NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1508 Linear Algebra with Applications (2006/07 Semester 2)
Tutorial 11 Solutions

1. (a) Solving the linear system

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 1 & 1 & \frac{1}{2} & 0 \\ -1 & 0 & -\frac{1}{2} & 0 \end{array}\right) \xrightarrow{\text{Gauss-Jordan Elimination}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So a general solution of the linear system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, t \in \mathbb{R}.$$

Thus the solution space is a line in $\mathbb{R}^3$ that passes through the origin.

(b) An orthonormal basis for the subspace above is

$$\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\}.$$

So for any $\boldsymbol{x} = (x, y, z)^T \in \mathbb{R}^3$, the projection of $\boldsymbol{x}$ onto the subspace is

$$\begin{aligned} P\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) &= \left[(x, y, z)^T \cdot \frac{1}{\sqrt{5}}(1, 0, -2)^T\right] \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{x-2z}{5} \\ 0 \\ \frac{-2x+4z}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 0 \\ -\frac{2}{5} & 0 & \frac{4}{5} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

So P is a linear operator on $\mathbb{R}^3$ The standard matrix for P is $\begin{pmatrix} \frac{1}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 0 \\ -\frac{2}{5} & 0 & \frac{4}{5} \end{pmatrix}$ an the formula is

$$P\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \frac{x-2z}{5} \\ 0 \\ \frac{-2x+4z}{5} \end{pmatrix}.$$

2. (a) $$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \left\{\begin{array}{rcrcrcrcl} & & x & + & y & & & = & 0 \\ w & & & & & + & z & = & 0 \\ w & + & x & + & y & + & z & = & 0 \end{array}\right.$$

The general solution of the linear system is $w = -t$, $x = -s$, $y = s$, $z = t$ where $s, t \in \mathbb{R}$. So $\ker(T) = \{\, (-t, -s, s, t) \mid s, t \in \mathbb{R} \,\}$.

$$\begin{pmatrix} x+y \\ w+z \\ w+z+y+z \end{pmatrix} = w\begin{pmatrix}0\\1\\1\end{pmatrix} + x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}1\\0\\1\end{pmatrix} + z\begin{pmatrix}0\\1\\1\end{pmatrix}$$

So $\mathrm{Im}(T) = \mathrm{span}\{(0,1,1)^T,\ (1,0,1)^T\}$.

(b) Since $\mathrm{Ker}(T)$ is the solution space of $\boldsymbol{Ax} = \mathbf{0}$ and $\mathrm{Im}(T)$ is the column space of $\boldsymbol{A}$, they are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

(c) By the Dimension Theorem for Matrices,

$$\begin{aligned} n &= \mathrm{Rank}(\boldsymbol{A}) + \mathrm{Nullity}(\boldsymbol{A}) \\ &= \dim(\text{the colmun space of } \boldsymbol{A}) + \dim(\text{the nullspace of } \boldsymbol{A}) \\ &= \dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)). \end{aligned}$$

3. (a) $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(150^o) & \sin(150^o) \\ \sin(150^o) & -\cos(150^o) \end{pmatrix}$. So T is the reflection about the line $y = x\tan(75^o)$.

(b) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(30^o) & \sin(30^o) \\ \sin(30^o) & -\cos(30^o) \end{pmatrix}$. So T is the clockwise rotation about the origin through an angle of 30^o.

(c) Since $\boldsymbol{A}$ is symmetric, it can be orthogonally diagonalized. We find that

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

so T is the scaling along axes in the directions of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by factors 3 and 2 respectively.

4. (a) For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$\begin{aligned} P(\boldsymbol{u} + c\boldsymbol{v}) &= (\boldsymbol{u} + c\boldsymbol{v}) - [\boldsymbol{n} \cdot (\boldsymbol{u} + c\boldsymbol{v})]\boldsymbol{n} \\ &= (\boldsymbol{u} + c\boldsymbol{v}) - [(\boldsymbol{n} \cdot \boldsymbol{u}) + c(\boldsymbol{n} \cdot \boldsymbol{v})]\boldsymbol{n} \\ &= \boldsymbol{u} - (\boldsymbol{n} \cdot \boldsymbol{u})\boldsymbol{n} + c[\boldsymbol{v} - (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n}] = P(\boldsymbol{u}) + cP(\boldsymbol{v}) \end{aligned}$$

Thus P is a linear transformation.

(b) Let $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\begin{aligned} P\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \begin{pmatrix} x \\ y \end{pmatrix} - \left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{x}{2} - \frac{y}{2} \\ -\frac{x}{2} + \frac{y}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

So the standard matrix for P is $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.