

Chapter 3 Continuous Functions

Continuity is a very important concept in analysis. The tool that we shall use to study continuity will be sequences. There are important results concerning the subsets of the real numbers and the continuity of the function: the Extreme Value Theorem and the Intermediate Value Theorem. The next equally important and useful result is the uniform continuity of continuous function on a closed and bounded interval. We shall study also the properties of monotone functions. We shall assume the familiarity with the definition of a function, its domain, range and codomain and definition of injectivity, surjectivity and bijectivity of a function.

Definition 1. Let D be a subset of \mathbf{R} . A function $f: D \rightarrow \mathbf{R}$ is said to be *continuous* at a in D if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x in D ,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

We say f is *continuous* if f is continuous at every point in D .

Remark. Usually continuity of f at a point x is defined via limit. The function f is said to be continuous at a in D if $\lim_{x \rightarrow a} f(x) = f(a)$. Notice that this definition is equivalent to Definition 1 when a is a limit point of D .

We shall be using sequences to study continuity and so the following equivalent definition of continuity is particularly useful.

Definition 2. A function $f: D \rightarrow \mathbf{R}$ is said to be *continuous* at a in D if for any sequence (a_n) such that $a_n \rightarrow a$, we have that the sequence $(f(a_n))$ is convergent and converges to $f(a)$, i.e., $f(a_n) \rightarrow f(a)$.

Definitions 1 and 2 are equivalent. Usually it is easier to use a sequence to investigate continuity and Definition 2 is in the form that may be used readily. However, as in the proof of the Intermediate Value Theorem, Definition 1 is a better choice for effective use. We shall prove the equivalence of these two definitions later (see Theorem 12).

Example 3.

1. The Dirichlet function $f: \mathbf{R} \rightarrow \mathbf{R}$ is defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}.$$

Then f is discontinuous everywhere, i.e., at every point in \mathbf{R} .

This is because, by the density of the rational numbers, at every point x in \mathbf{R} , there is a sequence of rational numbers (a_n) such that $a_n \rightarrow x$. We can produce this sequence as follows. For each n in \mathbf{P} , by the density of the rational numbers (Corollary 4.4.7, Chapter 1), there exists a rational number a_n in \mathbf{R} such that $x - 1/n < a_n < x + 1/n$. Plainly, $a_n \rightarrow x$. Then obviously $f(a_n) = 1$ for each n in \mathbf{P} . Thus, $f(a_n) \rightarrow 1$. Also by the density of the irrational numbers (Corollary 4.4.7 and Corollary 4.4.8, Chapter 1), there exists an irrational number b_n in \mathbf{R} such that $x - 1/n < b_n < x + 1/n$. Then $b_n \rightarrow x$ and $f(b_n) = 0$ for each n in \mathbf{P} . Plainly, $f(b_n) \rightarrow 0$. Thus, we have two sequences (a_n) and (b_n) both converging to x but $(f(a_n))$ and

$(f(b_n))$ do not converge to the same value. Therefore, by Definition 2, f is not continuous at x , for any x . Hence, f is discontinuous everywhere.

2. Any polynomial function is continuous. Here is a different proof using sequences. We will just show an example. The function $f(x) = x^2 + 3x + 1$ is continuous. Take any a in \mathbf{R} . We shall show that f is continuous at a . Take any sequence (a_n) converging to a . Then $f(a_n) = a_n^2 + 3a_n + 1 \rightarrow a^2 + 3a + 1$ by properties of convergent sequences (Properties 7 Chapter 2). Hence, f is continuous at a . Plainly, the proof for the general polynomial function is similar.

The properties of convergent sequences translate to the following:

Theorem 4. Suppose $f: D \rightarrow \mathbf{R}$ and $g: D \rightarrow \mathbf{R}$ are two functions, continuous at a in D .

Then

- (1) the sum $f + g: D \rightarrow \mathbf{R}$ is continuous at a ,
- (2) the product $f \cdot g: D \rightarrow \mathbf{R}$ is continuous at a ,
- (3) if $g(a) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at a ,
- (4) for any real number λ , λf is continuous at a .

Proof. Theorem 4 follows from Definition 2 and Properties 7 of Chapter 2.

Remark. Theorem 4 may be proved directly from Definition 1 using $\varepsilon - \delta$ argument.

An easy consequence of Theorem 4. is the following:

Corollary 5. Any rational function is continuous on its domain of definition.

Proof. A rational function is a function of the form $\frac{p}{q}$, where p and q are polynomial functions. By theorem 4 part (3) $\frac{p}{q}$ is continuous at a where $q(a) \neq 0$. Therefore, $\frac{p}{q}$ is continuous on its domain of definition, which is $\{x : q(x) \neq 0\}$.

The next question we ask is: Is composition of continuous function continuous?

Theorem 6. Suppose $f: D \rightarrow \mathbf{R}$ and $g: A \rightarrow \mathbf{R}$ are two functions such that $f(D) \subseteq A$, i.e., range of f is contained in the domain of g . Hence the composite function $g \circ f: D \rightarrow \mathbf{R}$ is defined. If f is continuous at a and g is continuous at $f(a)$, then the composite $g \circ f$ is continuous at a .

Proof. We shall show that for any sequence (a_n) such that $a_n \rightarrow a$, $g \circ f(a_n) \rightarrow g \circ f(a)$. Since f is continuous at a , $f(a_n) \rightarrow f(a)$. This means the sequence $(f(a_n))$ is a convergent sequence converging to $f(a)$. Then, since g is continuous at $f(a)$, $g(f(a_n)) \rightarrow g(f(a))$. Therefore, $g \circ f(a_n) = g(f(a_n)) \rightarrow g(f(a)) = g \circ f(a)$. Hence, $g \circ f$ is continuous at a .

The next result is an important result and is an important tool in analysis. In its more general situation it enunciated the following: any continuous function on a compact topological spaces attains its maximum and minimum. We shall use the result from Chapter 2, particularly Theorem 39. Chapter 2 gives criteria for completeness and connection of completeness with compactness via the Bolzano Weierstrass Theorem. One can give a proof

using entirely the idea of (order) completeness. One may say the conclusion of the Extreme Value Theorem for closed and bounded interval is equivalent to completeness for \mathbf{R} .

First we shall consider the continuous image of sequentially compact subset of \mathbf{R} . Now since compactness is equivalent to sequential compactness, we shall state the results in terms of sequentially compact subset. However, the results are true with 'sequential compact' replaced by 'compact'.

Theorem 7. Suppose $f : K \rightarrow \mathbf{R}$ is a continuous function. Then, if K is sequentially compact the image $f(K)$ is also sequentially compact.

Proof. Suppose (y_n) is a sequence in $f(K)$. Then for each y_n , there exists an element x_n in K such that $f(x_n) = y_n$. Thus, (x_n) is a sequence in K . Since K is sequentially compact, (x_n) has a convergent subsequence (x_{n_k}) such that $x_{n_k} \rightarrow x$ for some x in K . Since f is continuous and hence continuous at x , (by Definition 2) $f(x_{n_k}) \rightarrow f(x)$. Then $(f(x_{n_k})) = (y_{n_k})$ is a subsequence of (y_n) which converges to $f(x)$ in $f(K)$. Thus, we conclude that any sequence in $f(K)$ has a convergent subsequence that converges to a point in $f(K)$. Therefore, $f(K)$ is sequentially compact.

Theorem 8. Extreme value Theorem. Suppose $f : K \rightarrow \mathbf{R}$ is a continuous function and K is sequentially compact. Then f attains its maximum and minimum, i.e., there exists c, d in K such that $f(c) \leq f(x) \leq f(d)$ for all x in K .

Proof. By Theorem 7, $f(K)$ is sequentially compact. By Theorem 39 Chapter 2 $f(K)$ is closed and bounded. Then, $M = \sup\{f(k) : k \in K\}$ exists in \mathbf{R} , since $f(K)$ is bounded above. We shall show that $M \in f(K)$. For each n in \mathbf{P} , by the definition of $\sup\{f(k) : k \in K\}$, there exists an element a_n in $f(K)$ such that $M - 1/n < a_n \leq M$. Then plainly, (a_n) converges to M . But since $f(K)$ is closed, by Proposition 33 Chapter 2, M is in $f(K)$. Therefore, there exists an element d in K such that $f(d) = M$ and $f(x) \leq f(d)$ for all x in K . Similarly, since $f(K)$ is bounded below, $m = \inf\{f(k) : k \in K\}$ exists in \mathbf{R} . We can find a sequence in $f(K)$ which converges to m as follows.. For each n in \mathbf{P} , by the definition of $\inf\{f(k) : k \in K\}$, there exists an element b_n in $f(K)$ such that $m \leq b_n \leq m + 1/n$. We deduce in the same way for supremum, that $m \in f(K)$. Hence, there exists an element c in K such that $f(c) = \inf\{f(k) : k \in K\} \leq f(x)$ for all x in K .

Corollary 9. Let $[a, b]$ be a closed and bounded interval. Then any continuous function $f : [a, b] \rightarrow \mathbf{R}$ attains its maximum and minimum.

Definition 10. Suppose $f : A \rightarrow \mathbf{R}$ is a function. A *maximizer* for the function f is an element d in A such that $f(x) \leq f(d)$ for all x in A . A *minimizer* for the function f is an element c in A such that $f(c) \leq f(x)$ for all x in A .

Thus Theorem 8 says that if K is sequentially compact and if $f : K \rightarrow \mathbf{R}$ is a continuous function, then f has a maximizer and a minimizer.

The next property that we will be investigating is the so called intermediate value property for a continuous function defined on an interval. Topologically, this is just the same as

saying the continuous image of a 'connected' set is 'connected' . We will not bring in this notion but instead we will use a characterization of an interval.

Definition 11. A subset I of \mathbf{R} is an *interval*, if whenever a, b are in I and $a < b$, then the set $[a, b] = \{x \in \mathbf{R} : a \leq x \leq b\} \subseteq I$.

Remark.

1. We have previously called $[a, b]$, a closed interval. Plainly, it is an interval in the sense of Definition 11. Obviously, it is also closed since its limit points are in $[a, b]$. (See Definition 32 Chapter 2). Note that all of the following are intervals in the sense of Definition 11: $[a, b]$, (a, b) , $[a, b)$, $(a, b]$, $[a, +\infty)$, $(a, +\infty)$, $(-\infty, b]$, $(-\infty, b)$, $(-\infty, +\infty) = \mathbf{R}$. Thus, what we called an interval to be any one of the above type coincides with this definition.
2. If I is an *interval*, then it must be one of the above types depending first of all whether the interval is bounded above or bounded below or unbounded and then on the existence of supremum or infimum and whether they reside in I or not. It is an easy exercise to show this.
3. The notion of connected set is a more general notion that applies to metric spaces and topological spaces and of course to \mathbf{R} , since \mathbf{R} is a metric space. The only connected subsets of \mathbf{R} are the intervals. For the proof of this statement, see for instance Theorem 17.7 K. G. Binmore Foundation of Analysis: A straightforward Introduction Book 2 Topological Ideas.
4. Note that by definition for each a in \mathbf{R} , $[a] = \{a\}$ is an interval called the trivial interval.

We shall next use definition 1 for continuity instead of the sequence definition. Below we furnish a proof that Definitions 1 and 2 are equivalent.

Theorem 12. Let D be a non-empty subset of \mathbf{R} and a a point in D . Let $f : D \rightarrow \mathbf{R}$ be a function defined on D . The following two statements are equivalent.

(A) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x in D ,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

(B) For any sequence (a_n) in D such that $a_n \rightarrow a$, $f(a_n) \rightarrow f(a)$.

Proof.

(A) \Rightarrow (B). Assume (A). Then suppose (a_n) is a sequence in D such that $a_n \rightarrow a$. We shall show that $f(a_n) \rightarrow f(a)$. Given any $\varepsilon > 0$, then by (A) there exists $\delta > 0$ such that for all x in D ,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \quad \text{-----} \quad (1)$$

Since $a_n \rightarrow a$, there exists a positive integer N such that $n \geq N$ implies that $|a_n - a| < \delta$.

Therefore, by (1), $n \geq N \Rightarrow |f(a_n) - f(a)| < \varepsilon$. Hence, $f(a_n) \rightarrow f(a)$.

(B) \Rightarrow (A). Assume (B). Suppose on the contrary that (A) does not hold. Then there exists an $\varepsilon > 0$ such that for each positive integer n , there exists an element a_n in D such that $|a_n - a| < 1/n$ but $|f(a_n) - f(a)| \geq \varepsilon$. Plainly, $a_n \rightarrow a$. Since $|f(a_n) - f(a)| \geq \varepsilon$ for all n in \mathbf{P} , the sequence $(f(a_n))$ does not converge to $f(a)$. But by assumption (B) $(f(a_n))$ converges to $f(a)$. This contradiction shows that (A) holds.

We shall use either Definition 1 or 2 depending on which ever is more efficient.

Theorem 13. Intermediate Value Theorem. Suppose $f : [a, b] \rightarrow \mathbf{R}$ is continuous. If γ is an intermediate value between $f(a)$ and $f(b)$, i.e. either $f(a) \leq \gamma \leq f(b)$ or $f(b) \leq \gamma \leq f(a)$, then there exists c in $[a, b]$ such that $f(c) = \gamma$.

Proof. If $a = b$, we have nothing to prove. Assume $a < b$. Without loss of generality we may assume that $f(a) < f(b)$. If $\gamma = f(a)$ or $f(b)$, we have nothing to prove. Now take any γ such that $f(a) < \gamma < f(b)$. Then define $g : [a, b] \rightarrow \mathbf{R}$ by $g(x) = f(x) - \gamma$ for x in $[a, b]$. Then g is a continuous function, $g(a) < 0$ and $g(b) > 0$. We are going to find a point κ in $[a, b]$ such that $g(\kappa) = 0$. We do this by using the completeness property of the real numbers \mathbf{R} . Let $F = \{ x \in [a, b] : g(x) < 0 \}$. Then $F \neq \emptyset$ since $a \in F$ because $g(a) < 0$. Obviously F is bounded above by b . Hence by the completeness property of \mathbf{R} , supremum of F exists. Let $\kappa = \sup F$. Since g is continuous at a and $g(a) < 0$, there exists $\delta > 0$ such that for all x with $a \leq x < a + \delta < b$, $g(x) < 0$. (Take $\varepsilon = -g(a)/2$. By continuity of g at a , there exists $\delta_1 > 0$ such that for all x in $[a, b]$ with $a \leq x < a + \delta_1$, $|g(x) - g(a)| < -g(a)/2$ or $3g(a)/2 < g(x) < g(a)/2 < 0$. Take $\delta = \min(\delta_1, (b-a)/2)$.) This means $\kappa \geq a + \delta' > a$ for any δ' with $0 < \delta' < \delta$ since $g(a + \delta') < 0$. Therefore, $\kappa > a$. Thus $a < \kappa \leq b$. Now by the continuity of g at b and the fact that $g(b) > 0$, there exists $\delta_2 > 0$ such that for all x with $a < b - \delta_2 < x \leq b$, $g(x) > 0$. This means for any q in F , $q \leq b - \delta_2$ because for any k with $b - \delta_2 < k \leq b$, $k \notin F$ and consequently $b - \delta_2$ is an upper bound for F . Thus $\kappa = \sup F \leq b - \delta_2 < b$. Hence $a < \kappa < b$.

We now claim that $g(\kappa) = 0$. That is $f(\kappa) = \gamma$.

Suppose $g(\kappa) < 0$. Then by the continuity of g at κ , there exists $\delta_3 > 0$ such that $[\kappa - \delta_3, \kappa + \delta_3]$ is a proper subset of $[a, b]$ and for any x with $a < \kappa - \delta_3 \leq x \leq \kappa + \delta_3 < b$, we have $g(x) < 0$. This means $\kappa + \delta_3 \in F$. Thus $\kappa + \delta_3 \leq \sup F = \kappa$, and $\delta_3 \leq 0$ contradicting $\delta_3 > 0$. Hence $g(\kappa) \geq 0$. Similarly if $g(\kappa) > 0$, then by the continuity of g at κ , there exists $\delta_4 > 0$ such that for any x with $a < \kappa - \delta_4 \leq x \leq \kappa + \delta_4 < b$, we have $g(x) > 0$. Note that for any x in $[a, b]$, $x > \kappa$ implies that $g(x) \geq 0$. Then any x in $[a, b]$ and $g(x) < 0$ would imply that $x < \kappa - \delta_4$. Thus $\kappa - \delta_4$ is an upper bound for F and hence $\kappa \leq \kappa - \delta_4$ giving $\delta_4 \leq 0$ contradicting $\delta_4 > 0$. Hence $g(\kappa) = 0$. We now take $c = \kappa$ and $f(c) = \gamma$.

If $f(a) > f(b)$, then multiply by -1 , we get $-f(a) < -f(b)$. Replace f above by $-f$, γ by $-\gamma$ and the proof proceed exactly the same manner as above to obtain a c in $[a, b]$ such that $-f(c) = -\gamma$ and that is the same as $f(c) = \gamma$. This completes the proof.

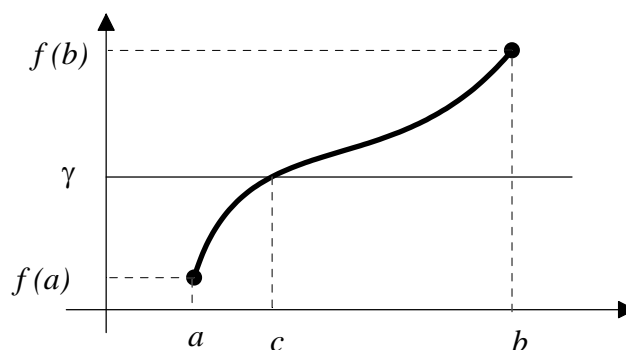


Fig. 2

Theorem 13 is a very useful tool in analysis. It is used in the proof of the first and second mean value theorems for integrals.

Continuity and intervals.

Some of the obvious questions we can ask is the following:

Is the continuous image of an interval, an interval?

Is a continuous and injective function defined on an interval, necessarily strictly monotone?

We shall answer these two and other similar questions.

Theorem 14. The continuous image of an interval is an interval.

Proof. Suppose J is an interval and f is a continuous function defined on J . We shall show that the image $f(J)$ is an interval by Definition 11. Let y_1 and y_2 with $y_1 \leq y_2$ be in $f(J)$. Then there exist x_1 and x_2 in J such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is continuous on J , f is continuous on $[x_1, x_2]$. Thus by the *Intermediate Value Theorem*, for any y with $y_1 \leq y \leq y_2$, there exists an element x in $[x_1, x_2]$ such that $f(x) = y$, that is, y is in $f(J)$. Hence $[y_1, y_2] \subseteq f(J)$. Therefore, $f(J)$ is an interval.

We can say more about the image of the interval if the function is also injective.

Theorem 15. If I is an open interval and f is an injective continuous function defined on I , then $f(I)$ is also an open interval.

Proof. By Theorem 14, $f(I)$ is an interval. Suppose that it is not open. Then $f(I)$ is of the form $[a, b]$, $[a, b)$, $(a, b]$, $[a, \infty)$, or $(-\infty, b]$. Suppose $f(I) = [a, b)$ or $[a, b]$, or $[a, \infty)$. Then since f is injective, there exists a unique x_0 in I such that $f(x_0) = a$, and for all x in I , $x \neq x_0 \Rightarrow f(x) > f(x_0)$. Since I is open, there exist elements c and d in I such that $c < x_0 < d$ and $[c, d] \subseteq I$. Now by the *Intermediate Value Theorem* $f([c, x_0]) \supseteq [a, f(c)]$ and $f([x_0, d]) \supseteq [a, f(d)]$. We may assume, without loss of generality, that $f(c) < f(d)$. (Rename c and d if necessary.) Then $f([c, x_0]) \cap f([x_0, d]) \supseteq [a, f(c)] \supseteq (a, f(c)) \neq \emptyset$. But f being injective implies that $f([c, x_0]) \cap f([x_0, d]) = \{f(x_0)\} = \{a\} \not\supseteq (a, f(c))$. This contradiction shows that $f(I)$ cannot be of the form $[a, b)$ or $[a, b]$ or $[a, \infty)$. By a similar argument we can show that $f(I)$ cannot be of the form $(a, b]$, or $(-\infty, b]$. Thus $f(I)$ must be an open interval.

The Intermediate Value Theorem is often used to demonstrate the existence of a root of a polynomial equation and forms the basis of many algorithms to extract root.

Example.

1. Here is a round about way to show the existence of square root. For any positive number C , there is a solution to the equation $x^2 = C$. The proof we shall give below is some what different from the existence proof given in Lemma 6 of Chapter 5 in "Real Numbers?"

Let f be defined on the interval $[0, C+1]$ by $f(x) = x^2$. Then f is continuous on $[0, C+1]$ and $f(C+1) = (C+1)^2 = C^2 + 2C + 1 > C > 0 = f(0)$. Thus by the Intermediate Value Theorem, there exists an element k in $(0, C+1)$ such that $f(k) = k^2 = C$.

2. Suppose f is continuous on $[a, b]$ and suppose either $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$. Then by the Intermediate Value Theorem, there exists an element c in $[a, b]$ such that $f(c) = 0$. This is the usual method to locate root. An algorithm can be devised to narrow the distance between a and b successively to obtain a nested sequence of intervals. For instance, suppose $f(a) > 0 > f(b)$. Let $c = (a + b)/2$. If $f(c) = 0$, then we have found the root and we are done. If $f(c) > 0$, then let $a_1 = c$, $b_1 = b$. If $f(c) < 0$, then let $a_1 = a$, $b_1 = c$. We then have $f(a_1) > 0 > f(b_1)$ and $(b_1 - a_1) = (b - a)/2$. We then repeat the process with the interval $[a_1, b_1]$. In this way, either we terminate when finding the root in a finite number of steps or we get a sequence of nested intervals,

$$[a, b] \supset [a_1, b_1] \supset \dots \supset [a_n, b_n]$$

whose end points are getting closer and closer to the actual root and hence approximate the root better and better as $(b_n - a_n) = (b - a)/2^n \rightarrow 0$.

Remark.

Theorems 13 and 14 are topological results. Theorem 14 is a special case of the following result: a continuous image of a connected set is connected.

We now examine continuous function defined on an interval. Our next result considers when a function is continuous by knowing if its image is an interval. First we formally define monotone function.

Definition 16. Let $f: D \rightarrow \mathbf{R}$ be a function defined on D .

- (1) f is said to be *monotonically increasing* or just simply *increasing*, if for all a, b in D , $a < b \Rightarrow f(a) \leq f(b)$.
- (2) f is said to be *strictly monotonically increasing* or *strictly increasing*, if for all a, b in D , $a < b \Rightarrow f(a) < f(b)$.
- (3) f is said to be *monotonically decreasing* or just simply *decreasing*, if for all a, b in D , $a < b \Rightarrow f(a) \geq f(b)$.
- (4) f is said to be *strictly monotonically decreasing* or *strictly decreasing*, if for all a, b in D , $a < b \Rightarrow f(a) > f(b)$.
- (5) f is said to be *monotone* if f is either increasing or decreasing.
- (6) f is said to be *strictly monotone* if f is either strictly increasing or strictly decreasing.

Recall the following definition of injectivity and surjectivity

Definition 17. Suppose $f: A \rightarrow B$ is a function. f is said to be *injective* if for all a and b in A , $f(a) = f(b) \Rightarrow a = b$. f is said to be *surjective* or *onto* if its image is B , i.e., $f(A) = B$. Recall that A is called the *domain* of f and B is called the *codomain* of f and the *range* of f is $f(A)$. Thus, f is *surjective* if range of f = codomain of f .

To facilitate the study of continuous functions using sequences, it is useful to know what a convergent sequence will possess as subsequences. First we recall the definition of peak index and trough index.

Definition 18. Suppose (a_n) is a sequence in \mathbf{R} . We say the sequence (a_n) has a *peak* at k if, for all $j \geq k$, $a_k \geq a_j$. a_k is called the *peak* and k the *peak index*. Similarly we say the sequence (a_n) has a *trough* at k if, for all $j \geq k$, $a_k \leq a_j$. a_k is called the *trough* and k the *trough index*.

We are going to construct convergent subsequences of a bounded sequence using the peak and the trough indices. But first we prove the following simple observation.

Proposition 19. Suppose (a_n) is a sequence in \mathbf{R} . If (a_n) converges to a in \mathbf{R} , then all subsequences are convergent and converge to the same limit a .

Proof. Suppose $a_n \rightarrow a$. Then given any $\varepsilon > 0$, there exists a positive integer N such that $n \geq N \Rightarrow |a_n - a| < \varepsilon$. Suppose (a_{n_k}) is a subsequence of (a_n) . Then, for all $k \geq N$, $n_k \geq k \geq N$ and so $|a_{n_k} - a| < \varepsilon$. Thus, by definition of convergence, $a_{n_k} \rightarrow a$.

Suppose (a_n) is a sequence converging to a in \mathbf{R} . Then (a_n) is bounded. Then (a_n) has either an infinite number of peak indices, say, k_1, k_2, k_3, \dots with $k_1 < k_2 < k_3 < \dots$ or (a_n) has finite or no peak indices. Then by definition of the peak,

$$a_{k_1} \geq a_{k_2} \geq a_{k_3} \geq \dots$$

Thus the subsequence (a_{k_j}) is a monotone decreasing sequence. Since (a_n) is bounded, (a_{k_j}) is also bounded and so bounded below and by the Monotone Convergence Theorem, $a_{k_j} \rightarrow \inf\{a_{k_j} : k \in \mathbf{P}\}$. By Proposition 19, $\inf\{a_{k_j} : k \in \mathbf{P}\} = a$. That means $a_{k_j} \geq a$ for all j in \mathbf{P} .

If there are only finite number of these peaks or no peak, then there is an index K , such that there are no peak indices $\geq K$. Let $n_1 = K$. Then since n_1 is not a peak index, there exists an index n_2 such that $n_2 > n_1$ but $a_{n_2} > a_{n_1}$. Similarly since a_{n_2} is not a peak, it means that it is not true that for all $j \geq n_2$, $a_j \leq a_{n_2}$. Hence there exists an index $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. Thus, in this way we recursively define $n_{k+1} > n_k$ such that that $a_{n_{k+1}} > a_{n_k}$. Therefore, (a_{n_k}) is a monotone increasing sequence. Moreover, for all $n \geq K$, $a_n \leq a$. This is because if there exists an integer $n_0 \geq K$ such that $a_{n_0} > a$, we can deduce a contradiction as follows. Then, there exists an integer $N \geq K$ such that

$$n \geq N \Rightarrow |a_n - a| < (a_{n_0} - a)/2 \Rightarrow a_n < a + (a_{n_0} - a)/2 = (a_{n_0} + a)/2 < a_{n_0}.$$

Consider the maximum of the set $\{a_K, a_{K+1}, \dots, a_N, a_{n_0}\}$. Then $\max\{a_K, a_{K+1}, \dots, a_N, a_{n_0}\} = a_h$ with $h \geq K$ and obviously for all $n \geq h$, $a_n \leq a_h$ and so we have a peak index $h \geq K$.

This contradicts that we have no peak index $\geq K$. Hence, for all $n \geq K$, $a_n \leq a$.

We can examine the trough indices in the same way. If (a_n) has an infinite number of trough indices, say, l_1, l_2, l_3, \dots with $l_1 < l_2 < l_3 < \dots$, then by definition of the trough,

$a_{l_1} \leq a_{l_2} \leq a_{l_3} \leq \dots$ and (a_{l_j}) is an increasing sequence. Plainly, (a_{l_j}) is bounded above and so by the Monotone Convergence Theorem, $a_{l_j} \rightarrow \sup\{a_{l_j} : j \in \mathbf{P}\}$. Thus, since $a_n \rightarrow a$, by Proposition 19, $\sup\{a_{l_j} : j \in \mathbf{P}\} = a$. Thus $a_{l_j} \leq a$ for all j in \mathbf{P} .

Now if there are only finite number of these troughs or no trough, then there is an index L , such that there are no trough indices $\geq L$. Let $n_1 = L$. Then since L is not a trough index, there exists an index n_2 such that $n_2 > n_1$ but $a_{n_2} < a_{n_1}$. Similarly since a_{n_2} is not a trough, it means that it is not true that for all $j \geq n_2$, $a_j \geq a_{n_2}$. Hence there exists an index $n_3 > n_2$ such that $a_{n_3} < a_{n_2}$. Thus, in this way we recursively define $n_{k+1} > n_k$ such that that $a_{n_{k+1}} < a_{n_k}$.

Therefore, (a_{n_k}) is a decreasing sequence. Moreover, for all $n \geq L$, $a_n \geq a$. Suppose there exists an integer $n_0 \geq L$ such that $a_{n_0} < a$. Then, there exists an integer $N \geq L$ such that

$$n \geq N \Rightarrow |a_n - a| < (a - a_{n_0})/2 \Rightarrow a_n > a - (a - a_{n_0})/2 = (a_{n_0} + a)/2 > a_{n_0}.$$

Then $\min\{a_L, a_{L+1}, \dots, a_N, a_{n_0}\} = a_h$ with $h \geq L$ and obviously, for all $n \geq h$, $a_n \geq a_h$ and so we have a trough index $h \geq L$. This contradicts that we have no trough index $\geq L$. Hence, for all $n \geq L$, $a_n \geq a$.

We have thus proved the following:

Proposition 20. Suppose (a_n) is a sequence in \mathbf{R} converging to a in \mathbf{R} . Then we have the following possibilities.

Either

(1) There is a monotonically decreasing subsequence (a_{k_j}) converging to a such that $a_{k_j} \geq a$ for all j in \mathbf{P} and that for each j in \mathbf{P} , $n \geq k_j \Rightarrow a_n \leq a_{k_j}$, and there is a monotonically increasing sequence (a_{l_j}) converging to a with $a_{l_j} \leq a$ and $n \geq l_j \Rightarrow a_n \geq a_{l_j}$, for all j in \mathbf{P} .

Or (2) There is a monotonically decreasing subsequence (a_{k_j}) converging to a such that $a_{k_j} \geq a$ for all j in \mathbf{P} and that for each j in \mathbf{P} , $n \geq k_j \Rightarrow a_n \leq a_{k_j}$, and there exists an integer L such that for all $n \geq L$, $a_n \geq a$.

Or (3) There is a monotonically increasing sequence (a_{l_j}) converging to a with $a_{l_j} \leq a$ and $n \geq l_j \Rightarrow a_n \geq a_{l_j}$, for all j in \mathbf{P} and there exists an integer K such that for all $n \geq K$, $a_n \leq a$.

Or (4) There exists a positive integer N such that $n \geq N \Rightarrow a_n = a$.

Proof. This is just a statement of the 4 possibilities depending on whether the sequence (a_n) has infinite peak or infinite trough indices or finite or none peak indices or trough indices. The argument preceding the proposition proves part (1) to (3) of the proposition. Part (4) is the remaining case when the sequence (a_n) have finite or no peak indices and finite or no trough indices. Thus there exists a positive integer K such that $n \geq K \Rightarrow a_n \leq a$ and a positive integer L such that $n \geq L \Rightarrow a_n \geq a$. Let $N = \max(K, L)$. Then $n \geq N \Rightarrow a_n = a$.

Remark. Proposition 20 describes the possible behaviour of a converging sequence that may be used.

We now investigate the behaviour of a monotone sequence under monotone function.

Theorem 21. Let A be a non-empty subset of \mathbf{R} . Suppose $f: A \rightarrow \mathbf{R}$ is a monotone function such that the image $f(A)$ is an interval. Let (a_n) be any monotone sequence in A converging to an element a in A . Then the sequence $(f(a_n))$ is convergent and converges to $f(a)$.

Proof. Suppose f is an increasing function. Suppose (a_n) is a monotone sequence in A converging to an element a in A . If (a_n) is increasing, then $a_n \leq a$ for all n in \mathbf{P} . Since f is increasing, it follows that $f(a_n) \leq f(a)$ for all n in \mathbf{P} . Therefore, the sequence $(f(a_n))$ is bounded above and is increasing since $i > k \Rightarrow a_i \geq a_k \Rightarrow f(a_i) \geq f(a_k)$. Thus, by the Monotone Convergence Theorem, $f(a_n) \rightarrow \sup \{f(a_k): k \in \mathbf{P}\} \leq f(a)$. Let $y = \sup \{f(a_k): k \in \mathbf{P}\}$. Now we claim that $y = f(a)$. Note that $f(a_1) \leq y \leq f(a)$. Since $f(A)$ is an interval, $[f(a_1), f(a)] \subseteq f(A)$. Suppose on the contrary that $y \neq f(a)$. Then $y < f(a)$. Hence, the mid point $(y + f(a))/2$ between y and $f(a)$ is in $[f(a_1), f(a)]$ and so is in $f(A)$ and so there exists an element b in A such that $f(b) = (y + f(a))/2 < f(a)$. Since f is increasing, $b < a$. Since $a = \sup \{a_n: n \in \mathbf{P}\}$, there exists a positive integer J such that $b < a_J \leq a$. Therefore, $f(b) \leq f(a_J) \leq y$. But this contradicts $f(b) = (y + f(a))/2 > y$. This contradiction shows that $y = f(a)$ and so $(f(a_n))$ converges to $f(a)$.

If (a_n) is decreasing, then $a_n \geq a$ for all n in \mathbf{P} . Since f is increasing, it follows that $f(a_n) \geq f(a)$ for all n in \mathbf{P} . Therefore, the sequence $(f(a_n))$ is bounded below and is decreasing since $i > k \Rightarrow a_i \leq a_k \Rightarrow f(a_i) \leq f(a_k)$. Thus, by the Monotone Convergence Theorem, $f(a_n) \rightarrow \inf \{f(a_k): k \in \mathbf{P}\} \geq f(a)$. Let $y = \inf \{f(a_k): k \in \mathbf{P}\}$. Now we claim that $y = f(a)$. As before we observe that $f(a_1) \geq y \geq f(a)$. Since $f(A)$ is an interval, $[f(a), f(a_1)] \subseteq f(A)$. Suppose on the contrary that $y \neq f(a)$. Then $y > f(a)$. Hence, the mid point $(y + f(a))/2$ between y

and $f(a)$ is in $[f(a), f(a_1)] \subseteq f(A)$ and so there exists an element b in A such that $f(b) = (y + f(a))/2 > f(a)$. Since f is increasing, $b > a$. Since $a = \inf\{a_n : n \in \mathbf{P}\}$, there exists a positive integer k such that $b > a_k \geq a$. Therefore, $f(b) \geq f(a_k) \geq y$. But $f(b) = (y + f(a))/2 < y$. This contradiction shows that $y = f(a)$ and so $(f(a_n))$ converges to $f(a)$.

If f is decreasing, then $-f$ is increasing. If $f(A)$ is an interval, then $(-f)(A)$ is also an interval. Thus, by what we have just proved, if (a_n) is a monotone sequence converging to a , then $-f(a_n) \rightarrow -f(a)$ and so $f(a_n) \rightarrow f(a)$. This completes the proof.

Theorem 22. Let A be a non-empty subset of \mathbf{R} . Suppose $f: A \rightarrow \mathbf{R}$ is a monotone function such that the image $f(A)$ is an interval. Then f is a continuous function.

Proof. We shall assume that f is increasing. Take any element a in A . We shall show that f is continuous at a by showing that if (a_n) is any sequence in A converging to a , then $f(a_n) \rightarrow f(a)$. Thus, suppose $a_n \rightarrow a$. Then by Proposition 20 we have the four possible consequences (1) to (4) as stated there.

For case (1), there is a decreasing subsequence (a_{k_j}) of (a_n) defined by the peaks of (a_n) and an increasing subsequence (a_{l_j}) of (a_n) defined by the troughs of (a_n) . Both subsequences converge to a . By Theorem 21, $f(a_{k_j}) \rightarrow f(a)$ and $f(a_{l_j}) \rightarrow f(a)$. Therefore, given any $\varepsilon > 0$, there exist positive integers N_1 and N_2 such that

$$j \geq N_1 \Rightarrow |f(a_{k_j}) - f(a)| < \varepsilon \quad \text{-----} \quad (1)$$

and

$$j \geq N_2 \Rightarrow |f(a_{l_j}) - f(a)| < \varepsilon. \quad \text{-----} \quad (2)$$

Let $N = \max\{k_{N_1}, l_{N_2}\}$. Then $n \geq N \Rightarrow a_n \leq a_{k_{N_1}}$ and $a_n \geq a_{l_{N_2}}$ by definition of peak and trough indices. For any $n \geq N$, either $a_n \geq a$ or $a_n < a$. If $a_n \geq a$, then $a \leq a_n \leq a_{k_{N_1}}$. Since f is increasing, $f(a) \leq f(a_n) \leq f(a_{k_{N_1}})$ and so $|f(a_n) - f(a)| \leq |f(a_{k_{N_1}}) - f(a)| < \varepsilon$ by (1). If $a_n < a$, then $a_{l_{N_2}} \leq a_n \leq a$. Hence, $f(a_{l_{N_2}}) \leq f(a_n) \leq f(a)$ and it follows that $|f(a_n) - f(a)| \leq |f(a_{l_{N_2}}) - f(a)| < \varepsilon$ by (2). Therefore, $n \geq N \Rightarrow |f(a_n) - f(a)| < \varepsilon$. This means $f(a_n) \rightarrow f(a)$.

For case (2) we have a decreasing subsequence (a_{k_j}) given by the peaks of (a_n) and there exists a positive integer L such that $n \geq L \Rightarrow a_n \geq a$. Then given any $\varepsilon > 0$, take $N = \max\{k_{N_1}, L\}$. Then $n \geq N \Rightarrow a \leq a_n \leq a_{k_{N_1}}$. It follows that $f(a) \leq f(a_n) \leq f(a_{k_{N_1}})$ and $|f(a_n) - f(a)| \leq |f(a_{k_{N_1}}) - f(a)| < \varepsilon$ by (1). Thus, $f(a_n) \rightarrow f(a)$.

For case (3) we have an increasing subsequence (a_{l_j}) of (a_n) defined by the troughs of (a_n) and there exists an integer K such that for all $n \geq K$, $a_n \leq a$. Let $N = \max\{l_{N_2}, K\}$. Then $n \geq N \Rightarrow a_{l_{N_2}} \leq a_n \leq a$ by the definition of trough index since $n \geq l_{N_2}$. It follows, as f is increasing, that $f(a_{l_{N_2}}) \leq f(a_n) \leq f(a)$ for $n \geq N$. Thus, $n \geq N \Rightarrow |f(a_n) - f(a)| \leq |f(a_{l_{N_2}}) - f(a)| < \varepsilon$ by (2). Hence, $f(a_n) \rightarrow f(a)$.

Case (4) is trivial since there exists an integer N such that $n \geq N \Rightarrow a_n = a \Rightarrow f(a_n) = f(a)$.

If f is decreasing, then $-f$ is increasing and the image $(-f)(A) = -f(A)$ is still an interval. Hence, by what we have just proved, $-f$ is continuous. Therefore, f is continuous.

This completes the proof.

Remark. A proof using the topological definition of continuity in terms of open sets is shorter. We only need to show that the inverse image of an open set, in this case an open interval, is (relatively) open.

Theorem 23. Suppose I is an interval and $f: I \rightarrow \mathbf{R}$ is a strictly monotone function. Then the inverse function $f^{-1}: f(I) \rightarrow I$ is continuous.

Proof. Since f is strictly monotone, its inverse f^{-1} is also strictly monotone. Since the image of the inverse function f^{-1} is I , which is an interval, by Theorem 22, f^{-1} is continuous.

Theorem 24. Suppose I is an interval and $f: I \rightarrow \mathbf{R}$ is a monotone function. Then f is continuous if and only if the range of f , $f(I)$ is an interval.

Proof. If f is continuous, then by Theorem 14, the image of I or the range of f is an interval.

Conversely, if the image $f(I)$ is an interval, then by Theorem 22, f is continuous.

The next result expresses that for a continuous function defined on an interval, injectivity is equivalent to strict monotonicity. We present a technical result to begin with.

Proposition 25. Suppose I is an interval and $f: I \rightarrow \mathbf{R}$ is continuous and injective. Then for any x, y and z in I with $x < y < z$ either $f(x) < f(y) < f(z)$ or $f(x) > f(y) > f(z)$. Hence we have

- (i) If $f(x) < f(y)$ or $f(x) < f(z)$ or $f(y) < f(z)$, then $f(x) < f(y) < f(z)$.
- (ii) If $f(x) > f(y)$ or $f(x) > f(z)$ or $f(y) > f(z)$, then $f(x) > f(y) > f(z)$.

Proof. Suppose $x < y < z$. Then $(x, y) \cap (y, z) = \emptyset$. Since f is injective, this implies that $f((x, y)) \cap f((y, z)) = \emptyset$. We have then the following possibilities regarding $f(x)$, $f(y)$ and $f(z)$:

Case (1) $f(x) < f(y)$ and $f(y) < f(z)$.

Case (2) $f(x) < f(y)$ and $f(y) > f(z)$.

Case (3) $f(x) > f(y)$ and $f(y) < f(z)$.

Case (4) $f(x) > f(y)$ and $f(y) > f(z)$.

Then by the Intermediate Value Theorem, since I is an interval, we have the following conclusions according to each case above:

(1) $(f(x), f(y)) \subseteq f((x, y))$ and $(f(y), f(z)) \subseteq f((y, z))$;

(2) $(f(x), f(y)) \subseteq f((x, y))$ and $(f(z), f(y)) \subseteq f((y, z))$;

(3) $(f(y), f(x)) \subseteq f((x, y))$ and $(f(y), f(z)) \subseteq f((y, z))$;

(4) $(f(y), f(x)) \subseteq f((x, y))$ and $(f(z), f(y)) \subseteq f((y, z))$;

Case (2) implies that $(f(x), f(y)) \cap (f(z), f(y)) = (\max(f(x), f(z)), f(y)) \neq \emptyset$. But $(f(x), f(y)) \cap (f(z), f(y)) \subseteq f((x, y)) \cap f((y, z)) = \emptyset$ and so $(f(x), f(y)) \cap (f(z), f(y)) = \emptyset$ contradicting $(f(x), f(y)) \cap (f(z), f(y)) \neq \emptyset$. Thus, Case (2) is not possible.

Similarly, case (3) implies that $(f(y), f(x)) \cap (f(y), f(z)) = (f(y), \min(f(x), f(z))) \neq \emptyset$. But $(f(y), f(x)) \cap (f(y), f(z)) \subseteq f((x, y)) \cap f((y, z)) = \emptyset$ and so $(f(y), f(x)) \cap (f(y), f(z)) = \emptyset$ contradicting $(f(y), f(x)) \cap (f(y), f(z)) \neq \emptyset$. Thus, Case (3) is not possible.

Therefore, we are left with cases (1) and (4). That is to say, $f(x) < f(y) < f(z)$ or $f(x) > f(y) > f(z)$. This completes the proof of the proposition.

Theorem 26. If I is an interval and $f: I \rightarrow \mathbf{R}$ is continuous. Then f is injective if and only if f is strictly monotone.

Proof. If f is strictly monotone, then plainly it is injective.

Suppose now f is injective and continuous.

Suppose for some x_1, x_2 in I with $x_1 < x_2$, we have that $f(x_1) < f(x_2)$. We shall show that then f is strictly increasing, i.e., for any y, z in I with $y < z$, $f(y) < f(z)$.

If $x_1 = y$ and $x_2 = z$, then we have nothing to show since $f(x_1) < f(x_2)$. If only one of y or z is equal to either x_1 or x_2 , then by Proposition 25 part (i) $f(y) < f(z)$. It remains to see the same conclusion can be reached when y and z are distinct from both x_1 or x_2 . By the total ordering on \mathbf{R} , we have the following six possibilities:

Case (1) $y < z < x_1 < x_2$;

Case (2) $y < x_1 < z < x_2$;

Case (3) $y < x_1 < x_2 < z$;

Case (4) $x_1 < y < z < x_2$;

Case (5) $x_1 < y < x_2 < z$;

Case (6) $x_1 < x_2 < y < z$.

For cases (1), (2) and (3), applying Proposition 25 Part (i), we obtained $f(y) < f(x_1)$ using the inequality $y < x_1 < x_2$ and the supposition $f(x_1) < f(x_2)$. Applying Proposition 4 Part (i) again, we have then $f(y) < f(z)$ since $f(y) < f(x_1)$ and either $y < x_1 < z$ or $y < z < x_1$.

For cases (4) and (5) since $x_1 < y < x_2$ and $f(x_1) < f(x_2)$, applying Proposition 4 Part (i), we get $f(y) < f(x_2)$. Then applying Proposition 25 Part (i) again we get $f(y) < f(z)$ since we now have $f(y) < f(x_2)$ and either $y < z < x_2$ or $y < x_2 < z$.

For case (6) Applying Proposition 25 part (i) gives us $f(x_2) < f(y)$. Therefore, applying again Proposition 25 Part (i) we get $f(y) < f(z)$ since $x_2 < y < z$. Hence f is strictly increasing.

Similarly, if $f(x_1) > f(x_2)$, we can show that for any y, z in I with $y < z$, we have that $f(y) > f(z)$. We only have to reverse the inequality in the images in the above proceeding and use Proposition 25 Part (ii) instead of Part (i). This means that f is strictly decreasing.

Therefore, f is strictly monotone. This completes the proof of Theorem 26.

Note that injectivity does not imply strict monotonicity as the following example will show. The theorem simply says that any example of an injective function defined on an interval and which is not monotone will have to be discontinuous.

Example

Define $g: \mathbf{R} \rightarrow \mathbf{R}$ by $g(x) = 2x$ if x is rational and $g(x) = -x$ if x is irrational. Then g is not continuous and g is injective but not monotone.

Corollary 27. If I is an interval and $f: I \rightarrow \mathbf{R}$ is continuous and injective, then the inverse function f^{-1} is continuous.

Proof. If f is continuous and injective, then by Theorem 26 f is strictly monotone and so by Theorem 23, its inverse f^{-1} is continuous.

Uniform continuity

Now we proceed to the idea of uniform continuity, which is often used in the development of integration. For a function $f: D \rightarrow \mathbf{R}$, continuity at a point by Definition 1 requires a δ to be found so that for all x in D ,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Thus, for different x , we may have different δ , i.e., δ depends on x . If we remove this dependency on x , then we get the idea of uniform continuity.

Definition 28. Let D be a subset of \mathbf{R} . A function $f: D \rightarrow \mathbf{R}$ is said to be *uniformly continuous* on D if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, y in D ,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Remark. Plainly, it follows from Definition 1 that if $f: D \rightarrow \mathbf{R}$ is uniformly continuous, then f is continuous.

Theorem 29. A continuous function defined on a closed and bounded interval $[a, b]$ is uniformly continuous.

Remark. In part inspired by the ideas of George Cantor and Weierstrass, Heine enunciated the notion of uniform continuity and proved the above Theorem using methods, which clearly spelt out the principle of being able to choose a finite covering from a countably infinite cover of $[a, b]$, a theorem independently stated by Emile Borel (1871-1956). We now recognize this as countable compactness. Heine used this principle in his proof of uniform continuity and the theorem on countable compactness is known as Heine-Borel Theorem. The extension of the Heine Borel Theorem to the case of uncountably infinite covering is credited to Lebesgue but was first published by Pierre Cousin (1867-1933) in 1895. We have proved the Heine Borel Theorem for countable compactness in Chapter 2 where we also proved the equivalence of sequential compactness and countable compactness for subsets of \mathbf{R} . The proof for the case of compactness without using the equivalence of compactness and countable compactness for metric spaces is given in my article "Closed and bounded sets, Heine Borel Theorem, Bolzano-Weierstrass Theorem, Uniform continuity and Riemann integrability".

Before we embark on the proof of Theorem 29, we shall show how a countable subfamily of open sets can be extracted from an arbitrary open cover. We shall be using the density of the rational numbers.

Any open set U is a countable union of intervals of the form (p, q) with $p < q$ and p and q are rational numbers. We deduce this as follows. For any a in U , there exists a $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$. Then, by the density of rational numbers, there exists rational numbers p and q such that $a - \delta < p < a$ and $a < q < a + \delta$ and so $a \in (p, q) \subseteq (a - \delta, a + \delta) \subseteq U$. Now the family $G = \{(p, q): p < q, p \text{ and } q \text{ are rational numbers}\}$ is a countable family of open sets. This is because it is indexed by the ordered pairs (p, q) with $p < q, p$ and $q \in \mathbf{Q}$ and so G is indexed by a subset of $\mathbf{Q} \times \mathbf{Q}$ which is countable. Hence, G is countable. Therefore any subset of G is countable. Consider the family $H = \{\text{open interval } (p, q): p, q \text{ rational and } (p, q) \subseteq U\}$. Then H is a subset of G and so is countable. Obviously $\cup \{V: V \in H\} = U$. Thus, any open set is a countable union of members from G , i.e., a countable union of open intervals of the form (p, q) with $p < q$ and p and q are rational numbers.

Theorem 30. Suppose A is a subset of \mathbf{R} and \mathcal{C} is any open cover of A by open subsets of \mathbf{R} . Then \mathcal{C} has a countable subcover.

Proof. Consider the family $F = \{ B \in \mathcal{C} : B \subseteq V, \text{ for some } V \in \mathcal{C} \}$. Then F is a cover for A . This is deduced as follows. Take any $x \in A$, then since \mathcal{C} is an open cover of A , there exists a member V in \mathcal{C} such that $x \in V$. Then since V is open, V is a union of members from \mathcal{C} . Hence, there exists B in F such that $x \in B \subseteq V$. Hence, $B \in F$. Thus, $A \subseteq \cup \{B : B \in F\}$ and so F is an open cover of A . F is countable since F is a subset of \mathcal{C} and so F is a countable open cover of A . We now use this cover to extract a countable subcover of \mathcal{C} . Now for each B in F choose a member V_B in \mathcal{C} such that $B \subseteq V_B$. Let now $H = \{V_B : B \in F\}$. Then obviously H is a subfamily of \mathcal{C} . H is countable because it is indexed by F and F is countable. Note that $\cup \{B : B \in F\} \subseteq \cup \{V_B : B \in F\} = \cup \{U : U \in H\}$. Hence, $A \subseteq \cup \{U : U \in H\}$ and so H is a cover for A . Therefore, H is a countable subcover of \mathcal{C} .

Remark.

The proof given above can be adapted to a proof that for a C_2 topological space, every open cover has a countable subcover (Lindelöf Theorem). A C_2 topological space is a topological space having a countable base for its topology. For the real numbers, \mathbf{R} , G is a countable base for the usual metric topology on \mathbf{R} . The density of the rational numbers plays an important part in the proof of Theorem 30.

Proof of Theorem 29.

Let $f : [a, b] \rightarrow \mathbf{R}$ be a continuous function. The most important fact here is that the interval $[a, b]$ is countably compact. Take any $\varepsilon > 0$. Then for each x in $[a, b]$, there exists $\delta_x > 0$ such that for all y in $[a, b]$,

$$|y - x| < \delta_x \Rightarrow |f(y) - f(x)| < \varepsilon/2. \text{ ----- (1)}$$

Then the family $F = \{(x - \delta_x/2, x + \delta_x/2) : x \in [a, b]\}$ is an open cover for $[a, b]$. Then by Theorem 30, F has a countable subcover. Then since $[a, b]$ is countably compact, the countable subcover has a finite subcover, i.e., it has n members for some positive integer n for a subcover. Let the subcover be $E = \{I(x_1, \delta_{x_1}/2), I(x_2, \delta_{x_2}/2), \dots, I(x_n, \delta_{x_n}/2)\}$, where we denote the open interval $(x_j - \delta_{x_j}/2, x_j + \delta_{x_j}/2)$ by $I(x_j, \delta_{x_j}/2)$ for $1 \leq j \leq n$. Let $\delta = \min\{\delta_{x_1}/2, \delta_{x_2}/2, \dots, \delta_{x_n}/2\}$.

Now take x, y in $[a, b]$ such that $|x - y| < \delta$. Since E covers $[a, b]$, $x \in I(x_k, \delta_{x_k}/2)$ for some k such that $1 \leq k \leq n$. Therefore, by (1) we have

$$|f(x) - f(x_k)| < \varepsilon/2 \text{ ----- (2)}$$

Now $|y - x_k| \leq |y - x| + |x - x_k| < \delta + \delta_{x_k}/2 \leq \delta_{x_k}/2 + \delta_{x_k}/2 = \delta_{x_k}$. Therefore, by (1),

$$|f(y) - f(x_k)| < \varepsilon/2 \text{ ----- (3)}$$

Hence, if $|x - y| < \delta$, $|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ by (2) and (3).

This means f is uniformly continuous.

Remark.

1. Note that the proof of Theorem 29 is easily adapted almost word for word to a proof of the general result: Suppose $f : D \rightarrow \mathbf{R}$ is a continuous function and D is a sequentially compact or countably compact or compact subset of \mathbf{R} . Then f is uniformly continuous.

2. One can use the sequential compactness of $[a, b]$ to prove Theorem 29. One proceeds by contradiction. Suppose f is not uniformly continuous. Then there exists a $\varepsilon > 0$ such that for each positive integer n , there exist x_n, y_n in $[a, b]$ such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Then since $[a, b]$ is sequentially compact, (x_n) has a convergent subsequence (x_{n_k}) converging to a point x in $[a, b]$. Then since $|x_n - y_n| < 1/n$, we see that $|y_{n_k} - x_{n_k}| < 1/n_k$ and so, (y_{n_k}) converges to the same limit x . Since f is continuous at x , both $(f(x_{n_k}))$ and $(f(y_{n_k}))$ converge to $f(x)$ and so $f(x_{n_k}) - f(y_{n_k}) \rightarrow 0$. But $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$ for all positive integer k and so $(f(x_{n_k}) - f(y_{n_k}))$ cannot converge to 0 and we have arrived at a contradiction.

We have used sequences to investigate if a function is continuous. We have the methods of sequences to decide if a sequence is convergent. This is particularly useful to test continuity by using various sequences to see if the corresponding image sequences under the function converge. It affords some way to look for a non convergent image sequence if the function is not continuous. The following is a similar criterion for uniform continuity using sequences.

Proposition 31. Suppose $f: D \rightarrow \mathbf{R}$ is a function. The following two statements are equivalent.

- (1) f is uniformly continuous.
- (2) For any two sequences (x_n) and (y_n) in $[a, b]$,
 $|x_n - y_n| \rightarrow 0 \Rightarrow |f(x_n) - f(y_n)| \rightarrow 0$.

Proof. (1) \Rightarrow (2) is easy just use Definition 28 (definition of uniform continuity) as in the proof of (A) \Rightarrow (B) of Theorem 12.

(2) \Rightarrow (1). We prove this by contradiction. Suppose on the contrary f is not uniformly continuous. That means there exists a $\varepsilon > 0$ such that for each positive integer n , there exist x_n, y_n in $[a, b]$ such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Plainly $|x_n - y_n| \rightarrow 0$ but $|f(x_n) - f(y_n)| \not\rightarrow 0$. This contradicts the assumption (2) that $|f(x_n) - f(y_n)| \rightarrow 0$. Hence, f must be uniformly continuous.

Example.

1. The following example is well known. The function $f: (0, 1) \rightarrow \mathbf{R}$ defined by $f(x) = 1/x$, is not uniformly continuous. Let $x_n = 1/(n+1)$ and $y_n = 1/(n+2)$. Then plainly $|x_n - y_n| \rightarrow 0$. But $|f(x_n) - f(y_n)| = |(n+1) - (n+2)| = 1$ and cannot converge to 0. Therefore, f is not uniformly continuous by Proposition 31. This is an example of a non uniformly continuous function defined on a bounded domain, which is not closed.

2. The function $f: \mathbf{R} \rightarrow \mathbf{R}$, defined by $f(x) = x^2$, is not uniformly continuous. Let $x_n = n$ and $y_n = n + 1/n$ for each positive integer n . Then $|x_n - y_n| = 1/n \rightarrow 0$. But $|f(x_n) - f(y_n)| = |n^2 - (n^2 + 2 + 1/n^2)| = 2 + 1/n^2 \rightarrow 2 \neq 0$. Thus, f is not uniformly continuous.

3. Plainly $f: \mathbf{R} \rightarrow \mathbf{R}$, defined by $f(x) = 3x$, is uniformly continuous.

Limits.

In Chapter 2 we have discussed limit points of a subset S of \mathbf{R} . The limit points may be or may not be in S . However, by Proposition 31 of Chapter 2, if a is a limit point of S , then there exists a sequence (x_n) in $S - \{a\}$ such that $x_n \rightarrow a$. So if f is a function defined on S ,

then we have the sequence $(f(x_n))$. Thus, this allows us to talk about the limit of a function f at a point not in the domain but is a limit point of the domain S .

Definition 32. Suppose $f: D \rightarrow \mathbf{R}$ is a function. Let a be a limit point of D . If there exists a number L such that for any sequence (x_n) in $D - \{a\}$, with $x_n \rightarrow a$, we have $f(x_n) \rightarrow L$, then we say the *limit* of f as x tends to a exists and equals L . We write $\lim_{x \rightarrow a} f(x) = L$.

Equivalently, $\lim_{x \rightarrow a} f(x) = L$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that
for all x in D , $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Remark.

1. It is an easy exercise to prove that if the limit exists, then it must be unique.
2. An examination of the definition of limit and the definition of continuity at a leads us to the following: Suppose a is a limit point of D , it is clear that f is continuous at a if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.

The following properties of limits are consequences of the properties of sequences.

Theorem 33. Suppose $f: D \rightarrow \mathbf{R}$ and $g: D \rightarrow \mathbf{R}$ are two functions and a is a limit point of D . Suppose that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

- (1) $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$,
- (2) $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = (\lim_{x \rightarrow a} f(x)) \cdot (\lim_{x \rightarrow a} g(x))$ and
- (3) if further $\lim_{x \rightarrow a} g(x) \neq 0$, then there exists $\delta > 0$ such that $g(x) \neq 0$ for all x in

$$(D - \{a\}) \cap (a - \delta, a + \delta) \text{ and } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}.$$

Proof. This follows from the convergence properties of sequences. For part 3, we need to have the quotient f/g defined in a small punctured neighbourhood $(D - \{a\}) \cap (a - \delta, a + \delta)$, hence the requirement that $g(x) \neq 0$ there. The proof is left as an easy exercise.

Hence, we can compute limit of a function as we would compute the limit of a sequence.

The next result that is often used is the property of limit with respect to composition.

Theorem 34. Suppose $f: D \rightarrow \mathbf{R}$ and $g: V \rightarrow \mathbf{R}$ are two functions such that $f(D - \{a\}) \subseteq V$. Thus, the composite $g \circ f: D - \{a\} \rightarrow \mathbf{R}$ is defined. Suppose a is a limit point of D and b is a limit point of V .

- (A) Suppose $\lim_{x \rightarrow a} f(x) = b$ and $\lim_{y \rightarrow b} g(y) = c$. Suppose that for $x \in D$ and $x \neq a$, $f(x) \neq b$.
Then $\lim_{x \rightarrow a} g \circ f(x) = c$.
- (B) Suppose $\lim_{x \rightarrow a} f(x) = b$ and $\lim_{y \rightarrow b} g(y) = c$. Suppose g is continuous at b , i.e., $g(b) = c$,
then $\lim_{x \rightarrow a} g \circ f(x) = c$.

Proof. Let $\varepsilon > 0$ be given. Then since $\lim_{y \rightarrow b} g(y) = c$, there exists $\delta_1 > 0$ such that

$$\text{for all } y \text{ in } V, 0 < |y - b| < \delta_1 \Rightarrow |g(y) - c| < \varepsilon. \text{ ----- (1)}$$

Since $\lim_{x \rightarrow a} f(x) = b$, there exists $\delta > 0$ such that

$$\text{for all } x \text{ in } D, 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \delta_1 \text{ ----- (2)}$$

- (A) Suppose $f(x) \neq b$ for $x \neq a$. Then by (2) for all x in D ,
 $0 < |x - a| < \delta \Rightarrow 0 < |f(x) - b| < \delta_1 \Rightarrow |g(f(x)) - c| < \varepsilon$ by (1).

Hence, $\lim_{x \rightarrow a} g \circ f(x) = c$.

(B) Suppose g is continuous at b , i.e., $g(b) = c$, then we have in place of (1), that there exists $\delta_1 > 0$ such that

$$\text{for all } y \text{ in } V, |y - b| < \delta_1 \Rightarrow |g(y) - g(b)| < \varepsilon. \text{ ----- (3)}$$

Therefore, by (2) for all x in D ,

$$0 < |x - a| < \delta \Rightarrow |f(x) - b| < \delta_1 \Rightarrow |g(f(x)) - g(b)| < \varepsilon \text{ by (3).}$$

Hence, $\lim_{x \rightarrow a} g \circ f(x) = g(b) = c$.

Remark

1. We can define left and right limits for functions defined on subset of \mathbf{R} in a similar way. Of course we would require a to be a limit point of $(a, \infty) \cap D$ for right limit and a limit point of $(-\infty, a) \cap D$ for left limit. One often used criterion is that limit exists if both left and right limits exist and are the same. It offers some technical help in computing limits.

2. The notion of limit is important for differentiation as derivative is a limit of a function. To investigate further the behaviour of a function near a limit point, other refined forms of limit, such as $\limsup_{x \rightarrow a} f(x)$ and $\liminf_{x \rightarrow a} f(x)$ and the one sided versions of these two limits are used. When applied to the difference quotient of a function at a point, they lead to the four Dini derivatives and to the Denjoy-Saks-Young Theorem on a real valued function defined on $[a, b]$, which gives possibilities for the values of Dini derivatives in the extended real numbers. We can use these derivatives to investigate the behaviour of function, which may not be differentiable everywhere. This belongs to advanced area of analysis.

Definition 35.

Suppose a is a limit point of $(a, \infty) \cap D$ and $f: D \rightarrow \mathbf{R}$ is a function. Then the limit of the function $f|_{(a, \infty) \cap D}: (a, \infty) \cap D \rightarrow \mathbf{R}$ is called the *right limit* of f at a . It is denoted by $\lim_{x \rightarrow a^+} f(x)$. Similarly suppose a is a limit point of $(-\infty, a) \cap D$. Then the limit of the function $f|_{(-\infty, a) \cap D}: (-\infty, a) \cap D \rightarrow \mathbf{R}$ is called the *left limit* of f at a . We denote this left limit by $\lim_{x \rightarrow a^-} f(x)$.

Plainly, if a is a limit point of $(a, \infty) \cap D$, then a is a limit point of D . Likewise a limit point of $(-\infty, a) \cap D$ is also a limit point of D .

Theorem 36. Suppose a is a limit point of $(a, \infty) \cap D$ and of $(-\infty, a) \cap D$ and $f: D \rightarrow \mathbf{R}$ is a function defined on D . Then $\lim_{x \rightarrow a} f(x)$ exists if and only if both the left and right limits of f at a exist and are the same.

Proof. Suppose $\lim_{x \rightarrow a} f(x)$ exists and equals L . This means for any sequence (x_n) in $D - \{a\}$ with $x_n \rightarrow a$, $f(x_n) \rightarrow L$. Therefore, for any sequence (x_n) in $(a, \infty) \cap D \subseteq D - \{a\}$ with $x_n \rightarrow a$ we have that $f(x_n) \rightarrow L$. Hence $\lim_{x \rightarrow a^+} f(x) = L$. Similarly, it follows that $\lim_{x \rightarrow a^-} f(x) = L$. Conversely suppose $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$. Take any sequence (x_n) in $D - \{a\}$. Consider the subsets $S = \{n : x_n > a\}$ and $T = \{n : x_n < a\}$. If S is finite, then T is infinite. Let $M = \max S$. Note that the limit of $(f(x_n))$ is the same as the limit of $(f(x_{n+M}))$ and that for all counting number n , $x_{n+M} < a$. Thus since $\lim_{x \rightarrow a^-} f(x) = L$, $f(x_{n+M}) \rightarrow L$. Therefore, $f(x_n) \rightarrow L$.

Similarly, if T is finite, then S must be infinite and we can show in the same way that $f(x_n) \rightarrow L$. Suppose now both S and T are infinite. Let $S = \{s_1, s_2, \dots\}$, where $s_1 < s_2 < \dots$ and let $T = \{t_1, t_2, \dots\}$, where $t_1 < t_2 < \dots$. Then since $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$, $f(x_{t_k}) \rightarrow L$ and $f(x_{s_k}) \rightarrow L$. This means given $\varepsilon > 0$, there exists a positive integer N_1 such that $k \geq N_1 \Rightarrow |f(x_{t_k}) - L| < \varepsilon$ and there exists a positive integer N_2 such that $k \geq N_2 \Rightarrow |f(x_{s_k}) - L| < \varepsilon$. Let $N = \max\{N_1, N_2\}$. Then $n \geq N \Rightarrow n \geq t_{N_1}$ and $n \geq s_{N_2}$. Now n is either equal to t_k or s_k for some k . If n is equal to t_k , then $t_k = n \geq t_{N_1}$ and consequently $k \geq N_1$ and so $|f(x_n) - L| = |f(x_{t_k}) - L| < \varepsilon$. If n is equal to s_k , then $s_k = n \geq s_{N_2}$ and consequently $k \geq N_2$ and we have $|f(x_n) - L| = |f(x_{s_k}) - L| < \varepsilon$. Hence, in either case we have $|f(x_n) - L| < \varepsilon$. Therefore, $f(x_n) \rightarrow L$. Thus $\lim_{x \rightarrow a} f(x)$ exists.

The next result is a useful tool in computing limits.

Theorem 37 Squeeze Theorem.

Suppose a is a limit point of D . Suppose f , g and h are functions defined on D such that $g(x) \leq f(x) \leq h(x)$ for any x in $D - \{a\}$. Suppose the limits of g and h at a exist and are the same, i.e., $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$. Then $\lim_{x \rightarrow a} f(x)$ exists and is equal to L .

Proof. This follows immediately from the Squeeze Theorem for sequences. For any sequence (x_n) in $D - \{a\}$ with $x_n \rightarrow a$, $g(x_n) \rightarrow L$ and $h(x_n) \rightarrow L$. Therefore, by the Squeeze Theorem for sequence (Theorem 13 Chapter 2), $f(x_n) \rightarrow L$. This means $\lim_{x \rightarrow a} f(x)$ exists and is equal to L .

Recall what it means for a sequence (a_n) of real numbers to converge to either $+\infty$ or $-\infty$. (see Definition 28 Chapter 2). We have the corresponding notion of the limit of a function tending to either $+\infty$ or $-\infty$. We shall be concerned mainly with this kind of behaviour either at a point or at $+\infty$ or $-\infty$. The definitions can be easily adapted from the definition of a limit of a function (see Definition 32).

Example 38

- Given that $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$, $\lim_{x \rightarrow 0} \frac{\sin(\sin(x))}{\sin(x)} = 1$. Let $g(x) = \frac{\sin(x)}{x}$ for $x \neq 0$ and $f(x) = \sin(x)$. Then $f(x) = \sin(x) \neq 0$ for $x \neq 0$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus by Theorem 34

$$(A), \lim_{x \rightarrow 0} \frac{\sin(\sin(x))}{\sin(x)} = \lim_{x \rightarrow 0} g(f(x)) = 1.$$

- $\lim_{x \rightarrow 0} \frac{x}{\sin^{-1}(x)} = 1$. Let $h(x) = \sin^{-1}(x)$. Then $h(x) \neq 0$ for $x \neq 0$. Therefore, by Theorem

$$34 (A), \lim_{x \rightarrow 0} g(h(x)) = \lim_{x \rightarrow 0} \frac{\sin(\sin^{-1}(x))}{\sin^{-1}(x)} = \lim_{x \rightarrow 0} \frac{x}{\sin^{-1}(x)} = 1, \text{ where } g \text{ is as defined in example 1 above.}$$

- Given that $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$, $\lim_{x \rightarrow 0} \frac{x}{\ln(1+x)} = 1$. Let $g(x) = \frac{e^x - 1}{x}$ and $f(x) = \ln(1+x)$

Note that $f(x) \neq 0$ for $x \neq 0$. Therefore, by Theorem 34 (A)

$$\lim_{x \rightarrow 0} g(f(x)) = \lim_{x \rightarrow 0} \frac{e^{\ln(1+x)} - 1}{\ln(1+x)} = \lim_{x \rightarrow 0} \frac{x}{\ln(1+x)} = 1.$$

4. Given that $\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0$, $\lim_{x \rightarrow 0} \sin\left(\frac{\cos(x) - 1}{x}\right) = 0$. Let $g(x) = \sin(x)$ and $f(x) = \frac{\cos(x) - 1}{x}$ for $x \neq 0$. Then since g is continuous at $x = 0$, Theorem 34 (B), $\lim_{x \rightarrow 0} \sin\left(\frac{\cos(x) - 1}{x}\right) = \lim_{x \rightarrow 0} g(f(x)) = 0$.
5. $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$
 For $x \neq 0$, $-x^2 \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq x^2$. Since $\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} -x^2 = 0$, by the Squeeze Theorem (Theorem 37), $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$.
6. In Theorem 34 (A), if g is not continuous at b , then it is necessary that there is a small interval $I = (a - \delta, a + \delta)$ such that for $x \in D \cap I$ and $x \neq a$, $f(x) \neq b$.
 For example let $g(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ and $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Then $\lim_{x \rightarrow 0} f(x) = 0$ by the Squeeze Theorem (Theorem 37) and $\lim_{y \rightarrow 0} g(y) = \lim_{y \rightarrow 0} \frac{\sin(y)}{y} = 1$. But $\lim_{x \rightarrow 0} g(f(x)) \neq 1$. Actually $\lim_{x \rightarrow 0} g(f(x))$ does not exist. We deduce this as follows. For each integer $n \geq 1$, let $x_n = \frac{1}{n\pi}$. Then for all integer $n \geq 1$, $x_n \neq 0$. The sequence (x_n) converges to 0. But for each integer $n \geq 1$, $g(f(x_n)) = g(0) = 0$ and so $g(f(x_n)) \rightarrow 0$ as $n \rightarrow \infty$. Take the sequence (y_n) where $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ for each integer $n \geq 1$. Then $y_n \neq 0$ for each integer $n \geq 1$ and $y_n \rightarrow 0$ as $n \rightarrow \infty$. Then $g(f(y_n)) = g(y_n) = \frac{\sin(y_n)}{y_n} \rightarrow 1$ as $n \rightarrow \infty$. Thus we have two sequences (x_n) and (y_n) in $\mathbf{R} - \{0\}$ both converging to 0 but their images $(g(f(x_n)))$ and $(g(f(y_n)))$ converge to distinct values. Therefore, by Definition 32, $\lim_{x \rightarrow 0} g(f(x))$ does not exist.

Infinity as limits

Definition 39.

Let D be a subset of \mathbf{R} . Suppose $f: D \rightarrow \mathbf{R}$ is a function. Let a be a limit point of D .

If for any sequence (x_n) in $D - \{a\}$, with $x_n \rightarrow a$, we have $f(x_n) \rightarrow +\infty$, then we say the *limit* of f as x tends to a is $+\infty$. We write $\lim_{x \rightarrow a} f(x) = +\infty$. It is important to note that $\lim_{x \rightarrow a} f(x) = +\infty$ implies that the limit at a does not exist in the ordinary sense (in the sense of Definition 32 Chapter 3) as $+\infty$ is not a real number. *Equivalently*, $\lim_{x \rightarrow a} f(x) = +\infty$ if given any $K > 0$, there exists $\delta > 0$ such that

$$\text{for all } x \text{ in } D, 0 < |x - a| < \delta \Rightarrow f(x) > K.$$

If for any sequence (x_n) in $D - \{a\}$, with $x_n \rightarrow a$, we have $f(x_n) \rightarrow -\infty$, then we say the *limit* of f as x tends to a is $-\infty$. We write $\lim_{x \rightarrow a} f(x) = -\infty$. Note again that $\lim_{x \rightarrow a} f(x) = -\infty$ implies that the limit at a does not exist in the ordinary sense (in the sense of Definition 32 Chapter 3) as $-\infty$ is not a real number. *Equivalently*, $\lim_{x \rightarrow a} f(x) = -\infty$ if given any $K < 0$, there exists $\delta > 0$ such that

$$\text{for all } x \text{ in } D, 0 < |x - a| < \delta \Rightarrow f(x) < K.$$

Remark.

Although it may be convenient to work with functions into the extended real numbers which include $+\infty$ and $-\infty$, the properties we seek concern largely with real numbers and it is best to stick to just the real numbers. Thus the formalism $\lim_{x \rightarrow a} f(x) = +\infty$ or $\lim_{x \rightarrow a} f(x) = -\infty$ is just a shorthand for the definition.

We can make similar definition for the left and right derivative.

Definition 40.

Suppose a is a limit point of $(a, \infty) \cap D$ and $f: D \rightarrow \mathbf{R}$ is a function. Then the limit of the function $f|_{(a, \infty) \cap D}: (a, \infty) \cap D \rightarrow \mathbf{R}$ is called the *right limit* of f at a .

It is denoted by $\lim_{x \rightarrow a^+} f(x)$. We write $\lim_{x \rightarrow a^+} f(x) = +\infty$ if $\lim_{x \rightarrow a^+} f|_{(a, \infty) \cap D}(x) = +\infty$.

Equivalently, $\lim_{x \rightarrow a^+} f(x) = +\infty$ if given any $K > 0$, there exists $\delta > 0$ such that for all x in D , $0 < x - a < \delta \Rightarrow f(x) > K$.

We write $\lim_{x \rightarrow a^+} f(x) = -\infty$ if $\lim_{x \rightarrow a^+} f|_{(a, \infty) \cap D}(x) = -\infty$.

Equivalently, $\lim_{x \rightarrow a^+} f(x) = -\infty$ if given any $K < 0$, there exists $\delta > 0$ such that for all x in D , $0 < x - a < \delta \Rightarrow f(x) < K$.

Similarly suppose a is a limit point of $(-\infty, a) \cap D$. Then the limit of the function $f|_{(-\infty, a) \cap D}: (-\infty, a) \cap D \rightarrow \mathbf{R}$ is called the *left limit* of f at a . We denote this left limit by $\lim_{x \rightarrow a^-} f(x)$.

We write $\lim_{x \rightarrow a^-} f(x) = +\infty$ if $\lim_{x \rightarrow a^-} f|_{(-\infty, a) \cap D}(x) = +\infty$.

Equivalently, $\lim_{x \rightarrow a^-} f(x) = +\infty$ if given any $K > 0$, there exists $\delta > 0$ such that for all x in D , $0 < a - x < \delta \Rightarrow f(x) > K$.

We write $\lim_{x \rightarrow a^-} f(x) = -\infty$ if $\lim_{x \rightarrow a^-} f|_{(-\infty, a) \cap D}(x) = -\infty$.

Equivalently, $\lim_{x \rightarrow a^-} f(x) = -\infty$ if given any $K < 0$, there exists $\delta > 0$ such that for all x in D , $0 < a - x < \delta \Rightarrow f(x) < K$.

Definition 41.

Let D be a subset of \mathbf{R} and $f: D \rightarrow \mathbf{R}$ is a function.

Suppose D is *not bounded above*. Suppose L is a real number.

If for any sequence (x_n) in D , with $x_n \rightarrow +\infty$, we have that $f(x_n) \rightarrow L$ then we say the *limit* of f as x tends to $+\infty$ is L . We write $\lim_{x \rightarrow +\infty} f(x) = L$.

Equivalently, $\lim_{x \rightarrow +\infty} f(x) = L$ if given any $\varepsilon > 0$, there exists a real number $K > 0$ such that for all x in D , $x > K \Rightarrow |f(x) - L| < \varepsilon$.

If for any sequence (x_n) in D , with $x_n \rightarrow +\infty$, we have that $f(x_n) \rightarrow +\infty$ then we say the *limit* of f as x tends to $+\infty$ is $+\infty$. We write $\lim_{x \rightarrow +\infty} f(x) = +\infty$.

Equivalently, $\lim_{x \rightarrow +\infty} f(x) = +\infty$ if given any $J > 0$, there exists a real number $K > 0$ such that for all x in D , $x > K \Rightarrow f(x) > J$.

Similarly, if for any sequence (x_n) in D , with $x_n \rightarrow +\infty$, we have that $f(x_n) \rightarrow -\infty$ then we say the *limit* of f as x tends to $+\infty$ is $-\infty$. We write $\lim_{x \rightarrow +\infty} f(x) = -\infty$.

Equivalently, $\lim_{x \rightarrow +\infty} f(x) = -\infty$ if given any $J < 0$, there exists a real number $K > 0$ such that

for all x in D , $x > K \Rightarrow f(x) < J$.

Definition 42.

Let D be a subset of \mathbf{R} and $f: D \rightarrow \mathbf{R}$ is a function.

Suppose D is *not bounded below*. Suppose L is a real number.

If for any sequence (x_n) in D , with $x_n \rightarrow -\infty$, we have that $f(x_n) \rightarrow L$ then we say the *limit* of f as x tends to $-\infty$ is L . We write $\lim_{x \rightarrow -\infty} f(x) = L$.

Equivalently, $\lim_{x \rightarrow -\infty} f(x) = L$ if given any $\varepsilon > 0$, there exists a real number $K < 0$ such that for all x in D , $x < K \Rightarrow |f(x) - L| < \varepsilon$.

If for any sequence (x_n) in D , with $x_n \rightarrow -\infty$, we have that $f(x_n) \rightarrow +\infty$ then we say the *limit* of f as x tends to $-\infty$ is $+\infty$. We write $\lim_{x \rightarrow -\infty} f(x) = +\infty$.

Equivalently, $\lim_{x \rightarrow -\infty} f(x) = +\infty$ if given any $J > 0$, there exists a real number $K < 0$ such that for all x in D , $x < K \Rightarrow f(x) > J$.

Similarly, if for any sequence (x_n) in D , with $x_n \rightarrow -\infty$, we have that $f(x_n) \rightarrow -\infty$ then we say the *limit* of f as x tends to $-\infty$ is $-\infty$. We write $\lim_{x \rightarrow -\infty} f(x) = -\infty$.

Equivalently, $\lim_{x \rightarrow -\infty} f(x) = -\infty$ if given any $J < 0$, there exists a real number $K < 0$ such that for all x in D , $x < K \Rightarrow f(x) < J$.

Remark.

Although it may be convenient to work with functions into the extended real numbers which include $+\infty$ and $-\infty$, the properties we seek concern largely with real numbers and it is best to stick to just the real numbers. Thus the formalism $\lim_{x \rightarrow a} f(x) = +\infty$ or $\lim_{x \rightarrow a} f(x) = -\infty$, where a may be replaced by a^+ , a^- , $+\infty$ or $-\infty$, is just a shorthand for the respective definition.

The next two theorems are the analogues of Theorem 33.

Theorem 43. Suppose $f: D \rightarrow \mathbf{R}$ and $g: D \rightarrow \mathbf{R}$ are two functions and D is not bounded above. Suppose $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow +\infty} g(x)$ both exist (*as finite number*). Then

- (1) $\lim_{x \rightarrow +\infty} (f(x) \pm g(x)) = \lim_{x \rightarrow +\infty} f(x) \pm \lim_{x \rightarrow +\infty} g(x)$,
- (2) $\lim_{x \rightarrow +\infty} (f(x) \cdot g(x)) = (\lim_{x \rightarrow +\infty} f(x)) \cdot (\lim_{x \rightarrow +\infty} g(x))$ and
- (3) if further $\lim_{x \rightarrow +\infty} g(x) \neq 0$, then $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow +\infty} f(x)}{\lim_{x \rightarrow +\infty} g(x)}$.

Proof. This follows from the convergence properties of sequences.

Similar results for limits of functions as x tends to $-\infty$ holds.

Theorem 44. Suppose $f: D \rightarrow \mathbf{R}$ and $g: D \rightarrow \mathbf{R}$ are two functions and D is not bounded below. Suppose $\lim_{x \rightarrow -\infty} f(x)$ and $\lim_{x \rightarrow -\infty} g(x)$ both exist (*as finite number*). Then

- (1) $\lim_{x \rightarrow -\infty} (f(x) \pm g(x)) = \lim_{x \rightarrow -\infty} f(x) \pm \lim_{x \rightarrow -\infty} g(x)$,
- (2) $\lim_{x \rightarrow -\infty} (f(x) \cdot g(x)) = (\lim_{x \rightarrow -\infty} f(x)) \cdot (\lim_{x \rightarrow -\infty} g(x))$ and
- (3) if further $\lim_{x \rightarrow -\infty} g(x) \neq 0$, then $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow -\infty} f(x)}{\lim_{x \rightarrow -\infty} g(x)}$.

Similarly we have the Squeeze Theorem too.

Theorem 45 Squeeze Theorem.

Suppose f , g and h are functions defined on D .

(1) Suppose D is not bounded above and there is a real number K such that $g(x) \leq f(x) \leq h(x)$ for all $x > K$ in D . Suppose $\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} h(x) = L$. Then $\lim_{x \rightarrow +\infty} f(x)$ exists and is equal to L .

(2) Suppose D is not bounded below and there is a real number K such that $g(x) \leq f(x) \leq h(x)$ for all $x < K$ in D . Suppose $\lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow -\infty} h(x) = L$. Then $\lim_{x \rightarrow -\infty} f(x)$ exists and is equal to L .

Proof.

(1) This follows immediately from the Squeeze Theorem for sequences. For any sequence (x_n) in D with $x_n \rightarrow +\infty$, $g(x_n) \rightarrow L$ and $h(x_n) \rightarrow L$. Since $x_n \rightarrow +\infty$, there exists an integer N such that for all $n \geq N$, $x_n > K$. Consider the subsequence (x_{n+N}) . Then by supposition $g(x_{n+N}) \leq f(x_{n+N}) \leq h(x_{n+N})$ for all $n \geq 1$. Therefore, by the Squeeze Theorem for sequences (Theorem 13 Chapter 2), $f(x_{n+N}) \rightarrow L$. It follows that $f(x_n) \rightarrow L$. This means $\lim_{x \rightarrow +\infty} f(x)$ exists and is equal to L .

The proof of part (2) is similar.

The next result is a comparison theorem.

Theorem 46. Suppose f and g are functions defined on D .

(1) Suppose D is not bounded above and there is a real number K such that $g(x) \leq f(x)$ for all $x > K$ in D . If $\lim_{x \rightarrow +\infty} g(x) = +\infty$, then $\lim_{x \rightarrow +\infty} f(x) = +\infty$. If $\lim_{x \rightarrow +\infty} f(x) = -\infty$, then $\lim_{x \rightarrow +\infty} g(x) = -\infty$.

(2) Suppose D is not bounded below and there is a real number K such that $g(x) \leq f(x)$ for all $x < K$ in D . If $\lim_{x \rightarrow -\infty} g(x) = +\infty$, then $\lim_{x \rightarrow -\infty} f(x) = +\infty$. If $\lim_{x \rightarrow -\infty} f(x) = -\infty$, then $\lim_{x \rightarrow -\infty} g(x) = -\infty$.

Proof. We shall prove only part (1). The proof of part (2) is similar.

If $\lim_{x \rightarrow +\infty} g(x) = +\infty$, then for any sequence (x_n) in D with $x_n \rightarrow +\infty$, $g(x_n) \rightarrow +\infty$. Therefore, for any real number $J > 0$, there exists a positive integer L such that $n \geq L \Rightarrow g(x_n) > J$.

Since $x_n \rightarrow +\infty$, there exists a positive integer N such that for all integer $n \geq N$, $x_n > K$. Let $M = \max\{L, N\} \geq N$. Therefore, by supposition, for all integer $n \geq M$, $g(x_n) \leq f(x_n)$. Hence $n \geq M \Rightarrow f(x_n) \geq g(x_n) > J$. This means $f(x_n) \rightarrow +\infty$. Therefore, by Definition 40 $\lim_{x \rightarrow +\infty} f(x) = +\infty$.

If $\lim_{x \rightarrow +\infty} f(x) = -\infty$, then for any sequence (x_n) in D with $x_n \rightarrow +\infty$, $f(x_n) \rightarrow -\infty$. Therefore, for any real number $J < 0$, there exists a positive integer L such that $n \geq L \Rightarrow f(x_n) < J$.

Since $x_n \rightarrow +\infty$, there exists a positive integer N such that for all integer $n \geq N$, $x_n > K$. Let $M = \max\{L, N\} \geq N$. Therefore, by supposition, for all integer $n \geq M$, $g(x_n) \leq f(x_n)$. Hence $n \geq M \Rightarrow g(x_n) \leq f(x_n) < J$. This means $g(x_n) \rightarrow -\infty$. Therefore, by Definition 40,

$$\lim_{x \rightarrow +\infty} g(x) = -\infty.$$

We next present the following useful results.

Theorem 47. Suppose $f: D \rightarrow \mathbf{R}$ is a function a is a limit point of D .

(1) If $\lim_{x \rightarrow a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \rightarrow a} \frac{1}{f(x)} = 0$.

(2) If $\lim_{x \rightarrow a} f(x) = 0$ and for some $\delta > 0$, $f(x) > 0$ for all x in $D \cap (a-\delta, a+\delta)$ except possibly at a , then $\lim_{x \rightarrow a} \frac{1}{f(x)} = +\infty$.

(3) If $\lim_{x \rightarrow a} f(x) = 0$ and for some $\delta > 0$, $f(x) < 0$ on $D \cap (a-\delta, a+\delta)$ except possibly at a , then $\lim_{x \rightarrow a} \frac{1}{f(x)} = -\infty$.

The statements are also true with $x \rightarrow a$ replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$.

Statement (1) is also true with $x \rightarrow a$ replaced by $x \rightarrow \pm \infty$.

Statement (2) is also true with $x \rightarrow a$ replaced by $x \rightarrow a^+$ and the condition be replaced by there exists $\delta > 0$, $f(x) > 0$ for all x in $D \cap (a, a+\delta)$ or with $x \rightarrow a$ replaced by $x \rightarrow a^-$ and the condition be replaced by there exists $\delta > 0$, $f(x) > 0$ for all x in $D \cap (a-\delta, a)$.

Statement (2) is also true with $x \rightarrow a$ replaced by $x \rightarrow a^+$ and the condition be replaced by there exists $\delta > 0$, $f(x) < 0$ for all x in $D \cap (a, a+\delta)$ or with $x \rightarrow a$ replaced by $x \rightarrow a^-$ and the condition be replaced by there exists $\delta > 0$, $f(x) < 0$ for all x in $D \cap (a-\delta, a)$.

Proof.

(1). Suppose $\lim_{x \rightarrow a} f(x) = +\infty$. Given any $\varepsilon > 0$. By the Archimedean property of \mathbf{R} , there exists a positive integer N such that $0 < \frac{1}{N} < \varepsilon$. Since $\lim_{x \rightarrow a} f(x) = +\infty$, there exists $\delta > 0$ such that for all x in D ,

$$0 < |x - a| < \delta \Rightarrow f(x) > N.$$

Therefore, for all x in D ,

$$0 < |x - a| < \delta \Rightarrow \frac{1}{f(x)} < \frac{1}{N} < \varepsilon.$$

Hence, $\lim_{x \rightarrow a} \frac{1}{f(x)} = 0$.

(2). Suppose $\lim_{x \rightarrow a} f(x) = 0$. By assumption $f(x) > 0$ for all x in $D \cap (a-\delta, a+\delta)$ except possibly at a . Note that a is a limit point of $D \cap (a-\delta, a+\delta)$. Thus $\frac{1}{f(x)}$ is defined on $C = D \cap (a-\delta, a+\delta) - \{a\}$.

Take any real number $K > 0$. Since $\lim_{x \rightarrow a} f(x) = 0$, there exists $\delta_1 > 0$ such that for all x in D ,

$$0 < |x - a| < \delta_1 \Rightarrow |f(x)| < \frac{1}{K}.$$

Now let $\eta = \min\{\delta, \delta_1\}$. Then $0 < |x - a| < \eta \Rightarrow 0 < |x - a| < \delta$ and $0 < |x - a| < \delta_1$.

Therefore, for all x in D ,

$$\begin{aligned} 0 < |x - a| < \eta &\Rightarrow 0 < |f(x)| = f(x) < \frac{1}{K} \\ &\Rightarrow \frac{1}{f(x)} > K. \end{aligned}$$

This means $\lim_{x \rightarrow a} \frac{1}{f(x)} = +\infty$.

(3) The proof is similar to part (2). By assumption $f(x) < 0$ for all x in $D \cap (a-\delta, a+\delta)$ except possibly at a . Take any real number $K < 0$. Since $\lim_{x \rightarrow a} f(x) = 0$, there exists $\delta_1 > 0$ such that for all x in D ,

$$0 < |x - a| < \delta_1 \Rightarrow |f(x)| < -\frac{1}{K}.$$

Let $\eta = \min\{\delta, \delta_1\}$. Then $0 < |x - a| < \eta \Rightarrow 0 < |x - a| < \delta$ and $0 < |x - a| < \delta_1$. Therefore, for all x in D ,

$$\begin{aligned} 0 < |x - a| < \eta &\Rightarrow 0 < |f(x)| = -f(x) < -\frac{1}{K} \\ &\Rightarrow \frac{1}{f(x)} < K. \end{aligned}$$

This means $\lim_{x \rightarrow a} \frac{1}{f(x)} = -\infty$.

The statements with $x \rightarrow a$ replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$ may be proved similarly so is the statement (3) with $x \rightarrow a$ may be replaced by $x \rightarrow \pm \infty$. The proof is left as an exercise.

We have similar results for limits at infinity in the following theorem..

Theorem 48. Suppose $f: D \rightarrow \mathbf{R}$ is a function.

(1) Suppose D is not bounded above and $\lim_{x \rightarrow +\infty} f(x) = 0$.

If for some real number $K > 0$, $f(x) > 0$ for all x in D and $x > K$, then $\lim_{x \rightarrow +\infty} \frac{1}{f(x)} = +\infty$.

If for some real number $K > 0$, $f(x) < 0$ for all x in D and $x > K$, then $\lim_{x \rightarrow +\infty} \frac{1}{f(x)} = -\infty$.

(2) Suppose D is not bounded below and $\lim_{x \rightarrow -\infty} f(x) = 0$.

If for some real number $K < 0$, $f(x) > 0$ for all x in D and $x > K$, then $\lim_{x \rightarrow -\infty} \frac{1}{f(x)} = +\infty$.

If for some real number $K < 0$, $f(x) < 0$ for all x in D and $x > K$, then $\lim_{x \rightarrow -\infty} \frac{1}{f(x)} = -\infty$.

Proof.

We shall prove only part (1). The proof of part (2) is similar.

Suppose $\lim_{x \rightarrow +\infty} f(x) = 0$ and $f(x) > 0$ for all x in D and $x > K$. Thus $\frac{1}{f(x)}$ is defined on $C = D \cap (K, \infty)$.

Take any real number $J > 0$. Since $\lim_{x \rightarrow +\infty} f(x) = 0$, there exists a real number $L > 0$ such that for all x in D ,

$$x > L \Rightarrow |f(x)| < \frac{1}{J}.$$

Now let $M = \max\{K, L\}$. Then $x > M \Rightarrow x > L$ and $x > K$. Therefore, for all x in D ,

$$\begin{aligned} x > M &\Rightarrow 0 < |f(x)| = f(x) < \frac{1}{J} \\ &\Rightarrow \frac{1}{f(x)} > J. \end{aligned}$$

This means $\lim_{x \rightarrow +\infty} \frac{1}{f(x)} = +\infty$.

Suppose $f(x) < 0$ for all x in D and $x > K$. Thus $\frac{1}{f(x)}$ is defined on $C = D \cap (K, \infty)$.

Take any real number $J < 0$. Since $\lim_{x \rightarrow +\infty} f(x) = 0$, there exists a real number $L > 0$ such that for all x in D ,

$$x > L \Rightarrow |f(x)| < -\frac{1}{J}.$$

Now let $M = \max\{K, L\}$. Then $x > M \Rightarrow x > L$ and $x > K$. Therefore, for all x in D ,

$$\begin{aligned} x > M &\Rightarrow 0 < |f(x)| = -f(x) < -\frac{1}{J} \\ &\Rightarrow \frac{1}{f(x)} < J. \end{aligned}$$

This means $\lim_{x \rightarrow +\infty} \frac{1}{f(x)} = -\infty$.

We next have the results for limits of sums and products involving infinity.

Theorem 49. Suppose $f: D \rightarrow \mathbf{R}$ and $g: D \rightarrow \mathbf{R}$ are two functions. Suppose $\lim_{x \rightarrow a} g(x) = c$.

(1) If $\lim_{x \rightarrow a} f(x) = +\infty$ and c is either finite or $+\infty$, then $\lim_{x \rightarrow a} [f(x) + g(x)] = +\infty$.

(2) If $\lim_{x \rightarrow a} f(x) = -\infty$ and c is either finite or $-\infty$, then $\lim_{x \rightarrow a} [f(x) + g(x)] = -\infty$.

Here $x \rightarrow a$ may be replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$ or $x \rightarrow \pm\infty$.

Proof. We shall prove part (1) only.

Suppose $\lim_{x \rightarrow a} g(x) = c$. If c is finite, then taking $\varepsilon = \left| \frac{c}{2} \right| + 1 > 0$, we have that there exists $\delta_1 > 0$ such that for all x in D ,

$$\begin{aligned} 0 < |x - a| < \delta_1 &\Rightarrow |g(x) - c| < \left| \frac{c}{2} \right| + 1 \\ &\Rightarrow g(x) > c - \left| \frac{c}{2} \right| + 1 \end{aligned} \quad \text{----- (1)}$$

Take any real number $K > 0$. Since $\lim_{x \rightarrow a} f(x) = +\infty$, there exists $\delta_2 > 0$ such that for all x in D ,

$$0 < |x - a| < \delta_2 \Rightarrow f(x) > K - c + \left| \frac{c}{2} \right| - 1. \text{----- (2)}$$

Now let $\delta = \min \{ \delta_1, \delta_2 \}$. $0 < |x - a| < \delta \Rightarrow 0 < |x - a| < \delta_1, \delta_2$

$$\Rightarrow f(x) + g(x) > K - c + \left| \frac{c}{2} \right| - 1 + c - \left| \frac{c}{2} \right| + 1 = K$$

by (1) and (2).

Hence $\lim_{x \rightarrow a} [f(x) + g(x)] = +\infty$.

Now suppose $\lim_{x \rightarrow a} g(x) = +\infty$. Then there exists $\delta_3 > 0$ such that for all x in D ,

$$0 < |x - a| < \delta_3 \Rightarrow g(x) > 1 \text{----- (3)}$$

Since $\lim_{x \rightarrow a} f(x) = +\infty$, there exists $\delta_4 > 0$ such that for all x in D ,

$$0 < |x - a| < \delta_4 \Rightarrow f(x) > K - 1. \text{----- (4)}$$

Let $\delta = \min \{ \delta_3, \delta_4 \}$.

Then for all x in D , $0 < |x - a| < \delta \Rightarrow 0 < |x - a| < \delta_3, \delta_4$

$$\Rightarrow f(x) + g(x) > K - 1 + 1 = K$$

by (1) and (2)..

Therefore, $\lim_{x \rightarrow a} [f(x) + g(x)] = +\infty$.

Part (2) is proved similarly. The cases when $x \rightarrow a$ is replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$ or $x \rightarrow \pm \infty$ is proved similarly.

Theorem 50. Suppose $\lim_{x \rightarrow a} f(x) = +\infty$ and $\lim_{x \rightarrow a} g(x) = c \neq 0$.

(1) If $c > 0$ or $c = +\infty$, then $\lim_{x \rightarrow a} f(x)g(x) = +\infty$.

(2) If $c < 0$ or $c = -\infty$, then $\lim_{x \rightarrow a} f(x)g(x) = -\infty$.

Here $x \rightarrow a$ may be replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$ or $x \rightarrow \pm \infty$.

Proof.

We shall prove part (2) only. The proof of part (1) is similar.

Suppose $\lim_{x \rightarrow a} g(x) = c$. If $c < 0$, then taking $\varepsilon = \left| \frac{c}{2} \right| > 0$, we have that there exists $\delta_1 > 0$ such that for all x in D ,

$$\begin{aligned} 0 < |x - a| < \delta_1 &\Rightarrow |g(x) - c| < \left| \frac{c}{2} \right| \\ &\Rightarrow g(x) < c + \left| \frac{c}{2} \right| = c - \frac{c}{2} = \frac{c}{2} < 0 \end{aligned} \quad \text{----- (1)}$$

Take any real number $K < 0$. Then since $\lim_{x \rightarrow a} f(x) = +\infty$, there exists $\delta_2 > 0$ such that for all x in D ,

$$0 < |x - a| < \delta_2 \Rightarrow f(x) > \frac{2|K|}{|c|} = 2\frac{K}{c} (>0) \quad \text{----- (2)}$$

Let $\delta = \min \{ \delta_1, \delta_2 \}$.

Then for all x in D , $0 < |x - a| < \delta \Rightarrow 0 < |x - a| < \delta_1, \delta_2$
 $\Rightarrow f(x)g(x) < 2\frac{K}{c}g(x)$ and $2\frac{K}{c}g(x) < K$
 by (1) and (2)
 $\Rightarrow f(x)g(x) < K$.

Therefore, $\lim_{x \rightarrow a} f(x)g(x) = -\infty$.

The cases when $x \rightarrow a$ is replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$ or $x \rightarrow \pm \infty$ are proved similarly.

Theorem 51. Suppose $\lim_{x \rightarrow a} f(x) = -\infty$ and $\lim_{x \rightarrow a} g(x) = c \neq 0$.

(1) If $c > 0$ or $c = +\infty$, then $\lim_{x \rightarrow a} f(x)g(x) = -\infty$.

(2) If $c < 0$ or $c = -\infty$, then $\lim_{x \rightarrow a} f(x)g(x) = +\infty$.

Here $x \rightarrow a$ may be replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$ or $x \rightarrow \pm \infty$.

The proof of Theorem 51 is similar to that of Theorem 50.

Examples 52.

(1) $\lim_{x \rightarrow 0} \frac{1}{|x|} = +\infty$, $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$ and $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

We can deduce that $\lim_{x \rightarrow 0} \frac{1}{|x|} = +\infty$ by using the Definition 38. Take any sequence (x_n) in $\mathbf{R} - \{0\}$ with $x_n \rightarrow 0$. Take any real number $K > 0$. Since $x_n \rightarrow 0$, there exists a positive integer N such that $n \geq N \Rightarrow 0 < |x_n| < 1/K$. Therefore, $n \geq N \Rightarrow \frac{1}{|x_n|} > K$. This shows that $\frac{1}{|x_n|} \rightarrow \infty$.

Hence by Definition 38, $\lim_{x \rightarrow 0} \frac{1}{|x|} = +\infty$.

We may also use Theorem 46 (2) as follows. Since $\lim_{x \rightarrow 0} |x| = 0$ and $|x| > 0$ for all $x \neq 0$, by

Theorem 46 (2) $\lim_{x \rightarrow 0} \frac{1}{|x|} = +\infty$.

We deduce similarly that $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$. Take any sequence (x_n) in $(0, \infty)$ with $x_n \rightarrow 0$. Then for any real number $K > 0$, there exists a positive integer N such that $n \geq N \Rightarrow 0 < |x_n| = x_n < 1/K$. Therefore, $n \geq N \Rightarrow \frac{1}{x_n} > K$. This means $\frac{1}{x_n} \rightarrow \infty$. Hence by Definition 39,

$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$. In almost exactly the same way, we shall show that $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$. Take any sequence (x_n) in $(-\infty, 0)$ with $x_n \rightarrow 0$. Then for any real number $K < 0$, there exists a positive integer N such that $n \geq N \Rightarrow 0 < |x_n| = -x_n < -1/K$. Therefore, $n \geq N \Rightarrow \frac{1}{x_n} < K$. This means $\frac{1}{x_n} \rightarrow -\infty$. Hence by Definition 39, $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$.

We may use Theorem 46 (1) to deduce that $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$ since $\lim_{x \rightarrow 0^+} x = 0$ and in the interval $(0, \infty)$, $x > 0$. Likewise Theorem 46 (3) applies to give $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$.

(2) $\lim_{x \rightarrow 3} \frac{1}{|x-3|} = +\infty$.

Since $\lim_{x \rightarrow 3} |x-3| = 0$ and $|x-3| > 0$ for $x \neq 3$, by Theorem 46 (2) $\lim_{x \rightarrow 3} \frac{1}{|x-3|} = +\infty$.

(3) $\lim_{x \rightarrow 7^+} \frac{1}{x-7} = +\infty$.

Since $\lim_{x \rightarrow 7^+} x-7 = 0$ and $x-7 > 0$ for x in $(7, \infty)$, by Theorem 46 (2) $\lim_{x \rightarrow 7^+} \frac{1}{x-7} = +\infty$.

(4) $\lim_{x \rightarrow +\infty} \frac{1}{x-3} = 0$. This is immediate by Theorem 46 (1) since $\lim_{x \rightarrow +\infty} x-3 = +\infty$.

(5) $\lim_{x \rightarrow 1} \frac{-2}{(x-1)^2} = -\infty$

- By Theorem 46 (2) $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = +\infty$, By Theorem 59 (2), $\lim_{x \rightarrow 1} \frac{-2}{(x-1)^2} = +\infty$
- (6) $\lim_{x \rightarrow 7} [\frac{1}{(x-7)^2} + 2 + 5x] = +\infty$.
 $\lim_{x \rightarrow 7} \frac{1}{(x-7)^2} = +\infty$ by Theorem 46 (2) and $\lim_{x \rightarrow 7} [2 + 5x] = 37$ is finite. Therefore, by Theorem 48 (1) $\lim_{x \rightarrow 7} [\frac{1}{(x-7)^2} + 2 + 5x] = +\infty$
- (7) $\lim_{x \rightarrow 7} \frac{x-9}{(x-7)^2} = -\infty$.
 We can write $\lim_{x \rightarrow 7} \frac{x-9}{(x-7)^2} = \lim_{x \rightarrow 7} \frac{1}{(x-7)^2} \cdot (x-9)$. Because $\lim_{x \rightarrow 7} \frac{1}{(x-7)^2} = +\infty$ and $\lim_{x \rightarrow 7} (x-9) = -2 < 0$, by Theorem 49 (2), $\lim_{x \rightarrow 7} \frac{x-9}{(x-7)^2} = -\infty$.
- (8) $\lim_{x \rightarrow \infty} \frac{1}{x} \cos(\frac{1}{x^2}) = 0$
 For $x \neq 0$, $-\left|\frac{1}{x}\right| \leq \frac{1}{x} \cos(\frac{1}{x^2}) \leq \left|\frac{1}{x}\right|$. Since $\lim_{x \rightarrow \infty} \left|\frac{1}{x}\right| = \lim_{x \rightarrow \infty} -\left|\frac{1}{x}\right| = 0$, by the Squeeze Theorem (Theorem 45), $\lim_{x \rightarrow \infty} \frac{1}{x} \cos(\frac{1}{x^2}) = 0$.

We have extended the notion of limits to include formally $\pm \infty$. We now describe how this behaves with composition. The next result is an analogue of Theorem 34.

Theorem 53.

Suppose $f : D \rightarrow \mathbf{R}$ and $g : V \rightarrow \mathbf{R}$ are two functions such that $f(D - \{a\}) \subseteq V$. Thus, the composite $g \circ f : D - \{a\} \rightarrow \mathbf{R}$ is defined. Suppose a is a limit point of D and b is a limit point of V . Let c be a finite real number or the symbol $+\infty$ or $-\infty$.

- (1) Suppose $\lim_{x \rightarrow a} f(x) = +\infty$ and $\lim_{y \rightarrow b} g(y) = c$. Then $\lim_{x \rightarrow a} g \circ f(x) = c$.
 (2) Suppose $\lim_{x \rightarrow a} f(x) = -\infty$ and $\lim_{y \rightarrow b} g(y) = c$. Then $\lim_{x \rightarrow a} g \circ f(x) = c$.
 (1) and (2) hold when $x \rightarrow a$ is replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$.
 (1) and (2) also holds when we require that $f(D) \subseteq V$ and $x \rightarrow a$ is replaced by $x \rightarrow +\infty$ or $x \rightarrow -\infty$.

Proof. We shall prove part (1) only. The proof of part (2) is similar.

Suppose c is finite. Since $\lim_{y \rightarrow b} g(y) = c$, given any $\epsilon > 0$, there exists a real number $L > 0$ such that for all y in V ,

$$y > L \Rightarrow |g(y) - c| < \epsilon. \quad \text{----- (1)}$$

Because $\lim_{x \rightarrow a} f(x) = +\infty$, there exists a $\delta > 0$ such that for all x in D ,

$$\lim_{y \rightarrow b} g(y) = cf(x) > L. \quad \text{----- (2)}$$

Therefore, for all x in D ,

$$0 < |x - a| < \delta \Rightarrow f(x) > L \Rightarrow |g(f(x)) - c| < \epsilon$$

by (2) and (1).

This means $\lim_{x \rightarrow a} g \circ f(x) = c$.

If c is $+\infty$, then because $\lim_{y \rightarrow b} g(y) = +\infty$, given any real number $K > 0$, there exists a real number $L > 0$ such that for all y in V ,

$$y > L \Rightarrow g(y) > K. \quad \text{----- (3)}$$

It then follows from (2) and (3) that for all x in D ,

$$0 < |x - a| < \delta \Rightarrow f(x) > L \Rightarrow g(f(x)) > K.$$

Hence $\lim_{x \rightarrow a} g \circ f(x) = +\infty$.

If c is $-\infty$, we proceed in exactly the same manner. $\lim_{y \rightarrow b} g(y) = -\infty$ implies that given any real number $K < 0$, there exists a real number $L > 0$ such that for all y in V ,

$$y > L \Rightarrow g(y) < K. \quad \text{-----} \quad (3)$$

Therefore, it follows from (4) and (2) that for all x in D ,

$$0 < |x - a| < \delta \Rightarrow f(x) > L \Rightarrow g(f(x)) < K.$$

Thus $\lim_{x \rightarrow a} g \circ f(x) = -\infty$.

The other cases when $x \rightarrow a$ is replaced by $x \rightarrow a^+$ or $x \rightarrow a^-$ or $x \rightarrow +\infty$ or $x \rightarrow -\infty$ are proved similarly.

Remark. Note that the proof is just a careful handling with the definition 39 and Definition 41.

Theorem 54.

Suppose $f: D \rightarrow \mathbf{R}$ and $g: V \rightarrow \mathbf{R}$ are two functions such that $f(D) \subseteq V$. Thus, the composite $g \circ f: D \rightarrow \mathbf{R}$ is defined. Suppose b is a limit point of V . Let c be a finite real number or the symbol $+\infty$ or $-\infty$.

(1) Suppose $\lim_{x \rightarrow +\infty} f(x) = b$ and $\lim_{y \rightarrow b} g(y) = c$. If there exists a real number $K > 0$ such that for all x in D and $x > K$, $f(x) \neq b$. Then $\lim_{x \rightarrow +\infty} g \circ f(x) = c$.

(2) Suppose $\lim_{x \rightarrow -\infty} f(x) = b$ and $\lim_{y \rightarrow b} g(y) = c$. If there exists a real number $K < 0$ such that for all x in D and $x < K$, $f(x) \neq b$. Then $\lim_{x \rightarrow -\infty} g \circ f(x) = c$.

(3) If c is finite, $\lim_{x \rightarrow +\infty} f(x) = b$, $\lim_{y \rightarrow b} g(y) = c$ and g is continuous at b , then $\lim_{x \rightarrow +\infty} g \circ f(x) = c$.

(4) If c is finite, $\lim_{x \rightarrow -\infty} f(x) = b$, $\lim_{y \rightarrow b} g(y) = c$ and g is continuous at b , then $\lim_{x \rightarrow -\infty} g \circ f(x) = c$.

Proof. We shall prove parts (1) and (3) only. The proof of parts (2) and (4) is similar.

(1) Suppose c is a finite real number. Then since $\lim_{y \rightarrow b} g(y) = c$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all y in V ,

$$0 < |y - b| < \delta \Rightarrow |g(y) - c| < \varepsilon. \quad \text{-----} \quad (1)$$

Since $\lim_{x \rightarrow +\infty} f(x) = b$, there exists a real number $L > 0$ such that for all x in D ,

$$x > L \Rightarrow |f(x) - b| < \delta.$$

Now let $M = \max\{K, L\}$. Then by assumption for all x in D , $x > M \Rightarrow x > K \Rightarrow f(x) \neq b$. Hence we have that for all x in D ,

$$x > M \Rightarrow 0 < |f(x) - b| < \delta. \quad \text{-----} \quad (2)$$

Therefore, combining (1) and (2) we obtain that for all x in D ,

$$x > M \Rightarrow 0 < |g(f(x)) - c| < \varepsilon.$$

This means $\lim_{x \rightarrow +\infty} g \circ f(x) = c$.

Now for the case $c = +\infty$. $\lim_{y \rightarrow b} g(y) = +\infty$ implies that given any real number $J > 0$, there exists $\delta > 0$ such that for all y in V ,

$$0 < |y - b| < \delta \Rightarrow g(y) > J. \quad \text{-----} \quad (3)$$

Therefore, it follows from (2) and (3) that for all x in D ,

$$x > M \Rightarrow 0 < |f(x) - b| < \delta \Rightarrow g(f(x)) > J.$$

This means $\lim_{x \rightarrow +\infty} g \circ f(x) = +\infty$.

The case when $c = -\infty$ is similar. We note that $\lim_{y \rightarrow b} g(y) = -\infty$ implies that given any real number $J < 0$, there exists $\delta > 0$ such that for all y in V ,

$$0 < |y - b| < \delta \Rightarrow g(y) < J. \quad \text{-----} \quad (4)$$

It follows then by (2) and (4) that for all x in D ,

$$x > M \Rightarrow 0 < |f(x) - b| < \delta \Rightarrow g(f(x)) < J.$$

This means $\lim_{x \rightarrow +\infty} g \circ f(x) = -\infty$.

(3). If c is finite and g is continuous at b , for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all y in V ,

$$|y - b| < \delta \Rightarrow |g(y) - c| < \varepsilon. \quad \text{----- (5)}$$

Since $\lim_{x \rightarrow +\infty} f(x) = b$, there exists a real number $L > 0$ such that for all x in D ,

$$x > L \Rightarrow |f(x) - b| < \delta. \quad \text{----- (6)}$$

Thus combining (5) and (6) we get that for all x in D ,

$$x > L \Rightarrow |f(x) - b| < \delta \Rightarrow |g(f(x)) - c| < \varepsilon.$$

This means $\lim_{x \rightarrow +\infty} g \circ f(x) = c$.

Example 55.

1. $\lim_{x \rightarrow +\infty} x \sin(\frac{1}{x}) = \lim_{x \rightarrow +\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1$. Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{\sin(x)}{x}$ for $x \neq 0$. Since

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \text{ and } \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{1}{x} = 0, \text{ by Theorem 54 (1),}$$

$$\lim_{x \rightarrow +\infty} g(f(x)) = \lim_{x \rightarrow +\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1.$$

2. Given that $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$, $\lim_{x \rightarrow +\infty} x^2(e^{\frac{1}{x^2}} - 1) = 1$. Let $g(x) = \frac{e^x - 1}{x}$ and $f(x) = \frac{1}{x^2}$

Note that $f(x) \neq 0$ for $x \neq 0$. Therefore, by Theorem 54 (1)

$$\lim_{x \rightarrow +\infty} g(f(x)) = \lim_{x \rightarrow 0} x^2(e^{\frac{1}{x^2}} - 1) = 1.$$

3. Note that in Theorem 54 (1) the condition that $f(x) \neq b$ for large x , cannot be removed.

For instance, let $g(x) = \frac{\sin(x)}{x}$ for $x \neq 0$ and $g(0) = 0$ and let $f(x) = 0$ the constant

function. Obviously, $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x \rightarrow +\infty} f(x) = 0$. But $\lim_{x \rightarrow +\infty} g(f(x)) \neq 1$.