NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1508 Linear Algebra with Applications (2006/07 Semester 2)
Tutorial 6 Solutions

1. (a) Performing Gauss-Jordan elimination on $\boldsymbol{A}$, we have the reduced row-echelon form

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So a basis for the row space of $\boldsymbol{A}$ is

$$\{(1,0,1,0,1),(0,1,-2,0,3),(0,0,0,1,-5)\};$$

a basis for the column space of $\boldsymbol{A}$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 7 \\ 5 \end{pmatrix} \right\}.$$

To find a basis for the nullspace of $\boldsymbol{A}$, consider the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ where $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)^T$. A general solution is

$$\begin{cases} x_1 &= -s-t \\ x_2 &= 2s-3t \\ x_3 &= s \\ x_4 &= 5t \\ x_5 &= t, \qquad s,t \in \mathbb{R}. \end{cases}$$

So a basis for the nullspace of $\boldsymbol{A}$ is

$$\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{pmatrix} \right\}.$$

(b) Since the reduced row-echelon form of $\boldsymbol{A}$ has 3 non-zero rows, $\text{rank}(\boldsymbol{A}) = 3$. The nullspace of $\boldsymbol{A}$ is of dimension 2, so $\text{nullity}(\boldsymbol{A}) = 2$. So $\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = 3 + 2 = 5 = \dim(\mathbb{R}^5)$.

(c) Since columns 3 and 5 of $\mathbb{R}$ has no leading entries, we add $(0,0,1,0,0)$ and $(0,0,0,0,1)$ to the basis for the row space of $\boldsymbol{A}$ to obtain a basis for $\mathbb{R}^5$.

2. Let $\boldsymbol{v} \in L$, then $\boldsymbol{v} \in H \cap K$. Since $\boldsymbol{v} \in H$,

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} + b \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \qquad \text{for some } a, b \in \mathbb{R}.$$

Similarly, $\boldsymbol{v} \in K$ implies

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + d \begin{pmatrix} 0 \\ -12 \\ -28 \end{pmatrix} \quad \text{for some } c, d \in \mathbb{R}.$$

So we need to find $a, b, c, d \in \mathbb{R}$ such that

$$\begin{cases} 5a + b - 2c = 0 \\ 3a + 3b + c + 12d = 0 \\ 8a + 4b - 5c + 28d = 0 \end{cases}$$

Performing Gauss-Jordan elimination

$$\left(\begin{array}{cccc|c} 5 & 1 & -2 & 0 & 0 \\ 3 & 3 & 1 & 12 & 0 \\ 8 & 4 & -5 & 28 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{10}{3} & 0 \\ 0 & 1 & 0 & \frac{26}{3} & 0 \\ 0 & 0 & 1 & -4 & 0 \end{array}\right).$$

A general solution is

$$\begin{cases} a = \frac{10t}{3} \\ b = -\frac{26t}{3} \\ c = 4t \\ d = t, \quad t \in \mathbb{R}. \end{cases}$$

When $t = 3$, we have the particular solution $a = 10$, $b = -26$, $c = 12$, $d = 3$ and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 10 \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} - 26 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = 24 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

So $(1, -2, -1)^T \in L$ and a basis for L is $\{(1, -2, -1)^T\}$.

3. (a) False, since $\text{rank}(\boldsymbol{A}) \leq 10$, $\text{nullity}(\boldsymbol{A}) \geq 2$. So the nullspace of $\boldsymbol{A}$ cannot be of dimension 1.

   (b) True, consider the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (c) True, consider the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

4. (a) We shall show that the column space of $\boldsymbol{AB}$ is a subspace of the column space of $\boldsymbol{A}$. If we can do this, then

$$\text{rank}(\boldsymbol{AB}) = \dim(\text{column space of } \boldsymbol{AB}) \leq \dim(\text{column space of } \boldsymbol{A}) = \text{rank}(\boldsymbol{A}).$$

First, note that both the column spaces of $\boldsymbol{AB}$ and $\boldsymbol{A}$ are subspaces of $\mathbb{R}^m$. Consider the matrix $(\boldsymbol{AB})_{m\times p}$.

$$\boldsymbol{AB} = [\boldsymbol{Ab_1} \;\; \boldsymbol{Ab_2} \;\; ... \;\; \boldsymbol{Ab_p}]$$

where $\boldsymbol{b_i}$ is the i-th column of $\boldsymbol{B}$. Note that each $\boldsymbol{Ab_i}$ (that is, each column of $\boldsymbol{Ab}$) is a linear combination of the columns of $\boldsymbol{A}$. Since the column space of $\boldsymbol{AB}$ is the space spanned by the columns of $\boldsymbol{AB}$, each vector $\boldsymbol{x}$ in the column space of $\boldsymbol{AB}$ is also a linear combination of the columns of $\boldsymbol{A}$. Thus, $\boldsymbol{x}$ also belongs to the column space of $\boldsymbol{A}$ and we have (column space of $\boldsymbol{AB}$) $\subseteq$ (column space of $\boldsymbol{A}$), and we are done.

(b) If $\boldsymbol{P}$ is invertible, we can write $\boldsymbol{P} = \boldsymbol{E_n}...\boldsymbol{E_1}$. So $\boldsymbol{PA} = \boldsymbol{E_n}...\boldsymbol{E_1}\boldsymbol{A}$ and $\boldsymbol{A}$ are row equivalent and so they have the same row space. Consequently, they have the same rank.

5. Since $|i-j|$ is odd if and only if $|j-1|$ is odd, the adjacency matrix defined in the question is the same as the matrix S defined in Lecture 13. Considering the (i,i)-entry of the matrix S^3,

$$s_{3,ii} = \sum_{k=1}^{n} s_{2,ik} s_{1,ki}. \qquad (*)$$

Since $s_{1,ki} = 0$ if $|k-i|$ is even, we only need to consider k in $(*)$ where $|k-i|$ is odd. Consider $s_{2,ik}$ where $|k-i|$ is odd.

$$s_{2,ik} = \sum_{r=1}^{n} s_{1,ir} s_{1,rk}.$$

If r is such that $|i-r|$ is odd, then $|r-k|$ is even, implying $s_{1,rk} = 0$. If r is such that $|i-r|$ is even, then $s_{1,ir} = 0$. In either case $s_{1,ir}s_{1,rk} = 0$, so $s_{3,ii} = 0$ for all i. This implies that there are no vertices v_i that belongs to a clique and thus G has no cliques.