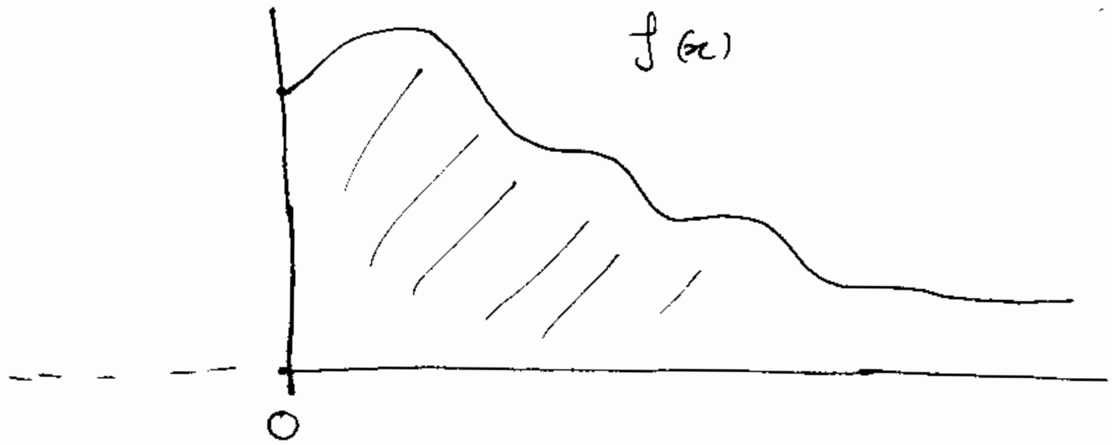
Lecture 11.
Improper Integrals.

The concept of Riemann integral of a function presupposes that the function is defined on a bounded subset of $\mathbb{R}$ and usually a connected one at that and the function is necessarily bounded on this bounded and connected subset. Hence this domain is a bounded interval. Since the function is bounded by arbitrary assigning values at the boundary or end points of the connected interval, we may always assume that the domain of the function is a closed and bounded interval $[a, b]$. Lebesgue's theorem then asserts that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous except for a set of measure zero in $[a, b]$. Hence we have two obvious situation when a function is not Riemann integrable. The obvious one is when f is discontinuous on a set of positive measure even though it is bounded. The other case is when either f is not bounded or the domain is not a bounded subset of $\mathbb{R}$. We shall be concern with this later case. Obviously this presents infinite possibilities and situations. We shall describle the more manageable situations.

We shall first of all consider unbounded domains. We shall adopt the usual requirement that the unbounded domain be connected.



For instance, if $f : [0, \infty) \longrightarrow \mathbb{R}$. is such that $f : [0, b] \longrightarrow \mathbb{R}$ is Riemann integrable on $[0, b]$, then Cauchy's definition of the improper integral is defined to be

$$\int_0^\infty f = \lim_{b \to \infty} \int_0^b f(t)\,dt .$$

This definition is equivalent to the Lebesgue integral $L\int_{[0,\infty)} f$ if f is non-negative.* We then say it is (improperly) integrable if the limit $\lim_{b \to \infty} \int_0^b f$ is finite.

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This then raises the question that if $f : [0, \infty) \to \mathbb{R}$ is such that $f : [0, b] \to \mathbb{R}$ is continuous except on a set of measure zero in $[0, b]$, hence $f : [0, \infty) \to \mathbb{R}$ is continuous on a set of measure zero, then it is measurable on $[0, \infty)$.

Similarly for non-negative function $f : (-\infty, 0] \to \mathbb{R}$ we can define the improper integral as:

$$\int_{-\infty}^{0} f = \lim_{b \to -\infty} \int_{b}^{0} f$$

Note that we have just looked at the case where the domain is unbounded at one end and bounded at the other and that the function f is non-negative. ~~It can~~ We can similarly define improper integrals on domain unbounded on one side but the function is non-positive.

For non-negative function f defined on all $\mathbb{R}$, the only connected unbounded (on both sides) interval we can define the improper integral

$$\int_{-\infty}^{\infty} f = \lim_{t \to \infty} \int_{-t}^{t} f$$

if $f : [-t, t] \to \mathbb{R}$ is Riemann integrable on $[-t, t]$ on each t.

Note that if $f : [0, \infty) \longrightarrow \mathbb{R}$ is such that $f : [0, b] \longrightarrow \mathbb{R}$ is Riemann integrable, then f is continuous on $[0, b]$ except for a subset E_b of measure zero in $[0, b]$. Therefore, ~~the~~ since $[0, \infty) = \bigcup_{n=1}^{\infty} [0, n]$, and so f is continuous everywhere except on the subset $E = \bigcup_{n=0}^{\infty} E_n$. Since each E_n is of measure zero, E is also of measure zero since ~~the E~~ the measure of E $\mu(E) \le \sum \mu(E_n) = 0$.

We would like to use convergence theorems in Lebesgue integration theory for our improper Riemann integrals. The notion of Lebesgue integrability is actually a notion about absolute integrability, meaning f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable. For instance for $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ and $f(0) = 1$, the improper Riemann integral exists, i.e. $\int_0^{\infty} f(x)\,dx$ exists, and equals $\frac{\pi}{2}$ (Reference see for e.g. Watson Fulks Theorem 17.3f p 612, Advanced Calculus) But $\int_0^{\infty} |f(x)|\,dx$ does not exist. Why?

Note that $\int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \geq \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\,dx = \frac{2}{k\pi}$

for each k. and so

$$\int_0^{n\pi} |f(x)|\,dx = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \geq \sum_{n=1}^{n} \frac{2}{k\pi}$$

Therefore, $\int_0^{\infty} |f(x)|\,dx$ does not exist.

Therefore, f is not Lebesgue integrable over $[0, \infty)$.
(Because if it did, then $|f|$ would be Lebesgue integrable over $[0, \infty)$. but we just show that it is not.)

<u>Definition</u> 1. Suppose $f : [a, \infty) \longrightarrow \mathbb{R}$ is such that f is Riemann integrable on every closed sub-interval of $[a, \infty)$. Then we define the <u>improper Riemann integral</u> to be

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

if the limit on the right exists (in $\mathbb{R}$).
Similarly, if $f : (-\infty, a] \longrightarrow \mathbb{R}$ satisfies that f is Riemann integrable on every closed sub-interval of $(-\infty, a]$, the <u>improper Riemann integrable</u> is defined to be

$$\int_{-\infty}^a f(x)\,dx = \lim_{r \to -\infty} \int_r^a f(x)\,dx$$

if the limit on the right exists.

Obviously if $\int_a^\infty f(x)\,dx$ exists, then $\int_r^\infty f(x)\,dx$ exists for any $r \geq a$.

Note that the previous examples shows that in general the existence of
$$\int_a^\infty f(x)\,dx$$
does not imply the existence of $\int_a^\infty |f(x)|\,dx$.

We also say that the integral $\int_a^\infty f(x)\,dx$ converges to mean that the limit exists.
And if $\int_a^\infty |f(x)|\,dx$ exists, then we say the

$\int_a^\infty f(x)\,dx$ is absolutely convergent.
Thus absolutely convergence of $\int_a^\infty f(x)\,dx$ amounts to the function f being Lebesgue integrable on $[a,\infty)$.

<u>Theorem 2</u>: Assume that $f : [a,\infty) \longrightarrow \mathbb{R}$ is Riemann integrable on every closed subinterval of $[a,\infty)$. Then $\int_a^\infty f(x)\,dx$ exists if and only if for every $\varepsilon > 0$, $\exists$ $N > 0$ (depending on ε) such that
$$\left| \int_s^t f(x)\,dx \right| < \varepsilon$$
for all $s, t \geq N$.

<u>Pf</u>: Suppose $I = \int_a^\infty f(x)\,dx$ exists.
Choose N s.t $\left| I - \int_a^r f(x)\,dx \right| < \varepsilon/2 \;\forall\, r > N$.
Then. $\left| \int_s^t f(x)\,dx \right| = \left| \int_a^t f(x)\,dx - \int_a^s f(x)\,dx \right|$
$$\leq \left| I - \int_a^t f(x)\,dx \right| + \left| I - \int_a^s f(x)\,dx \right|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{if} \quad t, s > N.$$

Conversely assume the condition is satisfied.
Let (a_n) be a sequence in $[a, \infty)$ such that $a_n \to \infty$. Then, $\exists$ M such that $n \geq M$
$\Rightarrow a_n > N$.
Therefore, the sequence $(\int_a^{a_n} f(x)\,dx)$ satisfies $n, m \geq M$

$$\Rightarrow \left| \int_a^{a_n} f(x)\,dx - \int_a^{a_m} f(x)\,dx \right|$$

$$= \left| \int_{a_n}^{a_m} f(x)\,dx \right| < \varepsilon \text{ by assumption.}$$

Then $\int_a^{a_n} f(x)\,dx \longrightarrow A$.
We shall show that $A = \int_a^\infty f(x)\,dx$.
So we shall show that for any other sequence (b_n) in $[a, \infty)$ with $b_n \to \infty$, $\int_a^{b_n} f(x)\,dx \to A$.
Let $\int_a^{b_n} f(x)\,dx \longrightarrow B$.

$$\text{Then } |A - B| \leq \left| A - \int_a^{a_n} f(x)\,dx \right| + \left| \int_{a_n}^{b_n} f(x)\,dx \right| + \left| \int_a^{b_n} f(x)\,dx - B \right|.$$

If we choose $n > N_1$ so that $\left|A - \int_a^{a_n} f(x)\,dx\right| < \varepsilon/3$
and $n > N_2$ so that $\left|\int_{a_n}^{b_n} f(x)\,dx\right| < \varepsilon/3$
(choose n so that $n > N_2 \Rightarrow a_n, b_n > N$
where $s, t > N \Rightarrow \left|\int_s^t f(x)\,dx\right| < \varepsilon/3$).
and $n > N_3$ so that $\left|\int_a^{b_n} f(x)\,dx - B\right| < \varepsilon/3$.
ie. choose $n > \max(N_1, N_2, N_3)$, then $|A - B| < \varepsilon$.
Since ε is arbitrary, $A = B$.

Define $f^+(x) = \frac{1}{2}\left(|f(x)| + f(x)\right) \geq 0$

and $f^-(x) = \frac{1}{2}\left(|f(x)| - f(x)\right) \geq 0$.

Then $f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0. \end{cases}$

& $f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{if } f(x) > 0 \end{cases}$

In particularly, $f(x) = f^+(x) - f^-(x)$.

In view of the fact that $|f|$ is measurable does not imply f is measurable. This of course, requires the existence of non-measurable subsets of $\mathbb{R}$. We can take $f(x) = \begin{cases} 1 & x \in E \\ -1 & x \notin E \end{cases}$ where E is a non-measurable subset. Obviously $|f|$ is measurable but f is not. [The existence will require the Axiom of Choice.].

Thus we have the following

<u>Theorem 2</u>: If $f : A \to \mathbb{R}$ is measurable, then f is Lebesgue integrable on A if and only if $|f|$ is Lebesgue integrable on A.

<u>Proof</u>: If f is measurable, then $|f|$, f^+ & f^- are measurable. Suppose f is Lebesgue integrable. Then there are simple functions u and l such that $l \leq f \leq u$

$\int_A u - \int_A l < \varepsilon$. Then $u^+ - l^+ \leq u - l$

and so $\int_A u^+ - \int_A l^+ < \varepsilon$. Since $l^+ \leq f^+ \leq u^+$

f^+ is Lebesgue integrable over A.

Note that $u = u^+ - u^-$ and $l = l^+ - l^-$.

Therefore $u - l = u^+ - u^- - (l^+ - l^-)$

$$= u^+ - l^+ + (l^- - u^-).$$

Note that $l \leq f \leq u$

$\Longrightarrow$ $l^+ \leq f^+ \leq u^+ \Longrightarrow u^+ - l^+ \geq 0$

Also $-l \geq -f \geq -u$

so that $(-l)^+ \geq (-f)^+ \geq (-u)^+$

i.e. $l^- \geq f^- \geq u^-$

and so $l^- - u^- \geq 0$.

Thus, $u^+ - l^+ \leq u - l$

and $l^- - u^- \leq u - l$.

$$\therefore \quad \int_A u^+ - \int_A l^+ \leq \int_A u - \int_A l < \varepsilon$$

Since u^+ & l^+ are also simple functions, and $l^+ \leq f^+ \leq u^+$, f^+ is Lebesgue integrable. Similarly $l^- \geq f^- \geq u^-$ and $\int_A l^- - \int_A u^- < \varepsilon \Rightarrow f^-$ is Lebesgue integrable. Therefore $|f| = f^+ + f^-$ is Lebesgue integrable.

Conversely if $|f|$ is Lebesgue integrable, since f^+ is measurable and $f^+ \leq |f|$.

Therefore f^+ is dominated by an integrable function in A and so f^+ is Lebesgue integrable.

[f^+ is measurable $\Rightarrow$ $\exists$ a monotone increasing sequence of non-negative simple functions ($\therefore$ integrable function) f_n, s.t $f_n \to f$. pointwise. Therefore, by Lebesgue convergence Theorem f^+ is Lebesgue integrable.]

Similarly since $f^- \leq |f|$, f^- is also Lebesgue integrable. Hence $f = f^+ - f^-$ is Lebesgue integrable. In particular

$$\int_A f = \int_A f^+ - \int_A f^- .$$

In view of Theorem 2, we have the following Definition.

Definition 3. Suppose $f : A \longrightarrow \mathbb{R}$ is measurable. Then f is Lebesgue integrable on A if $\int_A |f|$ is finite.

Or equivalently the non-negative functions $f^+ + f^-$

$$\int_A f^+ , \quad \int_A f^-$$

is finite and

$$\int_A f = \int_A f^+ - \int_A f^-$$

We now bring in improper Riemann integral.

Theorem 4.

Suppose $f : [a, \infty) \longrightarrow \mathbb{R}^+$ is a non-negative function such that f is Riemann integrable on every closed subinterval of $[a, \infty)$. Then f is measurable. f is Lebesgue integrable ~~and the Lebesgue integral~~ $\cancel{\int_{[a,\infty)} f =}$ if and only if the improper ~~integral~~ Riemann integral

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx \quad \text{exists}$$

and the Lebesgue integral $\int_{[a,\infty)} f = \int_a^\infty f(x)\,dx$.

Proof. That f is Riemann integrable on every closed subinterval of $[a, \infty)$ implies that f is measurable. For each integer $n \geq 1$ define $f_n : [a, \infty) \to \mathbb{R}^+$ by $f_n(x) = f(x)$ for $a \leq x \leq a+n$ and $f_n(x) = 0$ for $x > a+n$. Then $\{f_n\}$ is a monotone sequence converging pointwise to f. Each f_n is Riemann integrable and so is Lebesgue integrable and the Lebesgue integral $\int_{[a,\infty)} f_n = \int_a^{a+n} f(x)dx$ (the Riemann integral). By the Lebesgue monotone convergence theorem

$$\int_{[a,\infty)} f_n = \int_a^{a+n} f(x)\,dx \longrightarrow \int_{[a,\infty)} f$$

Thus $\lim_{n \to \infty} \int_a^{a+n} f(x)\,dx = \int_{[a,\infty)} f$.

Thus if the limit exists, f is Lebesgue integrable on $[a, \infty)$ and $L\int_{[a,\infty)} f = \int_a^\infty f(x)dx$.

Similarly, we have.

<u>Theorem 5</u>. Suppose $f : (-\infty, a] \longrightarrow \mathbb{R}^+$ is a non-negative function such that f is Riemann integrable on every closed subinterval of $(-\infty, a]$, ~~at~~ Then f is Lebesgue integrable if and only the improper Riemann integral

$$\int_{-\infty}^a f(x)\,dx = \lim_{t \to -\infty} \int_t^a f(x)dx$$

exists. and when it does the Lebesgue integral

$$L\int_{(-\infty,a]} f = \int_{-\infty}^a f(x)\,dx$$

<u>Theorem 6</u>. Suppose $f : (-\infty, \infty) \to \mathbb{R}^+$ is a non-negative function such that f is Riemann integrable on every closed subinterval in $(-\infty, \infty)$. Then f is Lebesgue integrable if and only if the improper Riemann integral $\int_{-\infty}^\infty f(x)dx$

$$= \lim_{t \to \infty} \int_{-t}^t f(x)dx$$

exists and moreover

$$L\int_{(-\infty,\infty)} f = \int_{-\infty}^\infty f(x)dx.$$

Finally

<u>Theorem 7</u>. Suppose $f : [a, \infty) \longrightarrow \mathbb{R}$ is Riemann integrable on every closed subinterval of $[a, \infty)$. Then f is Lebesgue integrable if and only if the improper Riemann integral ~~exists~~ $\int_a^\infty |f(x)|dx$ exists. Moreover

$$\int_{[a,\infty)} f = \text{the improper Riemann}$$

integral $\int_a^\infty f(x)\, dx$.

<u>Proof</u>. By Theorem 2 f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable.

$|f|$ is Lebesgue integrable $\Rightarrow$ f^+ and f^- are Lebesgue integrable on $[a, \infty)$ since $f^-, f^+ \leq |f|$. Therefore, by Theorem 4, the Lebesgue integral

$$L\int_{[a,\infty)} f^+ = \text{Improper Riemann integral} \int_a^\infty f^+(x)dx$$

Similarly, $L\int_{[a,\infty)} f^- = \int_a^\infty f^-(x)\, dx$.

Since $f = f^+ - f^-$ and $|f| = f^+ + f^-$ and so the improper Riemann integral $\int_a^\infty |f(x)|dx$ $= \int_a^\infty f^+(x)dx + \int_a^\infty f^-(x)dx$ exists and the Lebesgue integral $L\int_{[a,\infty)} f = L\int_{[a,\infty)} f^+ - L\int_{[a,\infty)} f^- = \int_a^\infty f^+(x)dx - \int_a^\infty f^-(x)dx$

$$= \int_a^\infty f(x)\, dx$$

Conversely suppose the improper Riemann integral $\int_a^\infty |f(x)|\,dx$ exists. Then by Theorem 4, $|f|$ is Lebesgue integrable. Since f is measurable, f is Lebesgue integrable by Theorem 2.

If the domain is unbounded on the left we then have

<u>Theorem 8</u>. Suppose $f: (-\infty, a] \longrightarrow \mathbb{R}$ is Riemann integrable on every closed subinterval of $(-\infty, a]$. Then f is Lebesgue integrable if and only if the improper Riemann integral $\int_{-\infty}^{a} |f(x)|\,dx$ exists. Moreover when it is Lebesgue integrable, the Lebesgue integral

$$L\int_{(-\infty, a]} f = \int_{-\infty}^{a} f(x)\,dx.$$

In view of the preceding Theorems we have the following definition.

<u>Definition 9</u>. Suppose $f: [a, \infty) \longrightarrow \mathbb{R}$ is such that f is Riemann integrable on every closed subinterval in $[a, \infty)$. If f is Lebesgue integrable, i.e. if both the improper Riemann integral $\int_a^\infty f^+(x)\,dx$ and $\int_a^\infty f^-(x)\,dx$ or equivalently $\int_a^\infty |f(x)|\,dx$ exists, we say $\int_a^\infty f(x)\,dx$ is <u>absolutely convergent</u>.

If the limit $\int_a^\infty f(x)\,dx$ exists but $\int_a^\infty |f(x)|\,dx$ does not (equivalently f is not Lebesgue integrable on $[a,\infty)$), then we say $\int_a^\infty f(x)\,dx$ converges conditionally.

Similarly suppose $f:(-\infty,a] \longrightarrow \mathbb{R}$ is such that f is Riemann integrable on every closed sub intervals in $(-\infty,a]$. If both the improper Riemann integrals $\int_{-\infty}^a f^+(x)\,dx$ and $\int_{-\infty}^a f^-(x)\,dx$ exists (equivalently $\int_{-\infty}^a |f(x)|\,dx$, or f is Lebesgue integrable), then we say $\int_{-\infty}^a f(x)\,dx$ converges absolutely. If $\int_{-\infty}^a |f(x)|\,dx$ does not exist and $\int_{-\infty}^a f(x)\,dx$ exists, we say $\int_{-\infty}^a f(x)\,dx$ converges conditionally. Note that conditional convergence implies non-Lebesgue integrability.

We say $f:(-\infty,\infty) \to \mathbb{R}$ has the improper integral $\int_{-\infty}^\infty f(x)\,dx$ converges absolutely if for some a (hence any a) $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ converge absolutely; and converges conditionally if either one of the above improper integrals converges conditionally.

Example 10. $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$ The convergence is conditional since $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ does not exist.

We now move on to a different kind of improper integral.

Suppose $f: [a,b] \longrightarrow \mathbb{R}$ is unbounded. Then f is not Riemann integrable on $[a,b]$. We first consider the case that f has only one singularity at one of the end point of $[a,b]$.

Definition 11. Suppose $f: [a,b] \longrightarrow \mathbb{R}$ is such that f is Rieman integrable on any sub-interval in $(a,b]$. Suppose that there is a sequence of points x_n in $[a,b]$ such that $|f(x_n)| \longrightarrow +\infty$ when $x_n \to a$. Then if the limit

$$\lim_{t \to a^+} \int_t^b f(x)\,dx$$

exists, this is defined to be the <u>improper Riemann integral</u> of f on $[a,b]$.

Now for simplicity we consider non-negative function $f : [a,b] \longrightarrow \mathbb{R}^+$. Suppose that f is measurable. What is the relation between Lebesgue integral on $[a,b]$ and improper Rieman integrable.

For each integer n define: $f_n : [a,b] \to \mathbb{R}^+$

by $f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ 0 & f(x) > n. \end{cases}$

Then (f_n) is an increasing ~~non-negative~~ sequence of non-negative measurable functions. In particular $f_n \nearrow f$ pointwise. Suppose f is Rieman integrable on each subinterval of (a,b), then each f_n is Riemann integrable. and

$$R.\int_a^b f_n(x)\,dx \longrightarrow L\int_{[a,b]} f$$

by the Monotone Convergence Theorem.

If the limit exists, then f is Lebesgue integrable and $\int_{[a,b]} f = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$.

We now apply the same argument to a different sequence of functions.

Suppose $f : [a, b] \longrightarrow \mathbb{R}^+$ is Riemann integrable on every sub-interval of $(a, b]$. and is non-negative. Suppose $\exists$ a strictly decreasing sequence (x_n) in (a, b) such that $x_n \downarrow a$ but $f(x_n) \to \infty$. That is f is unbounded in any neighbourhood containing a.

Then define $f_n(x) = \begin{cases} f(x) & \text{for } x \geq x_n \\ 0 & a < x < x_n \\ f(a) & x = a \end{cases}$

Then obviously (f_n) is an increasing sequence such that $f_n \longrightarrow f$ pointwise. Therefore, by the Lebesgue Monotone Convergence Theorem

$$L\int_{[a,b]} f_n \longrightarrow L\int_{[a,b]} f.$$

But since each f_n is Riemann integrable,

$$L\int_{[a,b]} f_n = \int_{x_n}^{b} f(x)\,dx.$$

$$\therefore \quad \lim_{n \to \infty} \int_{x_n}^{b} f(x)\,dx = L\int_{[a,b]} f.$$

Therefore if the limit exists, then f is Lebesgue integrable on $[a, b]$ and the Lebesgue integral = improper Riemann integral $\int_a^b f$

Note that the limit $\lim_{n\to\infty} \int_{x_n}^{b} f(x)\,dx$

is independent of the choice of the sequence (x_n) such that $x_n \downarrow a$.

<u>Theorem 12</u>. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on each subinterval in $(a, b]$. Then f is Lebesgue integrable iff the improper Riemann integral $\int_a^b |f|(x)\,dx$ exists. When it does $L\int_{[a,b]} f = \lim_{t\to a^+} \int_t^b f(x)\,dx = \int_a^b f(x)\,dx$ the improper Riemann integrable.

<u>Pf</u>: By Thm 2 f is Lebesgue integrable $\Longleftrightarrow$ $|f|$ is Lebesgue integrable. Hence as before we see that f^+ and f^- are Lebesgue integrable. By the preceding remark before Thm 12,

$$L\int_{[a,b]} f^+ = \text{improper Riemann integral } \int_a^b f^+(x)\,dx$$

$$\& \quad L\int_{[a,b]} f^- = \int_a^b f^-(x)\,dx$$

$$\therefore \quad L\int_{[a,b]} |f| = L\int_{[a,b]} f^+ + L\int_{[a,b]} f^-$$

$$= \int_a^b f^+ + \int_a^b f^- = \int_a^b |f|(x)\,dx$$

the improper Riemann integral.

In particular

$$L\int_{[a,b)} f = L\int_{[a,b)} f^+ - L\int_{[a,b)} f^-$$
$$= \int_a^b f^+ - \int_a^b f^-$$
$$= \int_a^b f$$ the improper

Rieman integral of f on $[a,b]$.

Conversely suppose the improper Rieman integral $\int_a^b |f|(x)\,dx$ exists. Then $|f|$ is Lebesgue integrable on $[a,b]$ and so since f is measurable, f is Lebesgue integrable.

<u>Theorem 13</u> Suppose $[a,b] \to \mathbb{R}$ is Rieman integrable on each subinterval in $[a,b)$. Then f is Lebesgue integrable iff the improper Rieman integral $\int_a^b |f|(x)\,dx$ exists. When it does, $L\int_a^b f =$ improper Rieman integral $\int_a^b f(x)\,dx$.

Proof is similar to Theorem 12.

Definition 14. The improper Rieman integral $\int_a^b f$ converges absolutely if $\int_a^b |f|$ exists equivalently if $\int_a^b f^+$ & $\int_a^b f^-$ exist or if f is Lebesgue integrable on $[a,b]$. If $\int_a^b f$ exists but $\int_a^b |f|$ does not, we say $\int_a^b f$ converges conditionally. If the limit $\int_a^b f$ does not exist, then we say it diverges.

Theorem 15. Cauchy Criteria

Suppose that $f: [a, b] \to \mathbb{R}$ is Riem integrable on every closed subinterval of $[a, b)$ Then the improper Rieman integral $\int_a^b f(x)\,dx$ exists if and only if for any $\varepsilon > 0$, $\exists$ an M depending on ε such that ~~$\forall M_\varepsilon < s < t < b$~~ $\forall \; M_\varepsilon < s < t < b$,

$$\left| \int_s^t f(x)\,dx \right| < \varepsilon.$$

Proof similar to Theorem 20.

This is the main test for absolute convergence.

Test For Convergence

Theorem 16. Let b be a finite pt or ∞, and f a non-negative function on $[a, b)$, integrable on $[a, c]$ for every c, $a < c < b$, with a singularity at b. Then $\int_a^b f(x)dx$ (the improper Riemann integral) exists iff the $\int_a^x f(t)dt$ is bounded for every $x \in [a, b)$.

Proof Obvious since $\int_a^x f(t)dt$ is non-decreasing.

Thm 17 Suppose f and g are non-negative functions on $I = [a, b)$ integrable on $[a, c]$ for every c, $a<c<b$. Suppose that each has a singularity at b and that $f(x) \leq g(x)$ on $[a, b)$. Then.

(i) If $\int_a^b g(x)\,dx$ converges then so does $\int_a^b f(x)dx$

(ii) If $\int_a^b f(x)\,dx$ diverges, then so does $\int_a^b g(x)dx$.

Thm 17 is a consequence of Th'm 16

<u>Theorem 18</u>. Suppose f & g are positive functions on $[a,b)$ and integrable on $[a,c]$ for every c between a & b, and suppose that

$$0 < \lim_{x\to b} \frac{f(x)}{g(x)} < \infty.$$

Then either both $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge or they both diverge.

Pf Suppose $\lim_{x\to b} \frac{f(x)}{g(x)} = \varepsilon$.

Then given $\varepsilon > 0$ $\exists$ δ such

$$b-\delta < x < b \Rightarrow \left| \frac{f(x)}{g(x)} - \varepsilon \right| < \frac{\varepsilon}{2}$$

Therefore taking $X = b-\delta$

$\forall$ $X < x < b$. $\frac{\varepsilon}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}\varepsilon$

$\therefore$

$$\frac{\varepsilon}{2} g(x) < f(x) < \frac{3}{2}\varepsilon g(x). \; \varepsilon$$

$\therefore$ By the Cauchy criterion since $\forall X < s < t < b$,

$$\frac{\varepsilon}{2}\int_s^t g(x)dx \leqslant \int_s^t f(x)dx \leqslant \frac{3\varepsilon}{2}\int_s^t g(x)dx.$$

So $\int_a^b g(x)$ converges $\Rightarrow$ $\int_a^b f(x)$ converges

and vice versa

Next we have a simple computational result for (Lebesgue) integrable functions whose integral is given by improper Riemann integrals. (Reference Theorem 12).

<u>Theorem 19</u>. (i) Suppose $f : [a, \infty) \longrightarrow \mathbb{R}$ is continuous and non-negative, i.e. $f(x) \geq 0$ for all $x \in [a, \infty)$. Suppose F is an anti-derivative of f. Then f is (Lebesgue) integrable on $[a, \infty)$ if and only if the limit $\lim_{x \to \infty} F(x)$ exists. In this case,

$$L\int_{[a,\infty)} f = \int_a^\infty f(x)\,dx$$

$$= \lim_{x \to \infty} F(x) - F(a).$$

(ii) Suppose $f : (a, b] \longrightarrow \mathbb{R}$ is continuous and non-negative on $(a, b]$. Then f is (Lebesgue) integrable if and only if the improper Riemann integrable ~~exists~~ $\int_a^b f$ exists. Suppose F is an anti-derivative of f. Then, in this case

$$L\int_a^b f = F(b) - \lim_{x \to a^+} F(x)$$

<u>Proof</u>. Part (i) is a consequence of Theorem 7 and the Fundamental Theorem of Calculus. Part (ii) is a consequence of Theorem 12 and the Fundamental Theorem of Calculus

For $x>0$ and $k>0$, an anti-derivative for $x^{-k}=\frac{1}{x^k}$ is $\frac{1}{-k+1}x^{-k+1}$ for $k\neq 1$ and $\ln(x)$ for $k=1$.

$$\therefore \int_a^x \frac{1}{t^k}\,dt = \begin{cases} \frac{1}{-k+1}x^{-k+1} - \frac{1}{-k+1}a^{-k+1} & \text{for } k>0 \;\&\; k\neq 1 \\ \ln(x)-\ln(a) & \text{for } k=1. \end{cases}$$

Now $(-k+1)<0 \iff k>1$

$\therefore \int_a^\infty \frac{1}{t^k}\,dt$ converges when $k>1$, since

$$\frac{1}{-k+1}x^{-k+1} \to 0 \text{ as } x\to\infty$$

and $$\int_a^\infty \frac{1}{t^k}\,dt = \frac{1}{k-1}a^{-k+1} = \frac{1}{k-1}\cdot\frac{1}{a^{k-1}}$$

for $a>0$ and $k>1$.

If $k<1$, then $-k+1=1-k>0$.

Thus $\int_a^\infty \frac{1}{t^k}dt$ diverges when $k<1$ since

$\frac{1}{-k+1}x^{-k+1} \to \infty$ as $x\to\infty$.

Similarly $\int_a^\infty \frac{1}{t}$ diverges since $\ln(x)\to\infty$ as $x\to\infty$.

Hence we have

(1) $\int_a^\infty \frac{1}{x^k}\,dx$ for $a>0$ converges if and only if $k>1$

(2) $\int_0^a \frac{1}{x^k}\,dx$ for $a>0$ converges if and only if $k<1$

Part (2) is a consequence of $\frac{1}{-k+1}x^{-k+1} \to \infty$ as $x\to 0$ if $k>1$, $\frac{1}{-k+1}x^{-k+1} \to 0$ as $x \to 0$ if $k<1$ and $\ln(x) \to -\infty$ as $x \to 0$.

Hence we have.

(3) $\int_a^\infty \frac{1}{(x-x_0)^k}\,dx$ for $a>x_0$ converges if and only if $k>1$

and

(4) $\int_{x_0}^a \frac{1}{(x-x_0)^k}$ for $a>x_0$ converges if and only if $k<1$.

Example

(1) $\int_1^\infty e^{-x} x^p \, dx$ converges for all p

The idea is to use a comparison test.

$$e^{-x} x^{p+2} = \frac{x^{p+2}}{e^x} \longrightarrow 0 \quad \text{as } x \to \infty .$$

$\therefore$ $\exists$ integer N such that $x \geq N \Rightarrow \frac{x^{p+2}}{e^x} < 1$.

i.e. $x \geq N \Rightarrow \frac{x^p}{e^x} < \frac{1}{x^2}$.

Since $\int_1^\infty \frac{1}{x^2} dx$ converges,

$$\therefore \quad \int_1^a e^{-x} x^p \, dx \leq \int_1^N e^{-x} x^p \, dx \int_N^a \frac{1}{x^2} dx$$

$$\leq \int_1^N e^{-x} x^p \, dx \int_1^\infty \frac{1}{x^2} dx$$

$\forall \; a \geq N$. $\therefore$ By Theorem 16, (or by Thm 17)

$\int_1^\infty e^{-x} x^p \, dx$ is convergent.

(2) $e^{\frac{1}{x}} x^{p+1} \longrightarrow \infty$ as $x \to 0^+$.

Therefore $\exists$ $\varepsilon > 0$ such that $0 < x < \varepsilon$

$\Longrightarrow e^{\frac{1}{x}} x^{p+1} > 1$ i.e. $e^{\frac{1}{x}} x^p > \frac{1}{x}$.

Thus for all $a > 0$ $\int_0^a e^{\frac{1}{x}} x^p \, dx$ diverges for all p since $\int_0^\varepsilon \frac{1}{x} dx$ is divergent, by Theorem 17.

(3) $\int_0^a \ln(x)\,dx$ converges for any $a \geq 0$.

For any $0 < t \leq a$

$$\int_t^a \ln(x)\,dx = \Big[x\ln(x)\Big]_t^a - \int_t^a 1\cdot dx$$

$$= \big(a\ln(a) - a\big) - \big(t\ln(t) - t\big).$$

$t\ln(t) \to 0$ as $t \to 0^+$

and so $\int_0^a \ln(x)\,dx = a\ln(a) - a$

(4) $\int_1^\infty \frac{1}{\ln(x)}\,dx$ diverges.

$\frac{x}{\ln(x)} \longrightarrow \infty$ as $x \longrightarrow \infty$

$\exists N > 0$ such that $x \geq N \Rightarrow \frac{x}{\ln(x)} > 1$.

$\therefore \frac{1}{\ln(x)} > \frac{1}{x}$

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges and since

$\frac{1}{\ln(x)} > 0$ for all $x > 1$, $\int_1^\infty \frac{1}{\ln(x)}\,dx$

is unbounded and so is not convergent.

For a function, not necessarily non-negative, we can, with sufficient condition satisfied by the function, give a convergent criterion along the line of the Alternating series test.

The condition for the function is crafted so that the Alternating series test can be applied.

<u>Theorem 20.</u> Suppose $f : [a, \infty) \to \mathbb{R}$ is such that f is (Riemann) integrable on $[a, b]$ for every $b > a$. Suppose that there is a sequence (a_n) with distinct terms such that $a_0 = a$ and $a_{n+1} > a_n$, $a_n \to \infty$ and satisfying:

(i) $f(x)$ has constant sign in each interval (a_n, a_{n+1})

(ii) $f(x)$ changes sign from (a_{n-1}, a_n) to (a_n, a_{n+1})

(iii) $$\left| \int_{a_{n-1}}^{a_n} f(x)\,dx \right| \geq \left| \int_{a_n}^{a_{n+1}} f(x)\,dx \right|$$

(iv) $$\int_{a_{n-1}}^{a_n} f(x)\,dx \to 0 \text{ as } n \to \infty.$$

Then the improper Riemann integral $\int_a^\infty f(x)\,dx$ converges.

Proof. Let $c_n = \int_{a_{n-1}}^{a_n} f(x)\,dx$.

Then by condition (iv) $c_n \to 0$. But condition (iii) $|c_n|$ is decreasing. By condition (ii) $\sum_{n=1}^{\infty} c_n$ is an alternating series. Therefore, by the Alternating Series Test $\sum c_n$ is convergent.

Now take any $b > a$. Then since $a_n \to \infty$, $\exists$ an integer n_0 such $a_{n_0} \geq b$. Let n_0 be the least such integer such that $a_{n_0} \geq b$. Hence $a_{n_0-1} < b \leq a_{n_0}$.

Thus
$$\int_a^b f(x)\,dx = \sum_{k=1}^{n_0-1} c_k + \int_{a_{n_0-1}}^{b} f(x)\,dx$$

Since $a_{n_0-1} < b \leq a_{n_0}$

$$\left| \int_{a_{n_0-1}}^{b} f(x)\,dx \right| \leq \left| \int_{a_{n_0-1}}^{a_{n_0}} f(x)\,dx \right| = |c_{n_0}|$$

Obviously as b tends to infinity $n_0 \to \infty$

Therefore, given $\varepsilon > 0$, $\exists\ K$ such that $b > K \implies$

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{\infty} c_k \right|$$

$$\leq \left| \int_a^b f(x)\,dx - \sum_{k=1}^{n_0-1} c_k \right| + \left| \sum_{k=n_0}^{\infty} c_k \right|$$

$$\leq \quad |c_{n_0}| + \left| \sum_{k=n_0}^{\infty} c_k \right|$$

$$< \quad \varepsilon/2 \quad + \quad \varepsilon/2 = \varepsilon$$

where K is chosen so that $K > a_N$ and $N \geq \max(N_1, N_2)$, where

N_1 is such that $n \geq N_1 \Rightarrow |c_n| < \varepsilon/2$

& N_2 is such that $k \geq N_2 \implies$

$$\left| \sum_{n=k}^{\infty} c_n \right| \leq \varepsilon/2 .$$

N_1 exists by (ii) and N_2 exists since $\sum_1^{\infty} c_k$ is convergent. (Therefore, $b > K$

$\implies n_0 - 1 \geq N \Rightarrow n_0 \geq N$.)

Hence $\int_a^b f(x)\,dx \to \sum_{k=1}^{\infty} c_k$ as $b \to \infty$

Use of the Lebesgue Dominated Convergence Theorem.

Example. $\lim_{n \to \infty} \int_0^1 \frac{1 - e^{-nx}}{\sqrt{x}} \, dx.$

$f_n(x) : (0, 1] \longrightarrow \mathbb{R}, \quad f_n(x) = \frac{1 - e^{-nx}}{\sqrt{x}} \, dx.$

$f_n(x) \to \frac{1}{\sqrt{x}}$ as $n \to \infty$ pointwise on $(0, 1]$.

Now since for $x \geq 0$ $e^{nx} \geq e^0 = 1$

$e^{-nx} \leq 1$ for $x \geq 0$.

Therefore $1. \geq 1 - e^{-nx} \geq 0$

$\therefore 0 \leq f_n(x) \leq \frac{1}{\sqrt{x}}$ for $x \in (0, 1]$.

Since the improper Riemann integral $\int_0^1 \frac{1}{\sqrt{x}} dx$

$= \lim_{t \to 0^+} \left[2\sqrt{x} \right]_t^1 = 2$.

Therefore since each $f_n(x)$ is non-negative, and bounded by $\frac{1}{\sqrt{x}}$ which is Lebesgue integrable on $[0, 1]$ and so the sequence $(f_n(x))$ is dominated by the Lebesgue integrable function and so $\int_0^1 f_n(x) \, dx \longrightarrow \int_0^1 \frac{1}{\sqrt{x}} dx = 2$.

by L.D.C.T.