

for $r > 0$. That is,

$$\ln\left[\frac{F(R)}{F(1)}\right] \leq (n-1)\int_1^R f^{1-n}(r)\int_0^r f^{n-3}(s)\left(g'^2(A(s)) + g(A(s))h(A(s))\right)\, ds\, dr$$

for $R \geq 1$. Let

$$G(R) = \exp\left[(n-1)\int_1^R f^{1-n}(r)\int_0^r f^{n-3}(s)\left(g'^2(A(s)) + g(A(s))h(A(s))\right) ds\, dr\right]$$

for $R \geq 1$. Then we obtain inequality (6.21). □

Theorem 6.34. *In equation (1.1), assume that $n \geq 3$, $f''(0) = g''(0) = 0$, $f''(r) \geq 0$ for $r > 0$, and $g''(y) \geq 0$ for $y > 0$. Assume also that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty\,.$$

If there exists an unbounded positive solution $\alpha_\tau \in C^2((0,\infty))$ of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2), then I is open on the right, that is, $I = [0, M)$ for some positive number M.

Proof. Lemma 6.19 shows that I is path connected. As the C^2-solution α_τ is unbounded, theorem 2.2 implies that I is bounded from above. Assume that $I = [0, M]$, where $0 \leq M < \alpha_\tau(1)$. As $M \in I$, by definition, there is a bounded non-negative solution $\alpha_M \in C^2((0,\infty)) \cap C^1([0,\infty))$ of equation (1.1) in $\mathbb{R}^+$, with

$$\alpha_M(0) = 0 \quad \text{and} \quad \alpha_M(1) = M\,.$$

There exists a positive constant C_o such that

$$\alpha_M \leq C_o \quad \text{in } \mathbb{R}^+. \tag{6.35}$$

As α_τ is unbounded from above, theorem 2.2 implies that

$$\alpha_M(r) < \alpha_\tau(r) \quad \text{for } r > 0\,. \tag{6.36}$$

Using theorem 4.23 we have $\alpha_\tau \in C^2((0,\infty)) \cap C^1([0,\infty))$. Let $c \in (M, \alpha_\tau(1))$. By theorem 3.45 and an extension of local solutions, there exists an $\alpha_c \in C^2((0,\infty)) \cap C^1([0,\infty))$ which is a positive solution of equation (1.1) in $\mathbb{R}^+$, with $\alpha_c(0) = 0$ and $\alpha_c(1) = c < \alpha_\tau(1)$. We also have

$$\alpha_c(r) < \alpha_\tau(r) \quad \text{for } r > 0\,. \tag{6.37}$$

By choosing α_τ as the function A in lemma 6.20, there exists a non-negative continuous function G on $[1,\infty)$, which is independent on c so far as $c \in (M, \alpha_\tau(1))$ (and hence (6.37) holds), such that

$$\alpha_c(R) - \alpha_M(R) \leq G(R)[\alpha_c(1) - \alpha_M(1)] = G(R)(c - M) \tag{6.38}$$

for $R \geq 1$. Using (6.35) we obtain

$$\alpha_c(R) \leq C_o + G(R)(c - M) \quad \text{for } R \geq 1\,. \tag{6.39}$$

Let

$$\theta_c(r) = (\alpha_c')^2(r) \quad \text{for } \ r > 0\,. \tag{6.40}$$

Given a positive number δ we have

$$\begin{aligned}
\theta_c'(r) &= [(\alpha_c')^2]'(r) = 2\alpha_c'(r)\alpha_c''(r) \\
&= 2\alpha_c'(r)\left[(n-1)\frac{g(\alpha_c(r))g'(\alpha_c(r))}{f^2(r)} - (n-1)\frac{f'(r)}{f(r)}\alpha_c'(r)\right] \\
&= 2\left(\delta^{\frac{1}{2}}\sqrt{\frac{f'(r)}{f(r)}}\,\alpha_c'(r)\right)\left(\frac{n-1}{\delta^{\frac{1}{2}}}\frac{g(\alpha_c(r))g'(\alpha_c(r))}{(f'(r))^{\frac{1}{2}}\,f^{\frac{3}{2}}(r)}\right) \\
&\quad - 2\,(n-1)\,\frac{f'(r)}{f(r)}(\alpha_c')^2(r) \\
&\le \frac{(n-1)^2}{\delta}\frac{g^2(\alpha_c(r)){g'}^2(\alpha_c(r))}{f^3(r)f'(r)} - (2n-2-\delta)\frac{f'(r)}{f(r)}\theta_c(r)
\end{aligned} \tag{6.41}$$

for $r > 0$. Here we make use of the assumption that $f''(r) \ge 0$ for $r > 0$ and hence $f'(r) \ge 1$ for $r > 0$. Therefore

$$\theta_c'(r) + (2n-2-\delta)\frac{f'(r)}{f(r)}\theta_c(r) \le \frac{(n-1)^2}{\delta}\frac{g^2(\alpha_c(r)){g'}^2(\alpha_c(r))}{f^3(r)f'(r)} \tag{6.42}$$

for $r > 0$. That is,

$$\left(f^{2n-2-\delta}(r)\theta_c(r)\right)' \le \frac{(n-1)^2}{\delta}f^{2n-5-\delta}(r)\frac{g^2(\alpha_c(r)){g'}^2(\alpha_c(r))}{f'(r)} \tag{6.43}$$

for $r > 0$. As $n \ge 3$, we may choose the positive constant δ such that

$$2n-5-\delta = 0\,, \quad \text{that is} \quad 2n-2-\delta = 3\,. \tag{6.44}$$

Using (6.44) and integrating both sides of (6.43), we obtain

$$f^3(R)\theta_c(R) - f^3(r_o)\theta_c(r_o) \le \frac{(n-1)^2}{\delta}\int_{r_o}^{R}\frac{g^2(\alpha_c(r)){g'}^2(\alpha_c(r))}{f'(r)}\,dr\,, \tag{6.45}$$

where $R > r_o > 0$. Thus we have

$$\theta_c(R) \le \frac{f^3(r_o)}{f^3(R)}\theta_c(r_o) + \frac{(n-1)^2}{\delta}f^{-3}(R)\int_{r_o}^{R}\frac{g^2(\alpha_c(r)){g'}^2(\alpha_c(r))}{f'(r)}\,dr \tag{6.46}$$

for $R > r_o > 0$. Given any constant $C > 1$, as $f'(0) = 1$, we have $f(r) \le Cr$ for any positive number r close to zero. As

$$\int_1^\infty \frac{dr}{f(r)} < \infty \quad \text{and} \quad \int_1^\infty \frac{dr}{Cr} = \infty\,, \tag{6.47}$$

there exists a point $r_1 > 0$ such that $f(r_1) > Cr_1$. Hence there exists a point $r_2 \le r_1$ such that $f'(r_2) = C_1 > C$. As $f'' \ge 0$ in $\mathbb{R}^+$, we have

$$f'(r) \ge C_1 \quad \text{for } \; r \ge r_2 \,.$$

We obtain

$$f(r) \ge C_1 r + f(r_2) - C_1 r_2 \quad \text{for } \; r \ge r_2 \,. \tag{6.48}$$

Thus we have, for any $C > 1$, there exists a number R_c depending on C such that

$$f(r) \ge Cr \quad \text{for } \; r \ge R_c \,. \tag{6.49}$$

Using (6.39) and by choosing c close to M, there is a point $r_o \le 2(C_o + 1)$ such that

$$\alpha_c'(r_o) \le 1 \,, \quad \text{that is,} \quad \theta_c(r_o) \le 1 \,. \tag{6.50}$$

Choose a positive number ϵ such that

$$\int_1^\infty \frac{dr}{f(r)} \le \frac{C_o}{\epsilon} \,. \tag{6.51}$$

If at $r > 0$ we have

$$\alpha_c(r) \le 5C_o \,, \tag{6.52}$$

then there is a positive constant C_5 such that

$$\frac{(n-1)^2}{\delta}\frac{g^2(\alpha_c(r)){g'}^2(\alpha_c(r))}{f'(r)} \le \frac{(n-1)^2}{\delta} g^2(\alpha_c(r)){g'}^2(\alpha_c(r)) \le C_5 \,. \tag{6.53}$$

By (6.49), there exists a number $R_o \ge 2(C_o + 1)$ (that depends on f, ϵ and C_o only) such that

$$\frac{f^3(r_o)}{f(R)}\theta_c(r_o) \le \frac{1}{2}\epsilon^2 \quad \text{and} \quad C_5 \frac{R - r_o}{f(R)} \le \frac{1}{2}\epsilon^2 \quad \text{for } \; R \ge R_o \,. \tag{6.54}$$

Using (6.39), if we choose c to be close to M, then we have

$$\alpha_c(R_o) \le 2C_o \,. \tag{6.55}$$

As α_c is non-decreasing, we have

$$\alpha_c(r) \le 2C_o \quad \text{for } \; r \in (0, R_o) \,. \tag{6.56}$$

Consider a positive number $R_1 > R_o$ such that

$$\alpha_c(r) \le 5C_o \quad \text{for } \; r \in (0, R_1) \,. \tag{6.57}$$

Using (6.46), (6.53) and (6.54) we have

$$(6.58)\qquad \theta_c(R) \;\le\; \frac{f^3(r_o)}{f^3(R)}\theta_c(r_o) + \frac{(n-1)^2}{\delta} f^{-3}(R) \int_{r_o}^{R} \frac{g^2(\alpha_c(r)) g'^2(\alpha_c(r))}{f'(r)}\, dr$$
$$\le\; \le \frac{f^3(r_o)}{f^3(R)}\theta_c(r_o) + C_5 f^{-3}(R)(R - r_o) \;\le\; \frac{\epsilon^2}{f^2(R)}$$

for $R \in (r_o\,, R_1)$. In particular, we have

$$(6.59)\;\; \theta_c(r) \le \epsilon^2 f^{-2}(r) \quad \text{for } r \in (R_o, R_1)\,, \;\text{ so } \;\text{ far } \;\text{ as } \alpha_c(r) \le 5C_o \text{ in } (0, R_1)\,.$$

Suppose that there exists a point $R' > R_o$ such that $\alpha_c(R') = 5C_o$. As $\alpha_c' > 0$ in $\mathbb{R}^+$, we have $\alpha_c(r) \le 5C_o$ on $(0, R')$. The above argument leading to (6.59) shows that

$$(6.60)\qquad \alpha_c(R') = \alpha_c(R_o) + \int_{R_o}^{R} \alpha_c'(r)\, dr \le 2C_o + \epsilon \int_1^\infty \frac{dr}{f(r)} \le 3C_o\,,$$

which is a contradiction. Hence we conclude that $\alpha_c < 5C_o$ in $\mathbb{R}^+$ if $c \in (M, \alpha_\tau(1))$ is close to M. That is, α_c is bounded and hence $c \in I$. This contradicts that $c > M$ and $I = [0, M]$. Hence $M > 0$ and $I = [0, M)$. □

The case $n = 2$ is similar.

Theorem 6.61. *In equation (1.1), assume that $n = 2$, $f''(0) = g''(0) = 0$ and $g''(y) \ge 0$ for $y > 0$. Assume also that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty\,.$$

If there exists an unbounded positive solution $\alpha_\tau \in C^2((0,\infty))$ of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2), then I is open on the right, that is, $I = [0, M)$ for some positive number M.

Proof. We follow the argument and notations as in the proof of theorem 6.34. In the present situation equation (1.1) is equivalent to equation (2.12), that is,

$$(6.62)\qquad \alpha'(r) = \frac{g(\alpha(r))}{f(r)} \quad \text{for } \; r > 0\,.$$

There exist positive constant C_1 and R_o such that

$$(6.63)\qquad g(y) \le C_1 \quad \text{for } \; 0 < y \le 5C_o\,,$$

and

$$(6.64)\qquad C_1 \int_{R_o}^\infty \frac{dr}{f(r)} \le C_o\,.$$

Using lemma 6.20, if we choose $c \in (M, \alpha_\tau(1))$ to be close to M, then we have

$$\alpha_c(R_o) \le 2C_o\,. \tag{6.65}$$

Assume that there is a point $R > R_o$ such that $\alpha_c(R) = 5C_o$. As α_c is non-decreasing, we obtain

$$\alpha_c(r) \le 5C_o \quad \text{for } \; r \in (0, R]\,. \tag{6.66}$$

Hence we have

$$\alpha_c'(r) \le \frac{C_1}{f(r)} \quad \text{for } \; r \in (0, R]\,. \tag{6.67}$$

Integrating both sides of (6.76) from R to R_o we have

$$\alpha_c(R) \le \alpha_c(R_o) + C_1 \int_{R_o}^{R} \frac{dr}{f(r)} \le 2C_o + C_o\,, \tag{6.68}$$

which is a contradiction. We may complete the proof as in the proof of theorem 6.34. □

Define

$$I_o \;=\; \Big\{\alpha'(0) \mid \alpha \in C^2((0,\infty) \cap C^1([0,\infty)) \text{ is a bounded non-negative solution of equation (1.1) in } \mathbb{R}^+ \text{ with } \alpha(0) = 0\Big\}\,.$$

and $\phi_o : \alpha'(0) \mapsto \alpha(\infty)$.

Theorem 6.69. *In equation (1.1), we assume that $n \ge 2$, $f''(0) = g''(0) = 0$ and $f''(r) \ge 0$, $g''(y) \ge 0$ for $r > 0$, $y > 0$. Assume also that*

$$\int_1^\infty \frac{dr}{f(r)} < \infty\,.$$

If there exists an unbounded positive C^2-solution α_τ of equation (1.1) in $\mathbb{R}^+$ with prescribed limit (1.2), then for every non-negative boundary value at infinity, the Dirichlet problem at infinity has a unique non-negative solution in $C^2((0,\infty)) \cap C^1([0,\infty))$ with prescribed limit (1.2). Furthermore the maps $\phi_1 : I = [0, M) \to [0,\infty)$ and $\phi_o : [0, m_o) \to [0,\infty)$ are homeomorphisms, where $m_o = \alpha_M'(0)$.

Proof. Using theorem 6.34 and theorem 6.61 we have $I = [0, M)$, where $0 < M \le \alpha_\tau(1)$. Suppose that the Dirichlet problem at infinity is not solvable for a positive number. Using lemma 6.1, remark 6.16 and remark 6.18, the map $\alpha(1) \mapsto \alpha(\infty)$ is continuous. Hence there exists a positive number C_o such that the Dirichlet problem at infinity has no solutions for any $c \ge C_o$. By theorem 3.45 and an extension, there is an $\alpha_M \in C^2((0,\infty)) \cap C^1([0,\infty))$ which is a positive solution of equation (1.1) in $\mathbb{R}^+$ such that

$$\alpha_M(0) = 0 \quad \text{and} \quad \alpha_M(1) = M \le \alpha_\tau(1)\,. \tag{6.70}$$

For $b \in I = [0, M)$, let $\alpha_b \in C^2((0,\infty)) \cap C^1([0,\infty))$ be a positive solution of equation (1.1) in $\mathbb{R}^+$ such that $\alpha_b(0) = 0$ and $\alpha_b(1) = b < M$. It follows that

$$\alpha_b(r) \le \alpha_M(r) \le \alpha_\tau(r) \quad \text{and} \quad \alpha_b(r) \le C_o \quad \text{for} \;\; r > 0\,. \tag{6.71}$$

This is because $\alpha_b(\infty) \le C_o$ and α_b is increasing. Note that C_o is independent on b. In lemma 6.20, using α_τ as the function A, there is a non-negative continuous function G defined on $[1,\infty)$, such that

$$\alpha_M(R) - \alpha_b(R) \le G(R)\,[\alpha_M(1) - \alpha_b(1)] \quad \text{for} \;\; R \ge 1\,. \tag{6.72}$$

Using the second inequality in (6.71) we obtain

$$\alpha_M(R) - C_o \le G(R)(M - b) \quad \text{for} \;\; R \ge 1\,. \tag{6.73}$$

The points are that G is independent on b so far as $b \in (0, M)$, and α_M is unbounded from above in $\mathbb{R}^+$ (otherwise $M \in I$). There exists a number $R_1 > 1$ such that $\alpha_M(R_1) > C_o$. When $b \to M^-$, (6.73) gives a contradiction. Hence the Dirichlet problem at infinity has a unique non-negative solution in $C^2((0,\infty)) \cap C^1([0,\infty))$ with prescribed limit (1.2) for every non-negative boundary value at infinity. The continuity and injectivity of the maps ϕ_1 and ϕ_o and their inverses follow from lemma 3.7, lemma 3.30, remark 3.43, lemma 6.1, remark 6.16 and remark 6.18. □

We observe that, if $f = g$, then the identity map is harmonic. Hence we may take $\alpha_\tau(r) = r$ for $r > 0$ in theorem 6.69.

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