

**ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS
OF THE EQUATION $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ in \mathbb{R}^n AND
POSITIVE SCALAR CURVATURE**

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Abstract. We study asymptotic behavior of smooth positive solutions of the conformal scalar curvature equation in \mathbb{R}^n , in connection with the global geometry of conformal metrics with positive scalar curvature outside a compact set. We show that the solution has slow decay if the radial change is controlled. For a smooth positive solution with slow decay, the corresponding conformal metric is found to be complete if and only if the total volume is infinite. We also determine the sign of the Pohozaev number in some situations and investigate the case when it is equal to zero.

1. Introduction. We consider smooth positive solutions u of the conformal scalar curvature equation

$$\Delta u + Ku^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

where $n \geq 3$ is an integer, Δ the standard Laplacian on \mathbb{R}^n , and K a smooth function. Equation (1.1) is studied extensively and remarkable results are obtained (cf. recent publications [1, 4, 5, 6, 8, 9, 10, 17], and the references within for descriptions of the development). Our concern here is when K is bounded between two positive numbers outside a compact subset of \mathbb{R}^n . This means that the conformal metric $g_c := u^{4/(n-2)}g_o$ has bounded positive scalar curvature outside a compact subset, where g_o is Euclidean metric. For smooth positive solutions of (1.1), the basic asymptotic properties we seek to consider are the following.

- (a) **Fast decay:** $u(x) \leq C|x|^{-(n-2)}$ for $|x| \gg 1$.
- (b) **Slow decay:** $u(x) \leq C|x|^{-(n-2)/2}$ for $|x| \gg 1$.
- (c) **Lower bound:** $u(x) \geq C|x|^{-(n-2)/2}$ for $|x| \gg 1$.

We also consider the global geometry of the conformal metric g_c , which is intimately linked to the asymptotic behavior. Basic global properties of (\mathbb{R}^n, g_c) include the total volume and completeness. One can classify smooth positive solutions of equation (1.1) according to whether the total volume is finite or not, or according to whether g_c is complete or not. Using a result of Brezis and Kato [2], it is shown in [14] that if the total volume of (\mathbb{R}^n, g_c) is finite, then u has fast decay (a). It can be seen that if lower bound (c) holds, then g_c is a complete metric. The asymptotic dimension of (\mathbb{R}^n, g_c) , which is propounded by Gromov (cf. [12]), is pertained to the asymptotic behavior of u . For slow decay indicates that the conformal metric g_c is asymptotically a bounded perturbation of the standard product metric on $\mathbb{R}^+ \times S^{n-1}$ (see [14]).

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A fundamental result of Caffarelli, Gidas and Spruck [3] shows that if K is equal to a positive constant outside a compact set of \mathbb{R}^n , then either g_c has fast decay, or

$$C_1|x|^{-(n-2)/2} \leq u(x) \leq C_2|x|^{-(n-2)/2} \quad \text{for } |x| \gg 1,$$

that is, u satisfies both (b) and (c). A series of deep works by C.-C. Chen and C.-S. Lin lead to the following, which we state by using the Kelvin transform and refer to [19] for the details: u has slow decay (b) if, among other things, K approaches a positive number at infinity and there exist positive constants α , c_1 and c_2 such that

$$c_1|x|^{1+\alpha} \leq |\nabla K(x)| \leq c_2|x|^{1+\alpha} \quad \text{for } |x| \gg 1. \quad (1.2)$$

In [14], the second author shows that if K is bounded between two positive numbers in \mathbb{R}^n , $x \cdot \nabla K(x) \leq 0$ for large $|x|$, u is bounded from above and u is *radially dominating* (see [14]), then u has slow decay.

Smooth positive solutions of equation (1.1) with K bounded between two positive numbers may not have slow decay. Using a method of Taliaferro in [20], unbounded positive solutions of equation (1.1) are constructed in [15] so that the conformal metrics are complete. Furthermore, we can make the conformal metric to have arbitrarily fast volume growth. In this paper we obtain slow decay (b) by imposing a control on the growth in the radial direction.

Theorem A. *Assume that K is bounded between two positive constants outside a compact subset of \mathbb{R}^n . Assume also that there exist positive constants r_o , c and C such that*

$$\frac{\partial K}{\partial r}(r, \theta) \geq -Ce^{-cr} \quad \text{for } \theta \in S^{n-1} \quad \text{and } r \geq r_o, \quad (1.3)$$

where (r, θ) is the polar coordinates on \mathbb{R}^n . Let u be a smooth positive solution of equation (1.1) in \mathbb{R}^n . Then u has slow decay (b) if there exist positive constants C' and r' such that

$$r \left| \frac{\partial u}{\partial r}(r, \theta) \right| \leq C' u(r, \theta) \quad \text{for } \theta \in S^{n-1} \quad \text{and } r \geq r'. \quad (1.4)$$

Fast decay (a) and lower bound (c) are related to the *Pohozaev number* of u , denoted by $P(u)$ (see (2.3) in section 2). For a smooth positive solution u of equation (1.1) with slow decay, $P(u) \neq 0$ implies lower bound (c). Using the results of Korevaar, Mazzeo, Pacard and Schoen in [13], if K is equal to a positive constant outside a compact set and g_c is complete, then $P(u) < 0$ and lower bound (c) holds (cf. also [3]). Chen and Lin [4, 7] show that, under similar conditions on $|\nabla K|$ as in (1.2) (with higher exponents on $|x|$; see [7] for the detailed conditions on K), then the conformal metric g_c can be realized as a smooth metric on S^n if and only if $P(u) = 0$. Moreover, it can be deduced that lower bound (c) holds, and hence g_c is complete, if and only if $P(u) < 0$. We show that the results can be generalized and the sign of the Pohozaev number can be determined in the following situation.

Theorem B. *Assume that $\lim_{|x| \rightarrow \infty} K(x) = K_\infty > 0$ and $|\nabla K| \leq C_o$ in \mathbb{R}^n for a positive number C_o . Let u be a smooth positive solution of (1.1) in \mathbb{R}^n with slow decay (b). Assume also that K satisfies one of the conditions in lemma 2.1 so that $P(u)$ exists. Then we have $P(u) \leq 0$. In addition, if it is also assumed that $x \cdot \nabla K(x) \leq 0$ for large $|x|$, then $P(u) = 0$ implies that u has fast decay.*

It can be seen that, under the conditions in the first part of theorem B, if $P(u) = 0$, then either u has fast decay, or the conformal metric g_c is *complete* and $\liminf_{|x| \rightarrow \infty} |x|^{\frac{n-2}{2}} u(x) = 0$. The examples in [7] and [16] suggest that the second situation in the above statement may occur. Our final result relates completeness to the total volume.

Theorem C. *Assume that K is bounded between two positive numbers outside a compact subset of \mathbb{R}^n . Let u be a smooth positive solution of equation (1.1) in \mathbb{R}^n with slow decay. Then either u has fast decay or the conformal metric g_c is complete. Furthermore, the conformal metric g_c is complete if and only if the total volume of (\mathbb{R}^n, g_c) is infinite.*

2. Preliminaries. Let u be a smooth positive solution of equation (1.1) in \mathbb{R}^n and

$$P(u, r) = \frac{n-2}{2n} \int_{B_o(r)} x \cdot \nabla K(x) u^{\frac{2n}{n-2}}(x) dx \quad \text{for } r > 0, \quad (2.1)$$

where $B_o(r)$ is the open ball in \mathbb{R}^n with center at the origin and radius $r > 0$. The Pohozaev identity (see, for examples, [9] and [14]) shows that

$$P(u, r) = \int_{S_r} \left[r \left(\frac{\partial u}{\partial r} \right)^2 - \frac{r}{2} |\nabla u|^2 + \frac{n-2}{2n} r K u^{\frac{2n}{n-2}} + \frac{n-2}{2} u \frac{\partial u}{\partial r} \right] dS \quad (2.2)$$

for $r > 0$, where $S_r = \partial B_o(r)$ is the sphere of radius r . The Pohozaev number of u is given by

$$P(u) = \lim_{r \rightarrow \infty} P(u, r) \quad (2.3)$$

provided the limit exists.

Lemma 2.1. *Let u be a smooth positive solution of equation (1.1) in \mathbb{R}^n with slow decay (b). Then the limit in (2.3) exists if we assume any one of the following conditions.*

- (I) *There exists a number $m > 1$ such that $x \cdot \nabla K \in L^m(\mathbb{R}^n \setminus B_o(1))$.*
- (II) *There exist positive numbers C, ε and r_o such that*

$$\left| \frac{\partial K}{\partial r} \right| \leq \frac{C}{r(\ln r)^{1+\varepsilon}} \quad \text{for } r \geq r_o.$$

- (III) *$x \cdot \nabla K(x)$ does not change sign for large $|x|$ and there exist positive constants α and β such that $-\alpha^2 \leq K \leq \beta^2$ in \mathbb{R}^n .*

Proof. Let $\Omega_R = \mathbb{R}^n \setminus B_o(R)$ for $R > 0$. We note that

$$\left| \int_{\Omega_R} |x| \frac{\partial K}{\partial r} u^{\frac{2n}{n-2}} dx \right| \leq C_o \left[\int_{\Omega_R} \left| r \frac{\partial K}{\partial r} \right|^m dx \right]^{\frac{1}{m}} \left[\int_{\Omega_R} |x|^{-nl} dx \right]^{\frac{1}{l}}$$

and the right hand side tends to zero as $R \rightarrow \infty$. Here $l > 1$ is the number such that $1/m + 1/l = 1$, and C_o is a positive constant. Likewise,

$$\left| \int_{\Omega_R} |x| \frac{\partial K}{\partial r} u^{\frac{2n}{n-2}} dx \right| \leq C' \int_R^\infty \frac{dr}{r(\ln r)^{1+\varepsilon}} \rightarrow 0$$

as $R \rightarrow \infty$, where C' is a positive constant. Hence the limit in (2.3) exists if we assume either (I) or (II). Assume that $x \cdot \nabla K(x) \leq 0$ for large $|x|$. The integral in

(2.1) is a non-increasing function for large r . For $R > R_o$ large enough, there exists a positive constant C_1 such that

$$\begin{aligned} & \int_{R_o}^R \int_{S_r} r \frac{\partial K}{\partial r} u^{\frac{2n}{n-2}} dS dr \geq C_1 \int_{R_o}^R \int_{S^{n-1}} \frac{\partial K}{\partial r} d\theta dr \\ & = C_1 \int_{R_o}^R \frac{d}{dr} \left(\int_{S^{n-1}} K(r, \theta) d\theta \right) dr \geq -C_1 \omega_n (\alpha^2 + \beta^2), \end{aligned}$$

where ω_n is the volume of the unit sphere in \mathbb{R}^n . Hence the integral in (2.1) is bounded from below and the limit in (2.3) exists. The other case that $x \cdot \nabla K(x) \geq 0$ for large $|x|$ is similar. \square

We assume that there exist positive constants a, b and r_o such that

$$0 < a^2 \leq K(x) \leq b^2 \quad \text{for } x \in \mathbb{R}^n \text{ with } |x| \geq r_o. \quad (2.4)$$

u is said to satisfy a spherical Harnack inequality if there exists a positive constant C_h such that the inequality

$$\sup_{S_r} u \leq C_h \inf_{S_r} u \quad (2.5)$$

holds for all $r > 0$. In [18], the following relations are observed (a version of the result is proved in [8]).

Lemma 2.2. *Assume that K satisfies (2.4). Let u be a smooth positive solution of (1.1) in \mathbb{R}^n . Then the following statements are equivalent.*

- (i) u has slow decay.
- (ii) u satisfies a spherical Harnack inequality.
- (iii) There exists a positive constant C_g such that $r |\nabla u(r, \theta)| \leq C_g u(r, \theta)$ for $r > 0$ and $\theta \in S^{n-1}$.

Lemma 2.3. *Let K and u be as in lemma 2.2. Assume that K satisfies one of the conditions in lemma 2.1 so that $P(u)$ exists. If $P(u) \neq 0$, then $u(x) \geq C_l |x|^{-(n-2)/2}$ for large $|x|$ and for a positive constant C_l .*

Proof. Suppose that the lower bound does not hold. It follows that there exists a sequence $\{x_i\} \subset \mathbb{R}^n$ such that $\lim_{i \rightarrow \infty} |x_i| = \infty$ and $u(x_i) |x_i|^{(n-2)/2} \rightarrow 0$ as $i \rightarrow \infty$, and by spherical Harnack inequality (2.5) and the gradient estimate in lemma 2.2,

$$r_i^{(n-2)/2} [\max_{S_{r_i}} u] \rightarrow 0 \quad \text{and} \quad r_i^{n/2} [\max_{S_{r_i}} |\nabla u|] \rightarrow 0 \quad \text{as } i \rightarrow \infty, \quad (2.6)$$

where $r_i = |x_i|$ for $i = 1, 2, \dots$. Hence $P(u) = 0$ by (2.2), which is a contradiction. \square

Without the assumption of slow decay, we obtain weaker estimates on volume growth (cf. [9] and [14]).

Theorem 2.4. *Assume that K satisfies (2.4). Let u be a smooth positive solution of equation (1.1) in \mathbb{R}^n . If there exists a positive constant δ such that $|P(u, r)| \geq \delta^2$ for large r , then there exist positive constants C' and C'' such that*

$$\int_{B_o(r)} u^{\frac{2n}{n-2}}(x) dx \geq C' \ln r \quad \text{and} \quad \int_{B_o(r)} |\nabla u|^2 dx \geq C'' \ln r \quad (2.7)$$

for large r . In particular, if $P(u)$ exists and is non-zero, then the total volume of (\mathbb{R}^n, g) is infinite.

Proof. Applying Young's inequality we have

$$\begin{aligned} \int_{S_r} u \frac{\partial u}{\partial r} dS &\leq \int_{S_r} r \left| \frac{\partial u}{\partial r} \right|^2 dS + \int_{S_r} \frac{u^2}{r} dS \\ &\leq \int_{S_r} r |\nabla u|^2 dS + C_1 \int_{S_r} r u^{\frac{2n}{n-2}} dS + \frac{\delta^2}{n-2} \end{aligned} \quad (2.8)$$

for large r , where C_1 is a positive constant depending on δ and n . From Pohozaev identity (2.2) and (2.8), there exist positive constant C_2 and C_3 such that

$$C_2 \int_{S_r} |\nabla u|^2 dS + C_3 \int_{S_r} u^{\frac{2n}{n-2}} dS \geq \frac{\delta^2}{2r}$$

for large r . It follows that

$$\int_{B_o(r)} |\nabla u|^2 dx + \int_{B_o(r)} u^{\frac{2n}{n-2}} dx \geq C_4 \ln r \quad (2.9)$$

for large r and for a positive constant C_4 . We can modify the argument in the proof of theorem 3.1 in [14] for our case so as to obtain positive constants C_5 and D_1 such that

$$C_5 \int_{B_o(2R)} u^{\frac{2n}{n-2}} dx + D_1 \geq \int_{B_o(R)} |\nabla u|^2 dx \quad (2.10)$$

for all large R . (2.9) and (2.10) imply that

$$\int_{B_o(r)} u^{\frac{2n}{n-2}} dx \geq C' \ln r$$

for large r and for a positive constant C' . Similarly we obtain the other estimate in (2.7). \square

Let

$$v(s, \theta) = r^{\frac{n-2}{2}} u(r, \theta), \quad r = e^s, \quad r > 0, \quad s \in \mathbb{R} \quad \text{and} \quad \theta \in S^{n-1}. \quad (2.11)$$

Then v satisfies the equation (see, for example, [14])

$$\frac{\partial^2 v}{\partial s^2} + \Delta_\theta v - \left(\frac{n-2}{2} \right)^2 v + K v^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R} \times S^{n-1}, \quad (2.12)$$

where Δ_θ is the standard Laplacian on S^{n-1} . The Pohozaev identity (2.2) becomes

$$\begin{aligned} P(v, s) := \frac{n-2}{2n} \int_{-\infty}^s \int_{S^{n-1}} \frac{\partial K}{\partial t} v^{\frac{2n}{n-2}} d\theta dt &= \int_{S^{n-1}} \left[\frac{1}{2} \left(\frac{\partial v}{\partial s} \right)^2 - \frac{1}{2} |\nabla_\theta v|^2 \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{n-2}{2} \right)^2 v^2 + \frac{n-2}{2n} K v^{\frac{2n}{n-2}} \right] d\theta \quad \text{for} \quad s \in \mathbb{R}. \end{aligned} \quad (2.13)$$

3. Proof of theorem A. By differentiating with respect to r the function

$$\int_{S_r} u^{\frac{2n}{n-2}} dS = \int_{S^{n-1}} u^{\frac{2n}{n-2}}(r, \theta) r^{n-1} d\theta \quad \text{for} \quad r > r_o$$

and using (1.4), there exist positive constants C_o and m such that

$$\int_{S_r} u^{\frac{2n}{n-2}} dS \leq C_o r^m \quad \text{for } r \geq r_o. \quad (3.1)$$

From (1.3) and (3.1) we have

$$\frac{n-2}{2n} \int_{B_o(R)} r \frac{\partial K}{\partial r} u^{\frac{2n}{n-2}} dx \geq -C_1 - C_2 \int_{r_1}^R e^{-cr} r^{m+1} dr \geq -C_3 \quad \text{for } R \geq r_1,$$

where C_1 , C_2 and C_3 are positive constants and $r_1 \geq \max\{r_o, r'\}$. Together with (1.9) and Pohozaev identity (2.2) we obtain

$$\int_{S_R} R |\nabla u|^2 dS \leq 2C_3 + C_4 \int_{S_R} \frac{u^2}{R} dS + \frac{n-2}{n} \int_{S_R} RK u^{\frac{2n}{n-2}} dS \quad (3.2)$$

for large R , where C_4 is a positive constant. By choosing a larger C_3 if necessary, we may assume that (3.2) holds for all $R > 0$. Similarly, as $u > 0$ in \mathbb{R}^n , we may assume that (1.4) holds for all $r > 0$. Integrating both sides of (3.2) we obtain

$$\int_{B_o(R)} r |\nabla u|^2 dx \leq 2C_3 R + C_4 \int_{B_o(R)} \frac{u^2}{r} dx + \frac{n-2}{n} \int_{B_o(R)} r K u^{\frac{2n}{n-2}} dx \quad (3.3)$$

for $R > 0$. Using equation (1.1) and (1.4) we have

$$\begin{aligned} \int_{B_o(R)} r K u^{\frac{2n}{n-2}} dx &= \int_{B_o(R)} r u \left(K u^{\frac{n+2}{n-2}} \right) dx = \int_{B_o(R)} (ru)(-\Delta u) dx \\ &= \int_{B_o(R)} r |\nabla u|^2 dx + \int_{B_o(R)} u \frac{\partial u}{\partial r} dx - \int_{S_R} R u \frac{\partial u}{\partial r} dS \\ &\leq \int_{B_o(R)} r |\nabla u|^2 dx + C_5 \int_{B_o(R)} \frac{u^2}{r} dx + C_6 \int_{S_R} u^2 dS \end{aligned} \quad (3.4)$$

for large R , where C_5 and C_6 are positive constants. It follows from (3.3) and (3.4) that

$$\begin{aligned} \frac{2}{n} \int_{B_o(R)} r K u^{\frac{2n}{n-2}} dx &\leq 2C_3 R + C_7 \int_{B_o(R)} \frac{u^2}{r} dx + C_8 \int_{S_R} u^2 dS \\ &\leq 2C_3 R + C_7 \varepsilon \int_{S_R} r^2 u^{\frac{2n}{n-2}} dS + C_8 \varepsilon \int_{B_o(R)} r u^{\frac{2n}{n-2}} dx + C_9 \varepsilon^{-(n-2)/2} R \end{aligned} \quad (3.5)$$

for large R , where we use Young's inequality and C_7 , C_8 and C_9 are positive constants. From (1.9) we have

$$\frac{d}{dr} \left(\int_{S_r} r^2 u^{\frac{2n}{n-2}} dS \right) \leq C_{10} \int_{S_r} r u^{\frac{2n}{n-2}} dS \quad \text{for } r \geq r_o,$$

where C_{10} is a positive constant. Hence

$$\begin{aligned} \int_{S_r} r^2 u^{\frac{2n}{n-2}} dS &= \int_0^r \left(\int_{S_s} s^2 u^{\frac{2n}{n-2}} dS \right)' ds \leq C_{11} + \int_{r_o}^r \left(\int_{S_s} s^2 u^{\frac{2n}{n-2}} dS \right)' ds \\ &\leq C_{11} + C_{10} \int_{r_o}^r \int_{S_s} s u^{\frac{2n}{n-2}} dS ds \leq C_{12} \int_{B_o(r)} r u^{\frac{2n}{n-2}} dx \quad \text{for } r \geq 2r_o, \end{aligned} \quad (3.6)$$

where C_{11} and C_{12} are positive constants that depends on r_o and u .

If $\lim_{R \rightarrow \infty} \int_{B_o(R)} u^{\frac{2n}{n-2}} dx < \infty$, then by using theorem 3.1 in [14], which can be modified for K satisfying (2.4) by making the constants there larger if necessary, together with a result of Brezis and Kato in [2], we have

$$u(x) \leq C_{13}|x|^{-(n-2)}$$

for large $|x|$ and for a positive constant C_{13} (cf. the proof of theorem 3.16 in [14]). In particular, u has slow decay. So we may assume that

$$\lim_{R \rightarrow \infty} \int_{B_o(R)} u^{\frac{2n}{n-2}} dx = \infty. \quad (3.7)$$

As $K(x) \geq a^2$ for large $|x|$, it follows from (3.7) that

$$\int_{B_o(R)} r K u^{\frac{2n}{n-2}} dx \geq \frac{a^2}{2} \int_{B_o(R)} r u^{\frac{2n}{n-2}} dx \quad (3.8)$$

for large R . If we choose ε to be small, then (3.5), (3.6) and (3.8) imply that

$$\int_{B_o(R)} r u^{\frac{2n}{n-2}} dx \leq C_{14}R \quad \text{for } R \geq r_o$$

for large R and

$$\int_{S_R} R^2 u^{\frac{2n}{n-2}} dS \leq C_{15}R$$

for large R , where C_{14} and C_{15} are positive constants. The last inequality can also be written as

$$\int_{S^{n-1}} u^{\frac{2n}{n-2}}(r, \theta) d\theta \leq C_{15}r^{-n} \quad (3.9)$$

for large r . It follows from lemma 4.39 and 4.40 in [14] that decay estimate (3.10) implies spherical Harnack inequality (2.5) for spheres of large radius. Together with (3.9), or by lemma 2.2, we conclude that u has slow decay. \square

Remark 3.1. Condition (1.3) in theorem A is used only to obtain a lower bound on $P(u, r)$. Therefore theorem A remains valid if we replace (1.3) by the assumption that $P(u, r) \geq -\delta^2$ for large r and for a constant δ .

4. Asymptotic behavior. We consider the sign of the Pohozaev number.

Theorem 4.1. *Assume that $\lim_{|x| \rightarrow \infty} K(x) = K_\infty > 0$ and $|\nabla K| \leq C_o$ in \mathbb{R}^n for a positive number C_o . Let u be a smooth positive solution of equation (1.1) in \mathbb{R}^n with slow decay. Assume also that K satisfies one of the conditions in lemma 2.1 so that $P(u)$ exists. Then we have $P(u) \leq 0$. If $P(u) = 0$, then either u has fast decay, or the conformal metric $g_c = u^{4/(n-2)}g_o$ is complete and*

$$\liminf_{|x| \rightarrow \infty} |x|^{\frac{n-2}{2}} u(x) = 0$$

and there exists a sequence $\{s_l\} \subset \mathbb{R}$, $s_l \rightarrow +\infty$ as $l \rightarrow +\infty$, such that the sequence $\{v_l\}$ defined by $v_l(s, \theta) = v(s_l + s, \theta)$ for $s \in \mathbb{R}$, $\theta \in S^{n-1}$ and $l = 1, 2, \dots$, converges uniformly in C^2 -norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to a standard spherical solution. Here v is defined in (2.11).

Remark 4.1. Standard spherical solutions on $\mathbb{R} \times S^{n-1}$ are the transforms of functions of the form

$$\left(\frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{n-2}{2}}$$

by (2.11), where λ is a positive number.

Proof. Suppose that $P(u) > 0$. Lemma 2.3 implies that there exist positive constants c_1, c_2 and s_o such that

$$c_1 \leq v(s, \theta) \leq c_2 \quad \text{for } s \geq s_o \quad \text{and } \theta \in S^{n-1}. \quad (4.1)$$

We have $P(u) = P(v) := \lim_{s \rightarrow +\infty} P(v, s)$, where $P(v, s)$ is given by (2.13). For $t > 0$, let $v_t(s, \theta) = v(s + t, \theta)$ for $\theta \in S^{n-1}$ and $s \geq s_o - t$. Using the equation

$$\frac{\partial^2 v_t}{\partial s^2} + \Delta_\theta v_t = F_t \quad \text{for } s \geq s_o - t,$$

where

$$F_t(s, \theta) = \left(\frac{n-2}{2} \right)^2 v_t(s, \theta) - K(s+t, \theta) v_t^{\frac{n+2}{n-2}}(s, \theta)$$

for $s \in \mathbb{R}$ and $\theta \in S^{n-1}$, (4.1), the boundedness of $|\nabla K|$, and elliptic estimates (cf. [11]), there exists a sequence of positive numbers $t_i \rightarrow \infty$ as $i \rightarrow \infty$ and a C^2 -function v_∞ defined on $\mathbb{R} \times S^{n-1}$ such that v_{t_i} converges in C^2 -norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to v_∞ as $i \rightarrow \infty$. Furthermore, v_∞ satisfies the equation

$$\frac{\partial^2 v_\infty}{\partial s^2} + \Delta_\theta v_\infty - \left(\frac{n-2}{2} \right)^2 v_\infty + K_\infty v_\infty^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R} \times S^{n-1},$$

and $c_1 \leq v_\infty \leq c_2$ in $\mathbb{R} \times S^{n-1}$. By a result of Caffarelli, Gidas, Spruck in [3], v_∞ is independent on θ . It follows that the Pohozaev number $P(v_\infty)$ is negative (cf. section 2.1 in [13]). Hence $P(v_\infty, s) < 0$ for large s . Fixed a large number s_o so that $P(v_\infty, s_o) < 0$. As v_{t_i} converges in C^2 -norm to v_∞ in a compact neighborhood of $s_o \times S^{n-1}$, we have $P(v_{t_i}, s_o) < 0$ for i large. But $P(v_{t_i}, s_o) = P(v, t_i + s_o) \rightarrow P(v) > 0$ as $i \rightarrow \infty$. Therefore we have a contradiction. Hence $P(u) \leq 0$. Assume that $P(u) = 0$. It follows from the above argument that v cannot be bounded away from zero in $\mathbb{R}^+ \times S^{n-1}$. Hence $\liminf_{|x| \rightarrow \infty} |x|^{\frac{n-2}{2}} u(x) = 0$. Define

$$\bar{v}(s) = \int_{S^{n-1}} v(s, \theta) d\theta \quad \text{for } s \in \mathbb{R}. \quad (4.2)$$

Then \bar{v} is bounded from above in \mathbb{R} . Either $\bar{v}'(s) \geq 0$ or $\bar{v}'(s) \leq 0$ for all large s , or \bar{v}' keeps on changing sign near positive infinity. In the first two cases we have $\lim_{s \rightarrow +\infty} \bar{v}(s) = \nu_o$ exists, as $\bar{v}(s)$ is bounded positive for large s . If $\nu_o > 0$, then using spherical Harnack inequality (2.5), there exists a positive constant c_3 such that $v(s, \theta) \geq c_3^2$ for large s and for $\theta \in S^{n-1}$. Hence we may apply the above argument to show that $P(u) < 0$, which is a contradiction. Assume that $\nu_o = 0$. Spherical Harnack inequality (2.5) implies that

$$\lim_{s \rightarrow +\infty} \max_{\theta \in S^{n-1}} v(s, \theta) = 0.$$

It is standard to conclude that u has fast decay (a) (cf. [13]). Therefore we need only to consider the case when \bar{v}' keeps on changing sign near positive infinity. If this is the case, then there exists a sequence of numbers $\{s_i\}$ such that $\lim_{i \rightarrow \infty} s_i = +\infty$ and \bar{v} achieves local maximum at s_i for $i = 1, 2, \dots$. It follows from equation (2.12) and spherical Harnack inequality (2.5) that

$$\bar{v}''(s) - \left(\frac{n-2}{2}\right)^2 \bar{v}(s) + C_2 \bar{v}^{\frac{n+2}{n-2}}(s) \geq 0 \quad (4.3)$$

for a positive constant C_2 and for large positive s (cf. [7]). Using (4.3) and the fact that $\bar{v}''(s_i) \leq 0$ we have

$$\bar{v}(s_i) \geq \left[\frac{1}{C_2} \left(\frac{n-2}{2}\right)^2 \right]^{\frac{n-2}{4}} \quad (4.4)$$

for large i . It follows from spherical Harnack inequality (2.5) that there exists a positive constant c_4 such that

$$v(s_i, \theta) \geq c_4^2 \quad \text{for } \theta \in S^{n-1} \quad (4.5)$$

and for large i . Define $v_i(s, \theta) = v(s+s_i, \theta)$ for $s \in \mathbb{R}$, $\theta \in S^{n-1}$ and $i = 1, 2, \dots$. As above, a subsequence of $\{v_i\}$ converges in C^2 -norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to a non-negative C^2 function v_∞ on $\mathbb{R} \times S^{n-1}$ which satisfies the equation

$$\frac{\partial^2 v_\infty}{\partial s^2} + \Delta_\theta v_\infty = v_\infty \left[\left(\frac{n-2}{2}\right)^2 - K_\infty v_\infty^{\frac{4}{n-2}} \right] \quad \text{in } \mathbb{R} \times S^{n-1}.$$

Furthermore, from (4.5) we have $v_\infty(0, \theta) \geq c_4^2 > 0$ for $\theta \in S^{n-1}$. Let

$$u_\infty(x) = u_\infty(r, \theta) = r^{-(n-2)/2} v_\infty(s, \theta) \quad \text{for } |x| > 0,$$

where $x = (r, \theta)$ and $r = e^s$. Then u_∞ satisfies the equation

$$\Delta u_\infty = -K_\infty u_\infty^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (4.6)$$

In addition, $u_\infty(x) \geq 0$ for $|x| > 0$ and $u_\infty(x) \geq c_4^2 > 0$ for $|x| = 1$. It follows from equation (4.6) and the maximum principle that $u_\infty(x) > 0$ for $|x| > 0$. Hence v_∞ is positive in $\mathbb{R} \times S^{n-1}$. Because v_∞ may not be bounded away from zero, we can only conclude that $P(v_\infty) \leq 0$. The above argument shows that if $P(v_\infty) < 0$, then $P(u) < 0$. Hence $P(v_\infty) = 0$. By a result of Caffarelli, Gidas and Spruck in [3] (see also [13]), v_∞ is a standard spherical solution. It remains to show that the conformal metric corresponding to u is complete in this case. From (4.3) and the fact that v is bounded from above, there exists a positive constant c_5 such that

$$\bar{v}''(s) \geq -c_5^2 \quad \text{for } s \in \mathbb{R}. \quad (4.7)$$

At s_i we have $\bar{v}'(s_i) = 0$, therefore $\bar{v}'(s)$ is not too negative for s close to s_i for $i = 1, 2, \dots$. By using (4.4) and (4.7), there exists a positive number ε independent on i such that

$$\bar{v}(s) \geq \frac{1}{2} \left[\frac{1}{C_2} \left(\frac{n-2}{2}\right)^2 \right]^{\frac{n-2}{4}} \quad \text{for } s \in [s_i, s_i + \varepsilon], \quad i = 1, 2, \dots$$

Using spherical Harnack inequality (2.5) there exists a positive constant c_6 such that

$$v(s, \theta) \geq c_6^2 \quad \text{for } s \in [s_i, s_i + \epsilon], \quad \theta \in S^{n-1} \quad (4.8)$$

and $i = 1, 2, \dots$. Without loss of generality, we may assume that $s_1 > 0$ and $s_{i+1} > s_i$ for $i = 1, 2, \dots$. For any fixed $\theta \in S^{n-1}$, the length of the curve $r \mapsto (r, \theta)$, $r \in (1, \infty)$, in the conformal metric $g_c = u^{4/(n-2)}g_o$ is given by

$$\int_1^\infty u^{\frac{2}{n-2}}(r, \theta) dr = \int_0^\infty v^{\frac{2}{n-2}}(s, \theta) ds \geq \sum_{i=1}^\infty \int_{s_i}^{s_i+\epsilon} v^{\frac{2}{n-2}}(s, \theta) ds = \infty.$$

Hence the conformal metric is complete. \square

Proof of Theorem C. Consider \bar{v} as defined in (4.2) and $\lim_{s \rightarrow \infty} \bar{v}(s)$. If the limit exists and is equal to zero, then u has fast decay (a) (cf. [13]). If the limit is non-zero, then $u(x) \geq c|x|^{(2-n)/2}$ for large $|x|$ and for a positive constant c , and hence the conformal metric g_c is complete. In case the limit does not exist, then \bar{v}' keeps on changing signs near positive infinity. The argument in the last part of the proof of theorem 4.1 shows that the conformal metric is also complete in this case. Finally, if the total volume of (\mathbb{R}^n, g) is infinite, then u does not have fast decay (a) and hence by above g_c is complete. If g_c is complete, then either $\lim_{s \rightarrow +\infty} \bar{v}(s)$ exists and is positive, or (4.8) holds. In either cases we have

$$\int_{\mathbb{R}^n \setminus B_o(1)} u^{\frac{2n}{n-2}}(x) dx = \int_0^\infty \int_{S^{n-1}} v^{\frac{2n}{n-2}} d\theta ds = \infty.$$

That is, the total volume of (\mathbb{R}^n, g) is infinite. \square

Theorem 4.2. *Assume that K satisfies the conditions in theorem 4.1 and $x \cdot \nabla K(x) \leq 0$ for large $|x|$. Let u be a smooth positive solution of equation (1.1) in \mathbb{R}^n with slow decay. If $P(u) = 0$, then u has fast decay.*

Proof. Let \bar{v} be defined as in (4.2). Suppose that $P(u) = 0$ and u does not have fast decay. It follows from the argument in the proof of theorem 4.1 that there exists a sequence $\{s_j\} \subset \mathbb{R}$ such that $s_j \rightarrow +\infty$ as $j \rightarrow \infty$ and each s_j is a local minimum for \bar{v} and $\lim_{j \rightarrow \infty} \bar{v}(s_j) = 0$. Let

$$w_j(s, \theta) = \frac{v(s + s_j, \theta)}{\bar{v}(s_j)} \quad \text{for } s \in \mathbb{R}, \quad \theta \in S^{n-1} \quad \text{and } j = 1, 2, \dots$$

Given $S > 0$, we claim that there exist positive numbers C_s and k_s such that

$$w_j(s, \theta) \leq C_s \quad \text{for } s \in [-S, S], \quad \theta \in S^{n-1} \quad \text{and } j \geq k_s. \quad (4.9)$$

Using Hölder's inequality, equation (2.12) and (4.3) we have

$$\left(\frac{n-2}{2}\right)^2 \bar{v}(s) - C_1 \bar{v}^{\frac{n+2}{n-2}}(s) \geq \bar{v}''(s) \geq \left(\frac{n-2}{2}\right)^2 \bar{v}(s) - C_2 \bar{v}^{\frac{n+2}{n-2}}(s)$$

for large s and for some positive constants C_1 and C_2 . It follows that there exists a positive constant ε_o such that

$$0 \leq \bar{v}''(s) \leq \frac{1}{2} \left(\frac{n-2}{2}\right)^2 \bar{v}(s) \quad \text{for } \bar{v}(s) \leq \varepsilon_o \quad \text{and large } s. \quad (4.10)$$

Choose the number k_s such that

$$\bar{v}(s_j) e^{\frac{1}{4}\left(\frac{n-2}{2}\right)^2 S^2} < \varepsilon_o \quad \text{for } j \geq k_s.$$

We may also assume that for $j \geq k_s$, s_j is large. For $j \geq k_s$ and s close to s_j , $\bar{v}''(s)$ satisfies (4.10). Hence we obtain

$$\bar{v}'(s) \leq \frac{1}{2} \left(\frac{n-2}{2} \right)^2 \int_{s_j}^s \bar{v}(t) dt \leq \frac{1}{2} \left(\frac{n-2}{2} \right)^2 \bar{v}(s)(s - s_j)$$

for $s > s_j$ close to s_j , as $v'(s) \geq 0$ for $s \geq s_j$ close to s_j . Therefore we have

$$\frac{\bar{v}(s)}{\bar{v}(s_j)} \leq e^{\frac{1}{4}\left(\frac{n-2}{2}\right)^2 (s-s_j)^2} \quad \text{for } s \in [s_j, s_j + S].$$

Likewise, we obtain a similar inequality on $[s_j - S, s_j]$. By using spherical Harnack inequality (2.5), we obtain the desired constant C_s and the bound in (4.9). It follows from (4.9) that there exists a subsequence of $\{w_j\}$ which converges in C^2 -norm on compact subsets of $\mathbb{R} \times S^{n-1}$ to a solution w of the equation

$$\frac{\partial^2 w}{\partial s^2} + \Delta_\theta w - \left(\frac{n-2}{2} \right)^2 w = 0 \quad \text{in } \mathbb{R} \times S^{n-1}.$$

The associated function h related to w as in (1.11) is, by the maximum principle and the above equation, a positive harmonic function on $\mathbb{R}^n \setminus \{0\}$. Therefore $h(x) = a|x|^{2-n} + b$ and

$$w(s, \theta) = a e^{-\frac{n-2}{2}s} + b e^{\frac{n-2}{2}s} \quad \text{for } s \in \mathbb{R} \text{ and } \theta \in S^{n-1},$$

where a and b are positive constants. As \bar{w} has a critical point at $s = 0$, we have $a = b > 0$. As in [13] we obtain

$$\begin{aligned} & \lim_{j \rightarrow \infty} \int_{S^{n-1}} \left[\frac{1}{2} \left(\frac{\partial w_j}{\partial s} \right)^2 (0, \theta) - \frac{1}{2} |\nabla_\theta w_j|^2 (0, \theta) - \frac{1}{2} \left(\frac{n-2}{2} \right)^2 w_j^2 (0, \theta) \right. \\ & \quad \left. + \bar{v}^{\frac{4}{n-2}}(s_j) \frac{n-2}{2n} K w_j^{\frac{2n}{n-2}} (0, \theta) \right] d\theta \\ &= \int_{S^{n-1}} \left[\frac{1}{2} \left(\frac{\partial w}{\partial s} \right)^2 (0, \theta) - \frac{1}{2} \left(\frac{n-2}{2} \right)^2 w^2 (0, \theta) \right] d\theta \\ &= -\omega_n \left(\frac{n-2}{2} \right)^2 ab < 0. \end{aligned} \tag{4.11}$$

Using the assumption that $(\partial K / \partial s)(s, \theta) \leq 0$ for large s and $\theta \in S^{n-1}$, Pohozaev identity (2.13) and $\lim_{s \rightarrow +\infty} P(v, s) = 0$, we see that $P(v, s) \geq 0$ for large s . On the other hand, by (4.11), $P(v, s_j) / \bar{v}^2(s_j)$ is strictly less than a negative number for all large j . We obtain a contradiction. Therefore if $P(u) = 0$, then u has fast decay. One can also prove theorem 4.3 by modifying the argument of Chen and Lin in section 3 of [7]. \square

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