NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1508 Linear Algebra with Applications (2006/07 Semester 2)

Tutorial 3 Solutions

1. (a) Yes. Consider $\boldsymbol{A} = \boldsymbol{I}_2$, $\boldsymbol{B} = -\boldsymbol{I}_2$.

   (b) Yes. Consider $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

2. If $a = 0$, then the first row of the matrix is a row of zeros, so the matrix is singular. We now consider $a \neq 0$. Denoting the matrix by $\boldsymbol{A}$,

$$\boldsymbol{A} \xrightarrow{R_3 - \frac{d}{a}R_1} \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix} = \boldsymbol{B}.$$

Now if $e = 0$, then the third row of $\boldsymbol{B}$ is a row of zeros. $\boldsymbol{B}$ is then singular, which implies $\boldsymbol{A}$ is also singular. If $e \neq 0$,

$$\boldsymbol{B} \xrightarrow{R_5 - \frac{h}{e}R_3} \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{C}.$$

Now $\boldsymbol{C}$ is singular, which implies that $\boldsymbol{A}$ is singular. Thus, $\boldsymbol{A}$ is singular for all values of a, b, c, d, e, f, g, h.

3. $\boldsymbol{A}(\boldsymbol{x_1} + \boldsymbol{x_2}) = \boldsymbol{A}\boldsymbol{x_1} + \boldsymbol{A}\boldsymbol{x_2} = \boldsymbol{y_1} + \boldsymbol{y_2} = \boldsymbol{w}$. Thus, $\boldsymbol{x_1} + \boldsymbol{x_2}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{w}$, which implies that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{w}$ is consistent.

4. (a) Consider the equation

$$\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} + b \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Solving the linear system:

$$\begin{cases} 2a + 3b - c = 2 \\ a - b = 3 \\ + 5b + 2c = -7 \\ 3a + 2b + c = 3 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -\frac{5}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Since the linear system is consistent, $(2, 3, -7, 3) \in \mathrm{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$.

(b) Performing the same elementary row operations as those in (a) on the vector $(x, y, z, w)^T$, we have

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y - \frac{x}{2} \\ z + 2y - x \\ w - \frac{7y}{3} - \frac{2z}{3} - \frac{x}{3} \end{pmatrix}.$$

So we choose x, y, z, w such that $w - \frac{7y}{3} - \frac{2z}{3} - \frac{x}{3} \neq 0$. For example, we can choose $\boldsymbol{w} = (1, 1, 1, 1)^T$ to be the vector not belonging to $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$.

5. (i) Since V_1 and V_2 are subspaces, $\mathbf{0} \in V_1$ and $\mathbf{0} \in V_2$. Thus, $\mathbf{0} = \mathbf{0} - \mathbf{0}$ belongs to $V_1 - V_2$ and $V_1 - V_2$ is non-empty.

   Let $\boldsymbol{u}, \boldsymbol{w} \in V_1 - V_2$. Then $\boldsymbol{u} = \boldsymbol{v_1} - \boldsymbol{v_2}$ and $\boldsymbol{w} = \boldsymbol{v_3} - \boldsymbol{v_4}$ for some $\boldsymbol{v_1}, \boldsymbol{v_3} \in V_1$ and $\boldsymbol{v_2}, \boldsymbol{v_4} \in V_2$. Now $\boldsymbol{u} + \boldsymbol{w} = (\boldsymbol{v_1} + \boldsymbol{v_3}) - (\boldsymbol{v_2} + \boldsymbol{v_4}) \in V_1 - V_2$ since $(\boldsymbol{v_1} + \boldsymbol{v_3}) \in V_1$ and $(\boldsymbol{v_2} + \boldsymbol{v_4}) \in V_2$. Thus, $V_1 - V_2$ is closed under addition.

   Let $\boldsymbol{u} \in V_1 - V_2$ and $k \in \mathbb{R}$. Then $\boldsymbol{u} = \boldsymbol{v_1} - \boldsymbol{v_2}$ for some $\boldsymbol{v_1} \in V_1$ and $\boldsymbol{v_2} \in V_2$. Now $k\boldsymbol{u} = (k\boldsymbol{v_1} - k\boldsymbol{v_2}) \in V_1 - V_2$ since $k\boldsymbol{v_1} \in V_1$ and $k\boldsymbol{v_2} \in V_2$. Thus, $V_1 - V_2$ is closed under scalar multiplication.

   Since $V_1 - V_2$ is a non-empty subset of $\mathbb{R}^n$ satisfying closure under addition and scalar multiplication, it is a subspace of $\mathbb{R}^n$.

   (ii) (a) We shall show that $\mathbb{R}^2 \subseteq V_1 - V_2$. This would imply $V_1 - V_2 = \mathbb{R}^2$. Let $(x, y)^T \in \mathbb{R}^2$. Then

   $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} - \begin{pmatrix} -x \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ y \end{pmatrix} \in V_1, \begin{pmatrix} -x \\ 0 \end{pmatrix} \in V_2.$$

   So $(x, y)^T \in V_1 - V_2$ and $\mathbb{R}^2 \subseteq V_1 - V_2$. Thus we have $V_1 - V_2 = \mathbb{R}^2$.

   (b) $V_1 = \{(t, t, t) | t \in \mathbb{R}\}$, $V_2 = \{(-s, s, 0) | s \in \mathbb{R}\}$.

   $$\begin{aligned} V_1 - V_2 &= \{(t, t, t) - (-s, s, 0) | s, t \in \mathbb{R}\} \\ &= \{(t, t, t) + (s, -s, 0) | s, t \in \mathbb{R}\} \\ &= \{t(1, 1, 1) + s(1, -1, 0) | s, t \in \mathbb{R}\} \\ &= \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

6. (a) For the series circuit,

   $$v_1 - v_2 = i_1 R_1 \Rightarrow v_2 = v_1 - i_1 R_1 \quad \text{and} \quad i_2 = i_1.$$

   So,

   $$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix},$$

   and the transfer matrix is $\boldsymbol{A_1} = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$. For the shunt circuit,

   $$v_2 = v_3 \quad \text{and} \quad (i_2 - i_3) R_2 = v_2.$$

So,

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix},$$

and the transfer matrix is $\boldsymbol{A_2} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$.

(b) $\boldsymbol{A_2 A_1} = \begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & 1 + \frac{R_1}{R_2} \end{pmatrix}$.

(c) Solving

$$\begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & 1 + \frac{R_1}{R_2} \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ -0.5 & 5 \end{pmatrix},$$

we have $R_1 = 8,\ R_2 = 2$.