NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1508 Linear Algebra with Applications

2006-2007 (Semester 2) **Tutorial 11**

1. (a) Find and describe the solution space of the following linear system:

$$\begin{cases} 2x & & + & z & = & 0 \\ x & + \; y & + & \frac{1}{2}z & = & 0 \\ -x & & - & \frac{1}{2}z & = & 0 \end{cases}$$

   (b) Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the solution space given above. Is P a linear operator on $\mathbb{R}^3$? If yes, find the standard matrix and formula for P. If not, explain why.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The *kernel* of T is defined to be a subset of $\mathbb{R}^n$ such that

$$\text{Ker}(T) = \{\boldsymbol{u} \in \mathbb{R}^n \mid T(\boldsymbol{u}) = \boldsymbol{0}\}$$

and the *image* of T is defined to be a subset of $\mathbb{R}^m$ such that

$$\text{Im}(T) = \{T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbb{R}^n\}.$$

   (a) Find Ker(T) and Im(T) if $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ w+z \\ w+x+y+z \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

   (b) Show that Ker(T) is a subspace of $\mathbb{R}^n$ and Im(T) is a subspace of $\mathbb{R}^m$.

   (c) Let $\boldsymbol{A}$ be the standard matrix for T. Identify Ker(T) and Im(T) with some of the vector spaces associated with $\boldsymbol{A}$. Hence, or otherwise, show that

$$\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = n.$$

   (The equation above is known as the *Dimension Theorem for Linear Transformations*.)

3. Describe geometrically the effect of each of the following linear operators $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$.

$$\text{(a) } \boldsymbol{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \quad \text{(b) } \boldsymbol{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \quad \text{(c) } \boldsymbol{A} = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix}.$$

4. Let $\boldsymbol{n}$ be an unit vector in $\mathbb{R}^n$. Define $P : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$P(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

   (a) Show that P is a linear transformation.

   (b) For the case where $n = 2$ and $\boldsymbol{n} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$, find the standard matrix for P.