NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1508 Linear Algebra with Applications (2006/07 Semester 2)

Tutorial 10 Solutions

1. If $\boldsymbol{A}$ is invertible, then every vector in $\mathbb{R}^3$ would be in the column space of $\boldsymbol{A}$. So we first find a such that $\boldsymbol{A}$ is singular.

$$\det(\boldsymbol{A}) = 2a + 1 \Rightarrow \det(\boldsymbol{A}) = 0 \Leftrightarrow z = -\frac{1}{2}.$$

Let us check for consistency of the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ when $a = -\frac{1}{2}$.

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 1 & -\frac{1}{2} & 1 & 2 \\ 0 & -1 & 1 & -\frac{1}{2} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

So indeed, $\boldsymbol{b}$ is not in the column space of $\boldsymbol{A}$ when $a = -\frac{1}{2}$.

To find all least squares solutions to $\boldsymbol{Ax} = \boldsymbol{b}$, we solve $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.

$$\begin{aligned} & \boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b} \\ \Rightarrow\ & \begin{pmatrix} 2 & 1 & 0 \\ 1 & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -\frac{1}{2} & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -\frac{1}{2} \end{pmatrix} \\ \Rightarrow\ & \begin{pmatrix} 5 & \frac{3}{2} & 1 \\ \frac{3}{2} & \frac{9}{4} & -\frac{3}{2} \\ 1 & -\frac{3}{2} & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}. \\ \Rightarrow\ & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{11}{12} \\ -\frac{7}{18} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

So the least squares solutions are given by the set

$$\left\{ \begin{pmatrix} \frac{11}{12} \\ -\frac{7}{18} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

2. (a) $[\boldsymbol{w}]_{S_1} = \boldsymbol{w}$, $[\boldsymbol{u}]_{S_1} = \boldsymbol{u}$. $[\boldsymbol{w}]_{S_2} = (-\frac{7}{3}, \frac{5}{3})^T$, $[\boldsymbol{u}]_{S_2} = (-1, 0)^T$. $[\boldsymbol{w}]_{S_1} \cdot [\boldsymbol{u}]_{S_1} = -1 + 4 = 3$. $[\boldsymbol{w}]_{S_2} \cdot [\boldsymbol{u}]_{S_2} = \frac{7}{3}$. The products are not equal.

   (b) Note that S_3 is an orthonormal basis for $\mathbb{R}^2$. So,

$$[\boldsymbol{w}]_{S_3} = (\boldsymbol{w} \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \boldsymbol{w} \cdot (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}))^T = (\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}})^T.$$

$$[\boldsymbol{u}]_{S_3} = (\boldsymbol{u} \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \boldsymbol{u} \cdot (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}))^T = (0, \frac{2}{\sqrt{2}})^T.$$

   Computing the dot product, $[\boldsymbol{w}]_{S_3} \cdot [\boldsymbol{u}]_{S_3} = 3$. We notice that this is the same as $[\boldsymbol{w}]_{S_1} \cdot [\boldsymbol{u}]_{S_1}$.

(c) Let $\boldsymbol{P}$ be the transition matrix from S to T. Note that $\boldsymbol{P}$ is orthogonal, since S and T are orthonormal bases. Now,

$$\begin{aligned}[\boldsymbol{w}]_T \cdot [\boldsymbol{u}]_T &= \boldsymbol{P}[\boldsymbol{w}]_S \cdot \boldsymbol{P}[\boldsymbol{u}]_S \\ &= (\boldsymbol{P}[\boldsymbol{w}]_S)^T \boldsymbol{P}[\boldsymbol{u}]_S \\ &= [\boldsymbol{w}]_S^T \boldsymbol{P}^T \boldsymbol{P}[\boldsymbol{u}]_S = [\boldsymbol{w}]_S^T[\boldsymbol{u}]_S = [\boldsymbol{w}]_S \cdot [\boldsymbol{u}]_S.\end{aligned}$$

3. (a) Since $\boldsymbol{A}$ is symmetric, there is an orthogonal matrix $\boldsymbol{P} = [\boldsymbol{v_1}\ \boldsymbol{v_2}\ \ldots\ \boldsymbol{v_n}]$ that orthogonally diagonalizes $\boldsymbol{A}$. In this case, the set $S = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is an orthonormal set and each $\boldsymbol{v_i}$ is an eigenvector of $\boldsymbol{A}$ corresponding to eigenvalue λ_i.

(b) Since S is an orthonormal basis for $\mathbb{R}^n$, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}\boldsymbol{x} &= (\boldsymbol{x}\cdot\boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{x}\cdot\boldsymbol{v_2})\boldsymbol{v_2} + \ldots + (\boldsymbol{x}\cdot\boldsymbol{v_n})\boldsymbol{v_n} \\ (\boldsymbol{x})_S &= (\boldsymbol{x}\cdot\boldsymbol{v_1}, \boldsymbol{x}\cdot\boldsymbol{v_2}, \ldots, \boldsymbol{x}\cdot\boldsymbol{v_n}) \\ \boldsymbol{Ax} &= (\boldsymbol{x}\cdot\boldsymbol{v_1})\boldsymbol{Av_1} + (\boldsymbol{x}\cdot\boldsymbol{v_2})\boldsymbol{Av_2} + \ldots + (\boldsymbol{x}\cdot\boldsymbol{v_n})\boldsymbol{Av_n} \\ &= \lambda_1(\boldsymbol{x}\cdot\boldsymbol{v_1})\boldsymbol{v_1} + \lambda_2(\boldsymbol{x}\cdot\boldsymbol{v_2})\boldsymbol{v_2} + \ldots + \lambda_n(\boldsymbol{x}\cdot\boldsymbol{v_n})\boldsymbol{v_n} \\ (\boldsymbol{Ax})_S &= (\lambda_1(\boldsymbol{x}\cdot\boldsymbol{v_1}), \lambda_2(\boldsymbol{x}\cdot\boldsymbol{v_2}), \ldots, \lambda_n(\boldsymbol{x}\cdot\boldsymbol{v_n}))\end{aligned}$$

(c) Note that

$$\begin{aligned}||\boldsymbol{x}||^2 &= (\boldsymbol{x}\cdot\boldsymbol{v_1})^2 + (\boldsymbol{x}\cdot\boldsymbol{v_2})^2 + \ldots + (\boldsymbol{x}\cdot v_n)^2 = 1 \\ \boldsymbol{x}^T\boldsymbol{Ax} &= \boldsymbol{x}\cdot\boldsymbol{Ax} \\ &= \lambda_1(\boldsymbol{x}\cdot\boldsymbol{v_1})^2 + \lambda_2(\boldsymbol{x}\cdot\boldsymbol{v_2})^2 + \ldots + \lambda_n(\boldsymbol{x}\cdot\boldsymbol{v_n})^2 \\ &\leq \lambda_1(\boldsymbol{x}\cdot\boldsymbol{v_1})^2 + \lambda_1(\boldsymbol{x}\cdot\boldsymbol{v_2})^2 + \ldots + \lambda_1(\boldsymbol{x}\cdot\boldsymbol{v_n})^2 \\ &= \lambda_1((\boldsymbol{x}\cdot\boldsymbol{v_1})^2 + (\boldsymbol{x}\cdot\boldsymbol{v_2})^2 + \ldots + (\boldsymbol{x}\cdot\boldsymbol{v_n})^2) \\ &= \lambda_1\end{aligned}$$

(d) The quadratic form can be written as

$$x_1^2 + x_2^2 + 4x_1x_2 = \boldsymbol{x}^T\boldsymbol{Ax} = (x_1\ \ x_2)\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

We find that the eigenvalues of $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ are 3 and -1. So the maximum value of $x_1^2 + x_2^2 + 4x_1x_2$ subject to $x_1^2 + x_2^2 = 1$ is 3.

4.

$$11x^2 + 24xy + 4y^2 = 15 \Leftrightarrow \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}11 & 12\\ 12 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 15.$$

Let $\boldsymbol{A} = \begin{pmatrix}11 & 12\\ 12 & 4\end{pmatrix}$. Find the eigenvalues of $\boldsymbol{A}$:

$$\begin{aligned}&\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0 \\ \Rightarrow\ &\begin{vmatrix}\lambda - 11 & -12\\ -12 & \lambda - 4\end{vmatrix} = 0 \\ \Rightarrow\ &(\lambda - 20)(\lambda + 5) = 0 \\ \Rightarrow\ &\lambda = 20, -5.\end{aligned}$$

Consider E_{-5}:

$$\begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = s \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \; s \in \mathbb{R}.$$

An orthonormal basis for E_{-5} is $\left\{ \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \right\}$.

Consider E_{20}:

$$\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = s \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \; s \in \mathbb{R}.$$

An orthonormal basis for E_{20} is $\left\{ \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \right\}$.

Let $\boldsymbol{P} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{P} \begin{pmatrix} x' \\ y' \end{pmatrix}$. So

$$\begin{pmatrix} x & y \end{pmatrix} \boldsymbol{A} \begin{pmatrix} x \\ y \end{pmatrix} = 15 \Leftrightarrow \begin{pmatrix} x' & y' \end{pmatrix} \boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P} \begin{pmatrix} x' \\ y' \end{pmatrix} = 15 \Leftrightarrow \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} -5 & 0 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 15.$$

So the equation of the conic using the new coordinates (x', y') is

$$-5x'^2 + 20y'^2 = 15 \Leftrightarrow -\frac{x'^2}{3} + \frac{y'^2}{\frac{3}{4}} = 1$$

and we see that it is a hyperbola.

5. Let $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix}$. By Gauss-Jordan elimination, we find that

$$\boldsymbol{B}^{-1} = \begin{pmatrix} -36 & 8 & 21 \\ 5 & -1 & -3 \\ 9 & -2 & -5 \end{pmatrix}.$$

So, the solution to

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -36 & 8 & 21 \\ 5 & -1 & -3 \\ 9 & -2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -36x + 8y + 21z \\ 5x - y - 3z \\ 9x - 2y - 5z \end{pmatrix}.$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-36x + 8y + 21z) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + (5x - y - 3z) \begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix} + (9x - 2y - 5z) \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

and

$$\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= (-36x+8y+21z)T\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + (5x-y-3z)T\begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix} + (9x-2y-5z)T\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \\
&= (-36x+8y+21z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (5x-y-3z)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + (9x-2y-5z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -41x+9y+24z \\ 14x-3y-8z \end{pmatrix} \\
&= \begin{pmatrix} -41 & 9 & 24 \\ 14 & -3 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{aligned}$$

So the standard matrix for T is $\begin{pmatrix} -41 & 9 & 24 \\ 14 & -3 & -8 \end{pmatrix}$ and

$$T\begin{pmatrix} 7 \\ 13 \\ 7 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$