NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics
MA1508 Linear Algebra with Applications (2006/07 Semester 2)
Tutorial 9 Solutions

1. (a) There are many such $\boldsymbol{A}$, for example $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   (b) (i) If $\boldsymbol{A}$ is invertible, then $\det(\boldsymbol{A}) \neq 0$ which implies $\det(\boldsymbol{A}^k) \neq 0$ for all k. This contradicts the fact that $\boldsymbol{A}$ is nilpotent. Hence $\boldsymbol{A}$ must be singular.

   (ii) Recall that λ is an eigenvalue of $\boldsymbol{A}$ if and only if λ^r is an eigenvalue of $\boldsymbol{A}^r$ for all positive integers r. Since $\boldsymbol{A}^k$ is the zero matrix, all the eigenvalues of $\boldsymbol{A}^k$ are zero, which implies that all the eigenvalues of $\boldsymbol{A}$ are zero (because if $\boldsymbol{A}$ has a non-zero eigenvalue, then $\boldsymbol{A}^k$ will have a non-zero eigenvalue).

2. (a) $\boldsymbol{A}$ has two eigenvalues 1 and -1.
   Considering E_1:

$$\begin{pmatrix} 0 & -2 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, t \in \mathbb{R}.$$

   Considering E_{-1}:

$$\begin{pmatrix} -2 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, t \in \mathbb{R}.$$

   Since $\boldsymbol{A}$ has only two linearly independent eigenvectors, it is not diagonalizable.

   (b)

$$\begin{vmatrix} \lambda - 2 & -2 & -2 \\ -2 & \lambda - 2 & -2 \\ -2 & -2 & \lambda - 2 \end{vmatrix} = \lambda^3 - 6\lambda^2 = \lambda^2(\lambda - 6).$$

   So 0 and 6 are the eigenvalues of $\boldsymbol{A}$.
   Considering E_0:

$$\begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, s, t \in \mathbb{R}.$$

   Considering E_6:

$$\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, s \in \mathbb{R}.$$

$\boldsymbol{A}$ is diagonalizable since it has three linearly independent eigenvectors. Let $\boldsymbol{P} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Then

$$\boldsymbol{A} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}.$$

3.

$$\begin{vmatrix} \lambda - 8 & 2 & -2 \\ 2 & \lambda - 5 & -4 \\ -2 & -4 & \lambda - 5 \end{vmatrix} = \lambda(\lambda - 9)^2.$$

So 0 and 9 are the eigenvalues of $\boldsymbol{B}$. Consider E_0:

$$\begin{pmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix}.$$

So,

$$\left\{ \frac{1}{3} \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} \right\}$$

is an orthonormal basis for E_0.

Consider E_9:

$$\begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, s, t \in \mathbb{R}.$$

So $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for E_9. To obtain an orthogonal basis for E_9, we apply Gram-Schmidt process.

$$\begin{aligned} \boldsymbol{w_1} &= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \boldsymbol{w_2} &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}{|| \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} ||^2} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix} \end{aligned}$$

So

$$\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{3\sqrt{5}} \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \right\}$$

is an orthonormal basis for E_9.

Thus we have

$$\boldsymbol{B} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix}^T.$$

4. (a) The line $y = -2x$ can be represented as

$$V = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

Then

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \right\}$$

is an orthonormal basis for V. The point on the line closest to $(4, -1)$ can be obtained be projecting $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ onto V.

$$\left[\begin{pmatrix} 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \right] \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix}.$$

Thus, the point on the line $y = -2x$ closest to $(4, -1)$ is $(\frac{6}{5}, -\frac{12}{5})$.

(b) Similar to (a), we first need to find an orthonormal basis for $V = \{(x, y, z) | x + y + 2z = 0\}$. A basis for V is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Applying Gram-Schmidt process, we find that

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

is an orthogonal basis for V. Normalizing,

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

is an orthonormal basis for V. Computing the projection of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ onto V, we have

$$\left[\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}+\left[\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{1}{3}\end{pmatrix}.$$

So the point on V closest to $(1,1,1)$ is $(1/3,1/3,-1/3)$.

(c) Since $y=-2x+1$ is not a subspace, we cannot apply the method in (a) directly. If we move the line $y=-2x+1$ and the point $(4,-1)$ down by 1 in the y-direction, the resultants are the line $y=-2x$ and the point $(4,-2)$. By the method in (a), $(8/5,-16/5)$ is the point on $y=-2x$ that is closest to $(4,-2)$, so $(8/5,-11/5)$ is the point on $y=-2x+1$ that is closest to $(4,-1)$.

5. Let $\boldsymbol{P}=[\boldsymbol{p_1}\ \boldsymbol{p_2}\ \ldots\ \boldsymbol{p_n}]$ where $\boldsymbol{p_i}$ is the i-th column of $\boldsymbol{P}$. Let λ_1, λ_2,..., λ_k be the non-zero eigenvalues of $\boldsymbol{A}$ while $\lambda_{k+1}=\ldots=\lambda_n=0$ are the remaining eigenvalues of $\boldsymbol{A}$. Since $\boldsymbol{P}$ diagonalizes $\boldsymbol{A}$, we know that each $\boldsymbol{p_i}$ is an eigenvector of $\boldsymbol{A}$ corresponding to λ_i. That is,

$$\boldsymbol{Ap_i}=\lambda_i\boldsymbol{p_i}\quad i=1,\ldots,k;$$
$$\boldsymbol{Ap_i}=\boldsymbol{0}\quad i=k+1,\ldots,n.$$

Since $\boldsymbol{A}$ is diagonalizable, we know that $\boldsymbol{P}$ is invertible and $\{\boldsymbol{p_1},\ldots,\boldsymbol{p_n}\}$ forms a basis for $\mathbb{R}^n$. So for any $\boldsymbol{x}\in\mathbb{R}^n$,

$$\begin{aligned}&\boldsymbol{x}=c_1\boldsymbol{p_1}+c_2\boldsymbol{p_2}+\ldots+c_n\boldsymbol{p_n}\quad c_i\in\mathbb{R}\\ \Rightarrow\ &\boldsymbol{Ax}=c_1(\boldsymbol{Ap_1})+c_2(\boldsymbol{Ap_2})+\ldots+c_n(\boldsymbol{Ap_n})\\ \Rightarrow\ &\boldsymbol{Ax}=(c_1\lambda_1)\boldsymbol{p_1}+(c_2\lambda_2)\boldsymbol{p_2}+\ldots+(c_k\lambda_k)\boldsymbol{p_k}\end{aligned}$$

This implies that every vector in the column space of $\boldsymbol{A}$, (given by $\boldsymbol{Ax}$) is a linear combination of $\boldsymbol{p_1}$,...,$\boldsymbol{p_k}$. So $\{\boldsymbol{p_1},\ldots,\boldsymbol{p_k}\}$ spans the column space of $\boldsymbol{A}$. On the other hand $\{\boldsymbol{p_1},\ldots,\boldsymbol{p_k}\}$ is a linearly independent set, so $\{\boldsymbol{p_1},\ldots,\boldsymbol{p_k}\}$ forms a basis for the column space of $\boldsymbol{A}$.

6. (a) No. Consider $\boldsymbol{A}=\begin{pmatrix}1&1\\0&1\end{pmatrix}$. In this case, all eigenvectors of $\boldsymbol{A}$ are of the form $k\begin{pmatrix}1\\0\end{pmatrix}$ while all eigenvectors of $\boldsymbol{A}^T$ are of the form $k\begin{pmatrix}0\\1\end{pmatrix}$.

(b) If $\boldsymbol{A}$ is diagonalizable, there exists an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A}=\boldsymbol{PDP}^{-1}\quad\text{where }\boldsymbol{D}\text{ is a diagonal matrix.}$$

Now this implies

$$\begin{aligned}\boldsymbol{A}^T&=(\boldsymbol{PDP}^{-1})^T\\&=(\boldsymbol{P}^{-1})^T\boldsymbol{D}^T\boldsymbol{P}^T\\&=(\boldsymbol{P}^{-1})^T\boldsymbol{DP}^T\\&=(\boldsymbol{P}^T)^{-1}\boldsymbol{DP}^T\end{aligned}$$

Thus $\boldsymbol{A}^T$ is diagonalizable and a matrix that diagonalizes $\boldsymbol{A}^T$ is $(\boldsymbol{P}^T)^{-1}$.