NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1508 Linear Algebra with Applications
2006-2007 (Semester 2) **Tutorial 8**

1. Let $\boldsymbol{A} = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$.

   (a) Show that 2 is an eigenvalue of $\boldsymbol{A}$.

   (b) Find a basis for the eigenspace associated with 2.

   (c) If $\boldsymbol{B}$ is another 3×3 matrix such that $\lambda = 3$ is an eigenvalue of $\boldsymbol{B}$ and the dimension of the eigenspace associated with $\lambda = 3$ is 2, prove that 5 is an eigenvalue of the matrix $\boldsymbol{A} + \boldsymbol{B}$.

2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be orthogonal matrices of order n. Prove or disprove the following statements.

   (a) $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are also orthogonal matrices.

   (b) $\boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{B} + \boldsymbol{A}$ are also orthogonal matrices.

   (c) $(\boldsymbol{AB})^T$ is a transition matrix from some orthonormal basis of $\mathbb{R}^n$ to some other orthonormal basis of $\mathbb{R}^n$.

3. Let $\boldsymbol{C}$ be a square matrix of order 3.

   (a) Suppose λ is an eigenvalue of $\boldsymbol{C}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{C}$ associated with λ. Show that $k\boldsymbol{x}$ is also an eigenvector of $\boldsymbol{C}$ associated with λ for any $k \in \mathbb{R}$.

   (b) If all eigenvectors of $\boldsymbol{C}$ are multiples of $\boldsymbol{x} = (1, 0, 0)^T$, is it true that $\boldsymbol{C}$ must be singular matrix?

   (c) Is it possible that $\boldsymbol{C}$ has eigenvalues 1, 0, -1 and corresponding eigenvectors $(-1, -1, 0)^T$, $(1, 1, 1)^T$ and $(1, 0, 0)^T$?

   (d) Is it possible that $\boldsymbol{C}$ has eigenvalues 1, 0, -1 and corresponding eigenvectors $(0, 1, 1)^T$, $(1, -1, 1)^T$ and $(1, 0, 0)^T$?

4. A total of n teams took part in a round-robin football tournament where each team plays all other teams once with each match ending with one winner and one loser (no draws). We can, of course, represent the results of the matches in this tournament with a digraph T with n vertices.

   (a) Let $\boldsymbol{A}$ be the adjacency matrix of T. Show that

   $$\sum_{k=1}^{n} \sum_{r=1}^{n} a_{kr} = \frac{n(n-1)}{2}.$$

   (b) T is said to be transitive if u, v and w are vertices in T such that $u \to v$ and $v \to w$, then $u \to w$. Show that T is transitive if and only if $(\boldsymbol{I} - \boldsymbol{A})$ is invertible.

(c) Suppose every team won at least one match. Prove that there are two teams with the same number of points.

(d) Suppose $n = 26$. If each team is awarded 1 point for a win and 0 for a loss, is it possible for **all** teams to end up with the same number of points?