NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

**MA1508 Linear Algebra with Applications**

**2006-2007 (Semester 2)** **Tutorial 7**

1. Let $\boldsymbol{v_1} = (2, 1, 0, -1, 0)$, $\boldsymbol{v_2} = (1, -1, 0, 0, 3)$, $\boldsymbol{v_3} = (0, 1, 3, -1, 1)$ and $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$.
   (a) Show that $S$ is a linearly independent set.
   (b) Let $W = \text{span}(S)$. Find $W^\perp$. (See Example 14.6 for definition of $W^\perp$.)
   (c) Find a basis for $W^\perp$ and state the dimension of $W^\perp$.
   (d) What do you observe about $\dim(W) + \dim(W^\perp)$?

2. Let $\boldsymbol{u_1} = (1, 2, 1, 0)$, $\boldsymbol{u_2} = (4, 1, 3, -1)$, $\boldsymbol{u_3} = (2, 3, 1, -1)$ and $V = \text{span}(\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3})$.
   (a) Write down a homogeneous linear system with 2 equations and 4 unknowns such that the solution space is $V$.
   (b) Find a basis for $V$.
   (c) Find an orthonormal basis for $V$.
   (d) Suppose the basis you found in (c) is $S$. Consider $\boldsymbol{w} = (0, 1, 2, 3)$. Determine if $\boldsymbol{w} \in V$. If yes, find $(\boldsymbol{w})_S$ and if not, find the projection of $\boldsymbol{w}$ on $V$.

3. Let $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 & 2 & 4 \\ -2 & 1 & -9 & -11 & -13 \\ 2 & -1 & 3 & 5 & 7 \end{pmatrix}$.
   (a) We have already seen that elementary row operations preserve the row space of a matrix but not the column space. Do elementary row operations preserve the nullspace of a matrix?
   (b) Show that $(-1, 2, -1, 0, 1)^T$, $(0, 2, 0, -1, 1)^T$ and $(-3, 0, -2, 1, 1)^T$ form a basis for the nullspace of $\boldsymbol{A}$.

4. Let $\boldsymbol{w_1} = (1, 1, 1)$, $\boldsymbol{w_2} = (1, 0, 1)$ and $\boldsymbol{w_3} = (1, 2, -1)$.
   (a) Show that $S_1 = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ is a basis for $\mathbb{R}^3$.
   (b) Using $S_1$, find an orthonormal basis $S_1'$ for $\mathbb{R}^3$.
   (c) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Show that $\boldsymbol{A}$ is invertible.
   (d) Show that $S_2 = \{\boldsymbol{A}\boldsymbol{w_1}, \boldsymbol{A}\boldsymbol{w_2}, \boldsymbol{A}\boldsymbol{w_3}\}$ forms a basis for $\mathbb{R}^3$.
   (e) Using $S_2$, find another orthonormal basis $S_2'$ for $\mathbb{R}^3$.
   (f) Find an orthogonal matrix $\boldsymbol{P}$ such that
   $$\boldsymbol{P}[\boldsymbol{w}]_{S_2'} = [\boldsymbol{w}]_{S_1'}$$
   for all $\boldsymbol{w} \in \mathbb{R}^3$.