NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA1508 Linear Algebra with Applications

2006-2007 (Semester 2) **Tutorial 9**

1. (a) Find a 3×3 non-zero matrix $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{0}$.
   (b) A square matrix is said to be *nilpotent* if $\boldsymbol{A}^k = \boldsymbol{0}$ for some positive integer k.
      (i) Show that a nilpotent matrix must be singular.
      (ii) Show that **all** eigenvalues of a nilpotent matrix are equal to zero.

2. For each of the following matrices $\boldsymbol{A}$, determine if it is diagonalizable. If it is, find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

$$\text{(a)} \quad \boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(b)} \quad \boldsymbol{A} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

3. Let $\boldsymbol{B} = \begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$. Find an orthogonal matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^T$.

4. Find (not by plotting a graph!)
   (a) The point on the line $y = -2x$ closest to $(4, -1)$.
   (b) The point of the plane $x + y + 2z = 0$ closest to $(1, 1, 1)$.
   (c) The point on the line $y = -2x + 1$ closest to $(4, -1)$.

5. Let $\boldsymbol{A}$ be a diagonalizable matrix and $\boldsymbol{P}$ be a matrix that diagonalizes $\boldsymbol{A}$. Show that the column vectors of $\boldsymbol{P}$ that correspond to the non-zero eigenvalues of $\boldsymbol{A}$ form a basis for the column space of $\boldsymbol{A}$.

6. We have seen (in Lecture 16) that if λ is an eigenvalue for $\boldsymbol{A}$, then it is also an eigenvalue of $\boldsymbol{A}^T$.
   (a) Is it true that if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with λ, then it is also an eigenvector of $\boldsymbol{A}^T$ associated with λ?
   (b) Prove that if $\boldsymbol{A}$ is diagonalizable, than $\boldsymbol{A}^T$ is also diagonalizable.