NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1508 Linear Algebra with Applications (2006/07 Semester 2)

Tutorial 8 Solutions

1. (a) $\det(2\boldsymbol{I}-\boldsymbol{A}) = \begin{vmatrix} -2 & 1 & -6 \\ -2 & 1 & -6 \\ -2 & 1 & -6 \end{vmatrix} = 0$ since it has two identical rows. Thus, 2 is an eigenvalue of $\boldsymbol{A}$.

(b)

$$\left(\begin{array}{ccc|c} -2 & 1 & -6 & 0 \\ -2 & 1 & -6 & 0 \\ -2 & 1 & -6 & 0 \end{array}\right) \underset{R_3-R_1}{\overset{R_2-R_1}{\longrightarrow}} \left(\begin{array}{ccc|c} -2 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{-\frac{1}{2}R_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A general solution is

$$\begin{cases} x_1 &= \frac{1}{2}s - 3t \\ x_2 &= s \\ x_3 &= t, \quad s, t \in \mathbb{R}. \end{cases}$$

$$E_2 = \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

So a basis for E_2 is

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(c) Let E_3 be the eigenspace associated with $\lambda = 3$. Note that E_3 is a subspace of $\mathbb{R}^3$ and $\dim(E_3) = 2$. Also E_2 is a subspace of $\mathbb{R}^3$ and $\dim(E_2) = 2$. By Test 1 (version 1) Question 9(c), $E_3 \cap E_2$ contains a non-zero vector $\boldsymbol{x}$. That is

$$\boldsymbol{A}\boldsymbol{x} = 2\boldsymbol{x} \quad \text{and} \quad \boldsymbol{B}\boldsymbol{x} = 3\boldsymbol{x}.$$

This implies

$$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{x} = 2\boldsymbol{x} + 3\boldsymbol{x} = 5\boldsymbol{x}.$$

Thus 5 is an eigenvalue of $(\boldsymbol{A}+\boldsymbol{B})$ and $\boldsymbol{x}$ is an eigenvector associated with 5.

2. (a) True. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal, $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$ and $\boldsymbol{B}^{-1} = \boldsymbol{B}^T$.

$$\begin{aligned} (\boldsymbol{AB})^T(\boldsymbol{AB}) &= (\boldsymbol{B}^T\boldsymbol{A}^T)(\boldsymbol{AB}) \\ &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}\boldsymbol{AB} = \boldsymbol{I} \end{aligned}$$

Therefore $\boldsymbol{AB}$ is orthogonal. Similarly

$$\begin{aligned} (\boldsymbol{BA})^T(\boldsymbol{BA}) &= \boldsymbol{A}^T\boldsymbol{B}^T(\boldsymbol{BA}) \\ &= \boldsymbol{A}^{-1}\boldsymbol{B}^{-1}\boldsymbol{BA} = \boldsymbol{I} \end{aligned}$$

(b) False. Consider $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

(c) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be orthogonal matrices of order n. In this case, $\boldsymbol{A}^T$ is also an orthogonal matrix. Let $S_1 = \{\boldsymbol{a_1}, ..., \boldsymbol{a_n}\}$ be the set of columns of $\boldsymbol{A}^T$. Then S_1 is an orthonormal basis for $\mathbb{R}^n$. Let $S_2 = \{\boldsymbol{e_1}, ..., \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{R}^n$. Note that S_2 is also an orthonormal basis for $\mathbb{R}^n$. Then,

$$\begin{aligned} & [\boldsymbol{a_1}]_{S_2} = \boldsymbol{a_1}, [\boldsymbol{a_2}]_{S_2} = \boldsymbol{a_2}, ..., [\boldsymbol{a_n}]_{S_2} = \boldsymbol{a_n} \\ \Rightarrow \quad & \boldsymbol{P}_{S_1 \to S_2} = \boldsymbol{A}^T \text{ is the transition matrix from } S_1 \text{ to } S_2 \end{aligned}$$

Likewise, $\boldsymbol{B}$ is an orthogonal matrix, let $S_3 = \{\boldsymbol{b_1}, ..., \boldsymbol{b_n}\}$ where $\boldsymbol{b_i}$ is the i-th column of $\boldsymbol{B}$. So S_3 is an orthonormal basis for $\mathbb{R}^n$. Similar to above,

$$\begin{aligned} & \boldsymbol{P}_{S_3 \to S_2} = \boldsymbol{B} \text{ is the transition matrix from } S_3 \text{ to } S_2 \\ \Rightarrow \quad & \boldsymbol{P}_{S_2 \to S_3} = \boldsymbol{B}^T \text{ is the transition matrix from } S_2 \text{ to } S_3 \end{aligned}$$

So

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T = \boldsymbol{P}_{S_2 \to S_3} \boldsymbol{P}_{S_1 \to S_2}$$

and for any $\boldsymbol{w} \in \mathbb{R}^n$,

$$\begin{aligned} (\boldsymbol{AB})^T [\boldsymbol{w}]_{S_1} &= \boldsymbol{P}_{S_2 \to S_3} \boldsymbol{P}_{S_1 \to S_2} [\boldsymbol{w}]_{S_1} \\ &= \boldsymbol{P}_{S_2 \to S_3} [\boldsymbol{w}]_{S_2} \\ &= [\boldsymbol{w}]_{S_3} \end{aligned}$$

So $(\boldsymbol{AB})^T$ is a transition matrix from S_1 (columns of $\boldsymbol{A}^T$) to S_3 (rows of $\boldsymbol{B}^T$ = columns of $\boldsymbol{B}$).

3. (a) $\boldsymbol{Cx} = \lambda \boldsymbol{x} \Rightarrow \boldsymbol{C}(k\boldsymbol{x}) = k(\boldsymbol{Cx}) = k(\lambda \boldsymbol{x}) = \lambda(k\boldsymbol{x})$. Therefore, $k\boldsymbol{x}$ is also an eigenvector of $\boldsymbol{C}$ associated with λ.

(b) No. $\boldsymbol{C}$ does not have to be singular. To satisfy the conditions that all eigenvectors of $\boldsymbol{C}$ are multiples of $\boldsymbol{x} = (1, 0, 0)^T$, we just need to find a $\boldsymbol{C}$ such that

(i) $\boldsymbol{C}$ has only one eigenvalue λ;

(ii) $\lambda \neq 0$;

(iii) $E_\lambda = \text{span}\{(1, 0, 0)^T\}$. This is equivalent to

* nullspace of $(\lambda \boldsymbol{I} - \boldsymbol{C}) = \text{span}\{(1, 0, 0)^T\}$;
* $\text{rank}(\lambda \boldsymbol{I} - \boldsymbol{C}) = 2$.

Since $(1, 0, 0)^T$ is an eigenvector associated with λ,

$$\boldsymbol{C} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}.$$

So the first column of $\boldsymbol{C}$ is $\begin{pmatrix}\lambda\\0\\0\end{pmatrix}$ and

$$\lambda\boldsymbol{I}-\boldsymbol{C}=\begin{pmatrix}0 & -c_{12} & -c_{13}\\ 0 & \lambda-c_{22} & -c_{23}\\ 0 & -c_{32} & \lambda-c_{33}\end{pmatrix}.$$

We choose $\lambda=1$ and values of c_{12}, c_{13}, c_{22}, c_{23}, c_{32}, c_{33} such that

(i') $\boldsymbol{C}$ has only $\lambda=1$ as its only eigenvalue;

(ii') columns 2 and 3 of $(\lambda\boldsymbol{I}-\boldsymbol{C})$ are linearly independent.

For example,

$$\boldsymbol{C}=\begin{pmatrix}1 & 1 & 0\\ 0 & 3 & -2\\ 0 & 2 & -1\end{pmatrix},\qquad \lambda\boldsymbol{I}-\boldsymbol{C}=\begin{pmatrix}\lambda-1 & -1 & 0\\ 0 & \lambda-3 & 2\\ 0 & -2 & \lambda+1\end{pmatrix}.$$

In this case,

$$\begin{aligned}\det(\lambda\boldsymbol{I}-\boldsymbol{C}) &= (\lambda\boldsymbol{I}-\boldsymbol{C})\begin{vmatrix}\lambda-3 & 2\\ -2 & \lambda+1\end{vmatrix}\\ &= (\lambda-1)^3.\end{aligned}$$

So $\boldsymbol{C}$ has only $\lambda=1$ as its only eigenvalue. It can now be easily verified that $E_1=\mathrm{span}\{(1,0,0)^T\}$ and $\det(\boldsymbol{C})\neq 0$ as required.

(c) Let $\boldsymbol{C}=[\boldsymbol{c_1}\ \ \boldsymbol{c_2}\ \ \boldsymbol{c_3}]$ where $\boldsymbol{c_i}$ is the i-th column of $\boldsymbol{C}$.

$$\begin{aligned}\boldsymbol{C}\begin{pmatrix}-1\\-1\\0\end{pmatrix} &= 1\begin{pmatrix}-1\\-1\\0\end{pmatrix}\Rightarrow -\boldsymbol{c_1}-\boldsymbol{c_2}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}\Rightarrow \boldsymbol{c_1}+\boldsymbol{c_2}=\begin{pmatrix}1\\1\\0\end{pmatrix}\\ \boldsymbol{C}\begin{pmatrix}1\\1\\0\end{pmatrix} &= 0\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow \boldsymbol{c_1}+\boldsymbol{c_2}+\boldsymbol{c_3}=\begin{pmatrix}0\\0\\0\end{pmatrix}\\ \boldsymbol{C}\begin{pmatrix}1\\0\\0\end{pmatrix} &= -1\begin{pmatrix}1\\0\\0\end{pmatrix}\Rightarrow \boldsymbol{c_1}=\begin{pmatrix}-1\\0\\0\end{pmatrix}\end{aligned}$$

Solving, we have

$$\boldsymbol{c_1}=\begin{pmatrix}-1\\0\\0\end{pmatrix}\quad \boldsymbol{c_2}=\begin{pmatrix}2\\1\\0\end{pmatrix}\quad \boldsymbol{c_3}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}.$$

So

$$\boldsymbol{C}=\begin{pmatrix}-1 & 2 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}.$$

(d) Using the same method as (c), we obtain

$$\boldsymbol{C} = \begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

4. (a) Since T is a tournament, in the adjacency matrix of $\boldsymbol{A}$,

$$a_{ij} = 1 \Leftrightarrow a_{ji} = 0 \text{ for all } i, j = 1, ..., n, i \neq j.$$

So there are precisely $\frac{1}{2}(n^2 - n)$ ones in $\boldsymbol{A}$. This implis

$$\sum_{k=1}^{n} \sum_{r=1} a_{kr} = \frac{n(n-1)}{2}.$$

(b) We claim that T is transitive if and only if T contains no cycles.

($\Rightarrow$) Suppose T contains at least one cycle. Let $C = v_0 v_1 ... v_k v_0$ be a smallest cycle in T. Since C is a smallest cycle, we must have $v_0 \to v_2$, $v_0 \to v_3$,...,$v_0 \to v_{k-1}$ in T. But now $v_0 \to v_{k-1}$, $v_{k-1} \to v_k$ and $v_k \to v_0$ in T implies that T is not transitive.

($\Leftarrow$) If T has no cycles, whenever $u \to v$ and $v \to w$ in T implies $u \to w$. So T is transitive.

Hence T is transitive $\Leftrightarrow$ T has no cycles $\Leftrightarrow$ $(\boldsymbol{I} - \boldsymbol{A})$ is invertible (by result in Lecture 13).

(c) For any team i, the number of wins it has is equal to the outdegree of vertex i in T. Since every vertex has outdegree ≥ 1 and $\leq n-1$, the outdegrees are integers from the set $\{1, ..., n-1\}$. Since there are $n-1$ integers in the set and n teams in total, there exists two teams with exactly the same outdegree.

(d) No it is not possible. Suppose that such an occurrence is possible, let

$$\text{outdegree of vertex } i = \sum_{k=1}^{n} a_{ik} = r \quad \text{for some integers } r \text{ and all } i = 1, ..., n$$

So by (a),

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} = nr = \frac{n(n-1)}{2}$$

Substitute $n = 26$ into the expression above, we have

$$26r = 13(25) \Rightarrow r = 12.5$$

which is a contradiction as r is an integer.