Set intersection problems: Supporting halfspaces and quadratic programming

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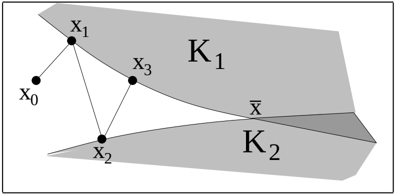
Problem

Let X be a Hilbert space,
Closed convex sets $K_i \subset X$, $i = 1, \ldots, r$, and $K := \cap_{i=1}^{r} K_i$.

- Set Intersection Problem (SIP): Find $x \in K$.
  - For starting iterate $x_0 \in X$,
    Alternating Projections converge to some $x \in K$.



- Typical convergence of alternating projections is linear
- This talk: "Fast convergence" refers to better than linear convergence.

Previous Results on Alternating Projections

For Alternating projections in a Hilbert space X, assume $K_l \subset X$ are closed convex sets, and $K := \cap K_l \neq \emptyset$.

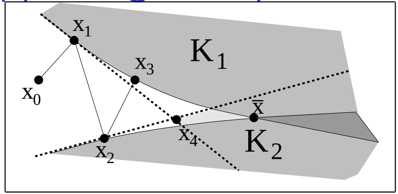
- Finds $P_K(x_0)$ for r subspaces
  - $r = 2$ (von Neumann '49), $r > 2$ (Halperin '62)
- Converges weakly in general (Bregman '65)
- May not converge strongly for 2 cones (Hundal '04)
- **Converges linearly (Gubin-Polyak-Raik '67) under bounded linear regularity (Bauschke-Borwein '93)**
- Accelerations (Gubin-Polyak-Raik '67, Gearhart-Koshy '89, Bauschke-Deutsch-Hundal-Park '03, Pierra '84, Bauschke-Combettes-Kruk '06)
- For nonconvex sets, MAP converges linearly under a CQ (Lewis-Malick '08, Lewis-Luke-Malick '09)
- Surveys (Bauschke-Borwein '96, Bauschke '01, Deutsch '01, Escalante-Raydan '11)

Applications of Set Intersection Problem

Applications of Alternating projection algorithm

1. Solving linear equations;
2. The Dirichlet problem which has in turn inspired the "domain decomposition" industry;
3. Probability and statistics;
4. Computing Bergman kernels;
5. Approximating multivariate functions by sums of univariate ones;
6. Least change secant updates;
7. Multigrid methods;
8. Conformal mapping;
9. Image restoration;
10. Computed tomography.

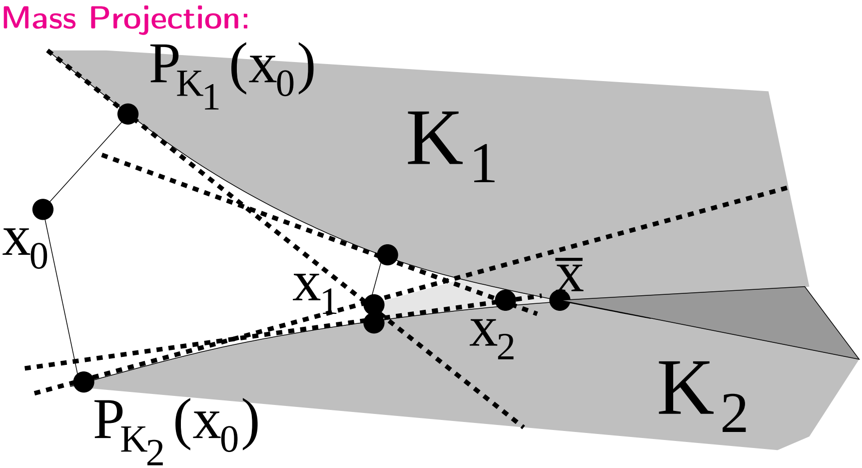
Supporting Halfspaces & Quadratic Programming



- **Supporting halfspaces** at x_1 and x_2
  - Intersection of 2 halfspaces approximate K better than each K_i taken singly.
    [Projection of x_1 onto polyhedron (x_4) better than x_3.]
  - Projection is a **QP**: $\min\{\|x_0 - x\|^2 \mid x \text{ in polyhedron}\}$.
  - (García-Palomares '98, '01): $K_i = \{x \mid f_i(x) \leq 0\}$.
    - Smoothness and LICQ gives superlinear convergence
  - Smooth boundaries should ensure superlinear/ quadratic convergence with a Newton-like method.

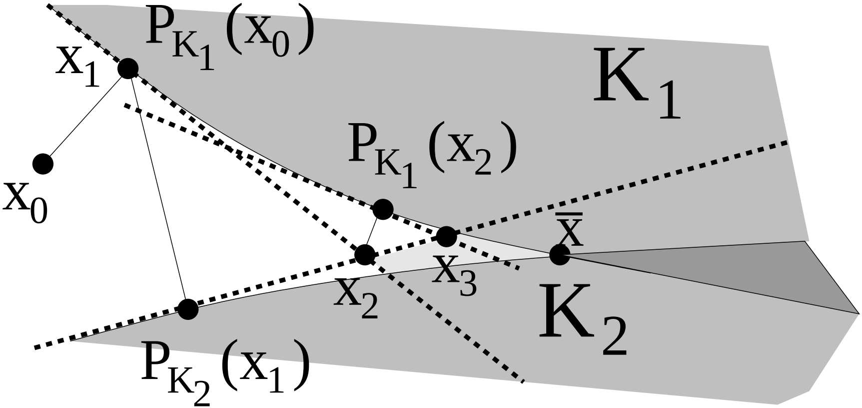
This talk:
Various improvements on an earlier result for
SHQP (Supporting halfspaces and quadratic programming).

Algorithm in Pictures: Mass Projection



Algorithm in Pictures: Cyclic Projection

Cyclic Projection:



The Algorithm

1. Start with $x_0 \in X$. Set $i = 1$
2. Project x_{i-1} onto K_j for all $j \in J_i \subset \{1, \ldots, r\}$, where J_i chosen beforehand, to **get** $|J_i|$ **halfspaces.**

   Examples:
   - $J_i = \{1, \ldots, r\}$: Mass projection
   - $J_i = \{[i \bmod r]\}$: Cyclic projection
3. **Create polyhedron** $F_i \subset K$ from some/ all of the previous supporting halfspaces.
   - More halfspaces: Better solution, but bigger QP
   - Less halfspaces: Smaller QP
4. **Obtain (better) estimate**: $x_i = P_{F_i}(x_{i-1})$.
5. $i \leftarrow i + 1$, and go back to step 2.

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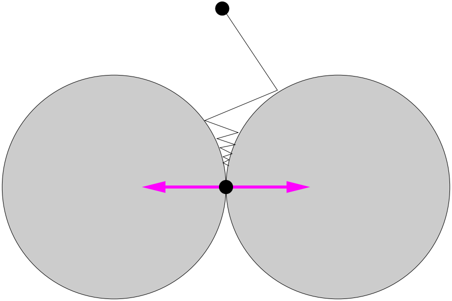
- Improving a multiple-term superlinear convergence result
  - Reducing size of QP
  - Quadratic convergence under additional regularity
  - Solving QP partially
- Practical to solve QP partially with inner GI steps
- Infeasibility detection: certificate in finitely many steps
- Comparable results in Convex Inequality Problems (CIP)
- Finite convergence of modified algorithm
  - Nonsmooth, nonconvex CIP
- Handling affine subspaces in general SIP
- Affine sets only: Strong convergence
- Nonconvex SIP
  - Example of non-monotone decrease in distance
  - Local conditions for SHQP to be effective

CQ for linear convergence of Alt. Proj.

Convergence of Alternating Projections need not be linear.



For linear convergence to $\bar{x}$, can use condition:

$$\sum_{l=1}^{r} v_l = 0 \text{ and } v_l \in N_{K_l}(\bar{x}) \text{ imply } v_1 = v_2 = \cdots = v_r = 0 \quad \text{(N-CQ)}$$

Under (N-CQ), there is $\kappa > 0$ s.t. $d(x, \bigcap_{l=1}^{r} K_l) \leq \kappa \max_l d(x, K_l)$.

An older result for convex **nonsmooth** pblms $K = \cap_{i=1}^{\mathbf{r}} K_i$:
p-term superlinear convergence, i.e.

$$\lim_{i\to\infty} \frac{\|x_{i+\mathbf{p}} - \bar{x}\|}{\|x_i - \bar{x}\|} = 0$$

if 1. Limit $\bar{x} \in K$ satisfies (N-CQ) (κ-linear regularity)
2. $n := \dim(X) < \infty$ and $\mathbf{p}$ is large enough ($\mathbf{p} \sim n\kappa^n$?)
3. Use mass projection and last $\mathbf{rp}$ halfspaces

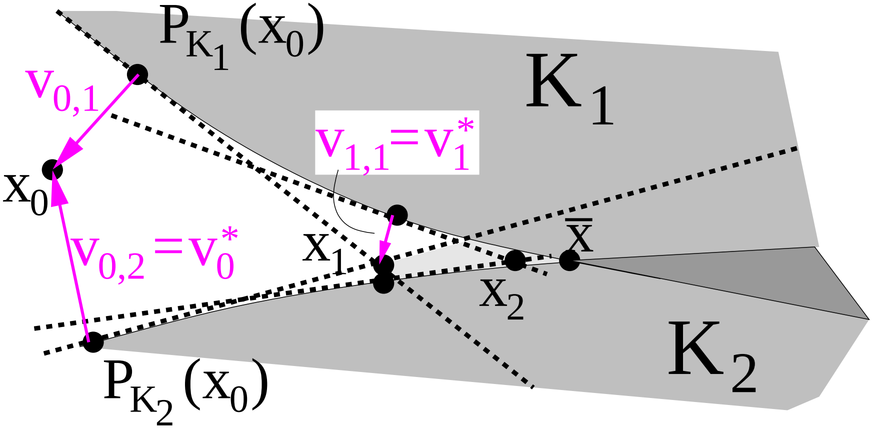
Better than linear convergence of alternating projections.
BUT **p** (the lag) is too large and **rp** halfspaces is too many.

Improvements (that can all hold concurrently) we'll present:

1. Less halfspaces and smaller **p**
    - $\mathbf{p} = 1$ when boundaries smooth, i.e., Newton-like.
2. Quadratic convergence under additional regularity
3. Partial projection onto polyhedron maintains fast convergence

Improvement 1: (Smaller lag, smaller QP size)

- Fejér monotonicity implies convergence to some $\bar{x} \in K$
  - Some regularity constant $\bar{\kappa} \in [0, \infty]$ at $\bar{x}$
- Let $v_{i,j} = x_i - P_{K_j}(x_i)$.
- Let $v_i^* = v_{i,j^*}$, where $j^* = \arg\max_j \|v_{i,j}\|$.



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Theorem:

For every $\epsilon > 0$, there exists I s.t. if $i > I$ and $i + k$ are s.t.

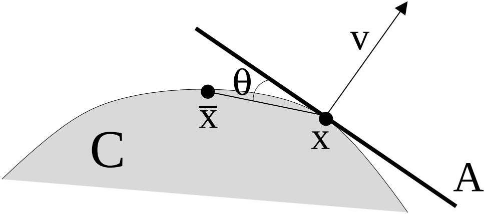
1. $\angle v_{i',j'} 0 v_{i+k}^* < \sin^{-1}(1/\bar{\kappa})$ for some
$i' \in [i, i+k-1]$ and $j \in \{1, \dots, r\}$
(i.e., normals $v_{i'j'}$, v_{i+k}^* formed by projections are close)
2. x_{i+k} lies in halfspace formed by projecting $x_{i'}$ onto $K_{j'}$

then $\frac{\|x_{i+k} - \bar{x}\|}{\|x_i - \bar{x}\|} < \epsilon$.

- This k is smaller than the p of the old result.
  - The older value p ensures such a k exists.
- When boundaries smooth: • $k = 1$ for mass projection
  • $k \leq r$ for cyclic projection

Improvement 2:

(Additional regularity for multiple-term quadratic convergence)



For $\bar{x}, x \in C \subset \mathbb{R}^n$, $v \in N_C(x)$,

let A be hyperplane through x with normal v.

Lemma: $\limsup_{x \xrightarrow[C]{} \bar{x}} \frac{d(\bar{x},A)}{\|x-\bar{x}\|} = 0$

(This result was crucial in original result)

2nd order supporting hyperplane (SOSH): $\limsup_{x \xrightarrow[C]{} \bar{x}} \frac{d(\bar{x},A)}{\|x-\bar{x}\|^2} < \infty$

- If all sets K_i have SOSH, then $\limsup_{i\to\infty} \frac{\|x_{i+k_i}-\bar{x}\|}{\|x_i-\bar{x}\|^2} < \infty$
  - i.e., multiple term quadratic convergence.

2nd order supporting hyperplane (SOSH): $\limsup_{x \xrightarrow[C]{} \bar{x}} \frac{d(\bar{x}, A)}{\|x-\bar{x}\|^2} < \infty$

SOSH appears often in applications:

- For $C = \{x : f(x) \leq 0\}$ where $f(\cdot)$ convex and smooth, C has SOSH at $\bar{x}$ if $f(\bar{x}) = 0$ and $\nabla^2 f(\bar{x}) \succ 0$.
- Intersection of convex sets SOSH at $\bar{x}$ is SOSH at $\bar{x}$.

Approximate methods to project onto polyhedron

Some preparation for improvement 3 (Partial projection):

Dual active set QP algorithm (Goldfarb-Idnani '81):
Consider the QP of finding $P_F(y)$, i.e.,

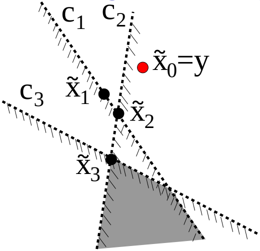
$$\begin{array}{ll} \min_{\tilde{x}\in\mathbb{R}^n} & \frac{1}{2}\|\tilde{x}-y\|^2 \\ \text{s.t.} & \tilde{x}\in F, \text{ where } F := \big\{x : \langle c_j, x\rangle \leq b_j \text{ for all } j \in \{1,\dots m\}\big\}. \end{array}$$

GI's algorithm finds a series of iterates $\{(\tilde{x}_i, A_i)\}_{i=1}^{\infty}$ s.t.

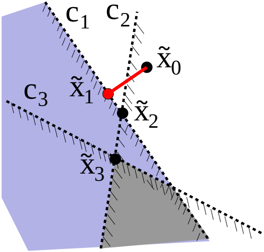
- $A_i \subset \{1,\dots,m\}$
- $\langle c_j, \tilde{x}_i\rangle = b_j$ for all $j \in A_i$ (i.e., A_i is the active set)
- $\tilde{x}_i = P_{F_i}(y)$, where $F_i = \{x : \langle c_j, \tilde{x}_i\rangle \leq b_j \text{ for all } j \in A_i\}$.
- (Improvement) If $\tilde{x}_i \neq P_F(y)$, then $\|\tilde{x}_{i+1}-y\| > \|\tilde{x}_i - y\|$.

GI algorithm terminates at $P_F(y)$ in finitely many steps.

Illustrating GI's algorithm



$A_0 = \emptyset$ (always)

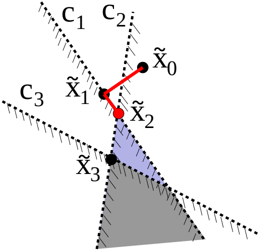


$A_1 = \{1\}$

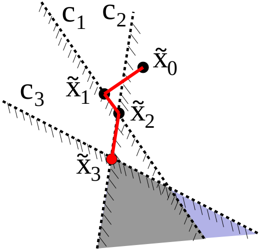
A_i : active set

$\tilde{x}_i$: approximate projection of y onto polyhedron

$\|\tilde{x}_i - y\|$ increases



$A_2 = \{1, 2\}$

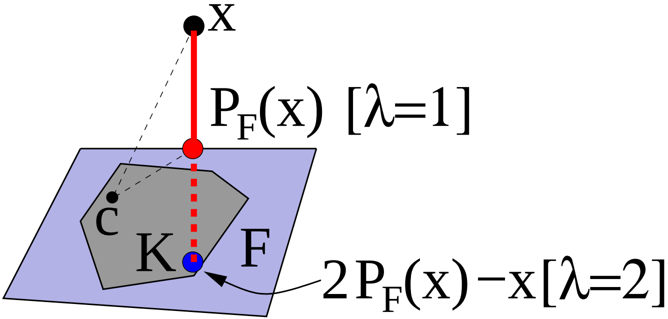


$A_4 = \{2, 3\}$

Fejér monotonicity:
If K and F convex sets, $K \subset F$, then for any x and $\lambda \in [1,2]$,

$$\| \overbrace{[x + \lambda[P_F(x) - x]]}^{\lambda=1 \text{ gives } P_F(x)} - c\|^2 \leq \|x - c\|^2 - \overbrace{\lambda[2-\lambda]\|x - P_F(x)\|^2}^{(*)}$$

for all $c \in F$ (and hence $c \in K$).



i.e., $[x + \lambda[P_F(x) - x]]$ closer to all $c \in K$ than x by factor $(*)$

Fejér monotonicity:

If K and F convex sets, $K \subset F$, then for any x and $\lambda \in [1,2]$,

$$\| \overbrace{[x + \lambda[P_F(x) - x]]}^{\lambda=1 \text{ gives } P_F(x)} - c\|^2 \le \|x - c\|^2 - \overbrace{\lambda[2-\lambda]\|x - P_F(x)\|^2}^{(*)}$$

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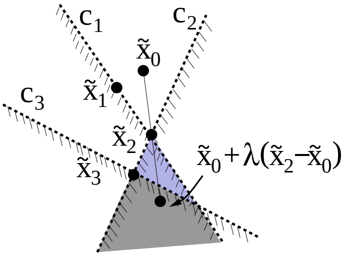


Iterate $\tilde{x}_3$ optimal, but consider iterate $\tilde{x}_2$.

$\tilde{x}_0 + \lambda(\tilde{x}_2 - \tilde{x}_0) \in F$ keeps Fejér monotonocity for $\lambda \in [1,2]$

- A point in polyhedron F maintaining Fejér monotonocity may be easier to obtain than solving QP.

Improvement 3 (Partial projections):
Instead of $x_{i+1} = P_{F_i}(x_i)$, x_{i+1} can be chosen s.t.

1. $x_{i+1} \in F_i$
2. Fejér monotonicity preserved
3. $x_i - x_{i+1}$ is a positive linear combination of normals used to generate the supporting halfspaces.

(Note that $P_{F_i}(x_i)$ satisfies (1)-(3),
but GI algorithm may find a point satisfying (1)-(3) earlier.)

Then still have fast convergence.

Next lesson from Fejér monotonicity

Next lesson: QP need not be solved to optimality in practice.
(Use as many GI steps as possible, but don't fret about not reaching optimality)

Set $F_i := \{x : \langle c_j, x\rangle \leq b_j \text{ for all } j \in \{1,\dots,m\}\}$
and $\tilde{F}_A := \{x : \langle c_j, x\rangle \leq b_j \text{ for all } j \in A\}$ for $A \subset \{1,\dots,m\}$

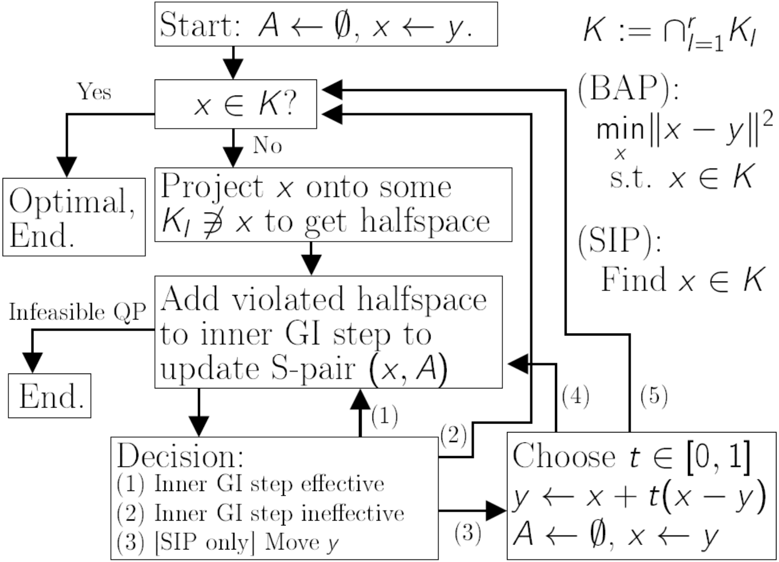
At iteration i, $\tilde{F}_A \supset F_i$ and $F_i \supset K := \cap_{l=1}^r K_l$.

Consider $\lambda = 1$. Use GI algorithm to try to find $P_{F_i}(x_i)$.
After j GI iterations, have $(\tilde{x}_j, A_j) = (P_{\tilde{F}_{A_j}}(x_i), A_j)$ s.t.

$$\underbrace{\|P_{\tilde{F}_{A_j}}(x_i) - c\|^2}_{(1)} \leq \|x_i - c\|^2 - \underbrace{\|x_i - P_{\tilde{F}_{A_j}}(x_i)\|^2}_{(2)} \text{ for all } c \in K.$$

After more GI iterations, (2) increases, so (1) decreases.
So more GI iterations improve choice of $x_{i+1} := P_{\tilde{F}_{A_j}}(x_i)$.

The SHDQP algorithm



$K := \cap_{l=1}^{r} K_l$

(BAP):

$$\min_x \|x - y\|^2 \quad \text{s.t. } x \in K$$

(SIP):

Find $x \in K$

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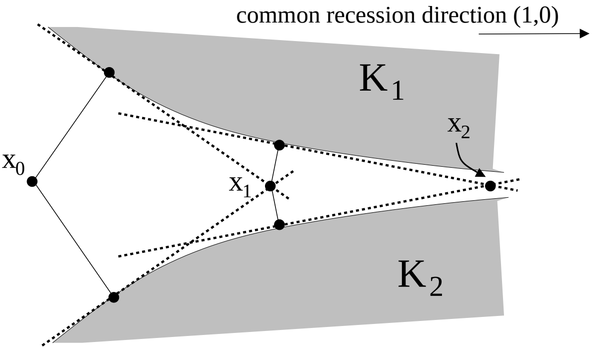
Infeasibility

When $K_1 \cap K_2 = \emptyset$, SHQP can run infinitely.



In this example, there is a common recession direction $(1, 0)$.

Infeasibility

Theorem: if $\cap_{i=1}^{r} K_i = \emptyset$ and

- no common recession directions,
- finite dimensional, and
- all previous halfspaces are collected,

then polyhedron will eventually be empty.

A QP algorithm (eg, GI algorithm) will give Farkas Lemma type certificate of infeasibility.

- i.e., for polyhedron $F := \{x : Ax \leq b\}$, the vector r s.t. $r \geq 0$, $r^T A = 0$ and $r^T b < 0$ certifies $F = \emptyset$.
  - In practice, can approximate $r^T A = 0$ by $r^T A \approx 0$.

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Extending to Convex Inequality Problems

Results extendible to the Convex Inequality Problem (CIP) of finding x s.t. $f(x) \leq 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is convex.

- At iterate x_i, let $y_i \in \partial f(x_i)$. We have
$$\underbrace{\{x : f(x) \leq 0\}}_{S} \subset \underbrace{\{x : f(x_i) + \langle y_i, x - x_i \rangle \leq 0\}}_{H_i,\text{ a halfspace}}.$$
- $P_{H_i}(x_i) := x_i - \frac{f(x_i)}{\|y_i\|^2} y_i$. (the usual one step iteration)
- Projecting iterates x_i onto polyhedra formed by some of the H_i speeds up convergence.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex.

- We say f is *semismooth at x* if
$$f(x+h) - f(x) - Vh = o(\|h\|\ ) \text{ for any } V \in \partial f(x+h).$$
[Fact: All convex functions are semismooth.]
- We say f is *strongly semismooth at x* if
$$f(x+h) - f(x) - Vh = O(\|h\|^2) \text{ for any } V \in \partial f(x+h).$$

Theorem: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ convex & $f^{-1}((-\infty, 0]) \neq \emptyset$. Then algorithm converges to a point $\bar{x}$ such that $f(\bar{x}) \leq 0$.

- At least linearly when $0 \notin \partial f(\bar{x})$ [Robinson, '76].
  - Use $x_{i+1} := P_{H_i}(x_i) = x_i - \frac{f(x_i)}{\|y_i\|^2} y_i$.
- [Many halfspaces] Suppose algorithm converges to a point $\bar{x}$ s.t. $0 \notin \partial f(\bar{x})$. Let $\bar{\alpha} < \sin^{-1}\left(\frac{d(0,\partial f(\bar{x}))}{\sup_{y\in\partial f(\bar{x})}\|y\|}\right)$.
  - For any $\epsilon > 0$, there is an $I > 0$ such that if $I < i < k$ and $\angle y_j 0 y_k \leq \bar{\alpha}$, where $y_j \in \partial f(x_j)$ and $y_k \in \partial f(x_k)$ are normals of halfspaces in polyhedral approximation, then

    Superlinear type convergence: $\dfrac{\|x_k - \bar{x}\|}{\|x_i - \bar{x}\|} \leq \dfrac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|} \leq \epsilon.$

  - If in addition f is strongly semismooth at $\bar{x}$, then there is $M > 0$ s.t.

    Quadratic type convergence: $\dfrac{\|x_k - \bar{x}\|}{\|x_i - \bar{x}\|^2} \leq \dfrac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|^2} < M.$

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Finite convergence

If $f(\hat{x}) < 0$ for some $\hat{x}$, then a modified algorithm known to give finite convergence to a point in $\{x : f(x) \leq 0\}$.

Projecting x_i on $\{x : f(x_i) + y_i^T(x - x_i) \leq 0\}$ gives $x_i - \frac{f(x_i)}{\|y_i\|^2} y_i$.

Finite convergence of a modified algorithm for convex $f(\cdot)$

- Smooth $f(\cdot)$: Polak-Mayne '79, Mayne-Polak-Heunis '81
- Nonsmooth $f(\cdot)$: Fukushima '82 and De Pierro-Iusem '88

Idea is to use $x_i - \frac{f(x_i)+\epsilon_i}{\|y_i\|^2} y_i$, where $\epsilon_i = 1/i$

- (i.e., try to solve $f(x) \leq -\epsilon_i$ instead at ith iteration)
- Eventually, $f(\hat{x}) < -\epsilon_i$

Alternating projection finds x s.t. $f(x) := \max_{1 \leq i \leq r} d(x, K_i) \leq 0$.

- No $\hat{x}$ s.t. $f(\hat{x}) < 0$, but consider $f_2(\cdot) := \max_{1 \leq i \leq r} \tilde{d}(x, K_i)$,

$$\text{where } \tilde{d}(x, K_i) := \begin{cases} d(x, K_i) & \text{if } x \notin K_i \\ \sup\{\alpha : \mathbb{B}(x, \alpha) \subset K_i\} & \text{if } x \in K_i \end{cases}$$

- $f_2(\cdot)$ is convex, and $f_2(\hat{x}) < 0$ if $\hat{x} \in \text{int}(\cap_{i=1}^r K_i)$.
- If $x \notin \cap_{i=1}^r K_i$ and $d(x, K_{i^*}) = f_2(x)$ then

$$\frac{x - P_{K_{i^*}}(x)}{\|x - P_{K_{i^*}}(x)\|} \in \partial f_2(x).$$

- Finite convergence with Fukushima's algorithm on $f_2(\cdot)$.
- Finite convergence of the ART (Algebraic reconstruction techniques), where the sets are hyperslabs $\{x : \langle c, x \rangle \in [a, b]\}$, seems to be based on this principle.

Finite convergence for nonsmooth, nonconvex CIP

A setting for finite convergence of CIP in nonconvex problems:

$f(\cdot)$ is *approximately convex at $\bar{x}$* if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$f(y) \geq f(x) + \langle s, y - x \rangle - \epsilon \|y - x\|$$

for all $x, y \in \mathbb{B}(\bar{x}, \delta)$ and $s \in \partial f(x)$.

(Ngai-Luc-Thera '00 & Daniilidis-Georgiev '04, in turn motivated by weak convexity in Vial '83 & Hiriart-Urruty '84)

Local finite convergence for nonconvex problems:

Theorem: Suppose

- $f(\cdot)$ approximately convex at $\bar{x}$,
- $\{\epsilon_i\} \subset \mathbb{R}_+$ decreasing and converges to 0 at sublinear rate
- $f(\bar{x}) = 0$ and $0 \notin \partial f(\bar{x})$,

There exists a $\bar{\epsilon}$ s.t. if $\|x_0 - \bar{x}\| < \bar{\epsilon}$ and $\epsilon_0 < \bar{\epsilon}$, then algorithm finds some x_i s.t. $f(x_i) \leq 0$.

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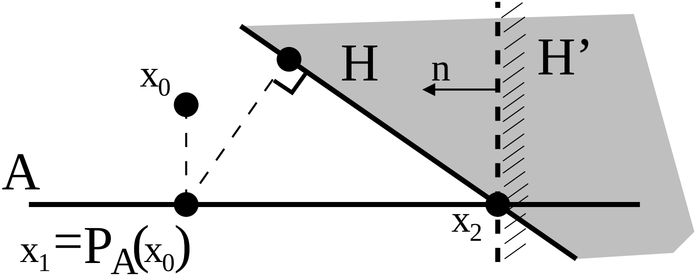
Affine sets in SHQP

Projecting onto intersection of affine space and halfspace easy. (Pierra '84, Bauschke-Combettes-Kruk '06)



$$P_{A\cap H}(x_0) = P_{A\cap H}\big(\overbrace{P_A(x_0)}^{=x_1}\big) = x_2,$$

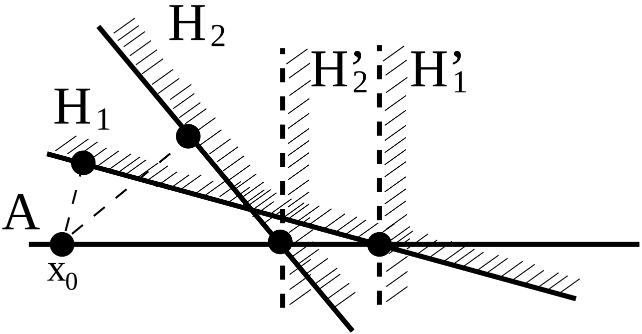
- x_2 can be found by elementary geometry

Construct H' s.t. $A \cap H' = A \cap H$ & its normal n is in $A - A$

- For $A = \{x : Q^T x = b\}$ and $H = \{x : \tilde{a}^T x \leq \tilde{b}\}$,
  can get $H' = \{x : \tilde{a}^T[I - QQ^T]x \leq \tilde{b} - Qb\}$.

Equality constraints in GI algorithm

For SIP with $K := \cap_{i=1}^{r} K_i \neq \emptyset$, if K_1 an affine space, we project onto rotated halfspaces so that all iterates lie in K_1.



In GI algorithm, prefer most violated halfspace to update.

- $P_{H_i'}(x)$ equivalent to $P_{A \cap H_i}(x)$ if $x \in A$.
- H_1' most violated halfspace, but $d(x_0, H_1) < d(x_0, H_2)$.

Outline/ Summary

- Improving a multiple-term superlinear convergence result
  - Reducing size of QP
  - Quadratic convergence under additional regularity
  - Solving QP partially
- Practical to solve QP partially with inner GI steps
- Infeasibility detection: certificate in finitely many steps
- Comparable results in Convex Inequality Problems (CIP)
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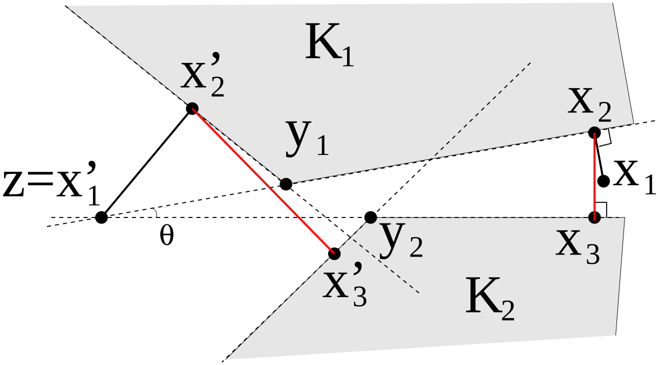
Nonconvex problems

Consider alternating projections for 2 sets, K_1 and K_2.

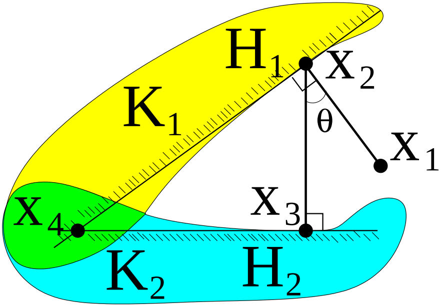
- $\|x_i - x_{i+1}\|$ estimates $d(K_1, K_2)$, and is decreasing.

This property is lost for SHQP, even for convex problems.



Note $\|x_2' - x_3'\| > \|x_2 - x_3\|$.
Above example in $\mathbb{R}^2$ s.t. $K_1 \cap K_2 = \emptyset$, but can easily use ideas here to get example $\mathbb{R}^3$ s.t. $K_1 \cap K_2 \neq \emptyset$.

2 step SHQP for nonconvex problems



$x_2 = P_{K_1}(x_1)$,
$x_3 = P_{K_2}(x_2)$,
$\theta = \angle x_1 x_2 x_3$,
$x_4 := P_{H_1 \cap H_2}(x_1)$.
(Note: $x_4 = P_{H_1 \cap H_2}(x_2)$
$= P_{H_1 \cap H_2}(x_3)$)

Theorem:
For all $\bar{\theta} \in (0, \pi/2)$ and $\zeta \in (0,1)$, we can find $\epsilon > 0$ s.t. if

- K_1 and K_2 are $\big(\sin\epsilon, (\kappa+2)\|x_2 - x_3\|\big)$-subregular at x_2 and x_3 respectively
- $\theta \in (0, \bar{\theta})$,

then $d(x_4, K_1 \cap K_2)^2 \leq -\zeta \|x_3 - x_4\|^2 + d(x_3, K_1 \cap K_2)^2$.

Wrapping up: The Outline/ Summary once more

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The end. Thanks!