Characterizing the generalized derivatives of a set-valued map: Extending the Aubin and Mordukhovich criterions

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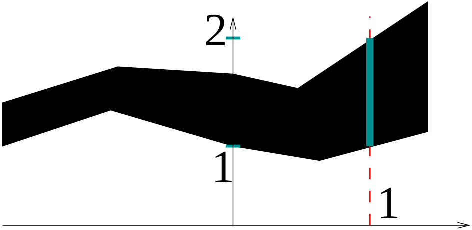
Set-valued maps

- $S : X \rightrightarrows Y$ means $S(x) \subset Y$.



Example: $S : \mathbb{R} \rightrightarrows \mathbb{R}$, and $S(1) = [1, 2]$.

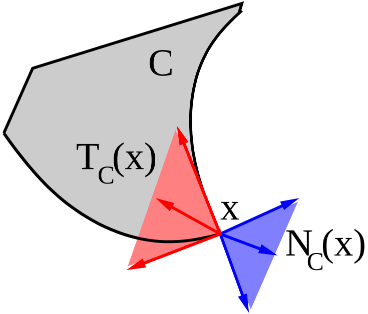
Consider

$$P(u, v) = \inf_{x \in S(u)} v^T x.$$

Set-valued map $S : U \rightrightarrows X$ useful for properties of $P(\cdot, \bar{v})$.

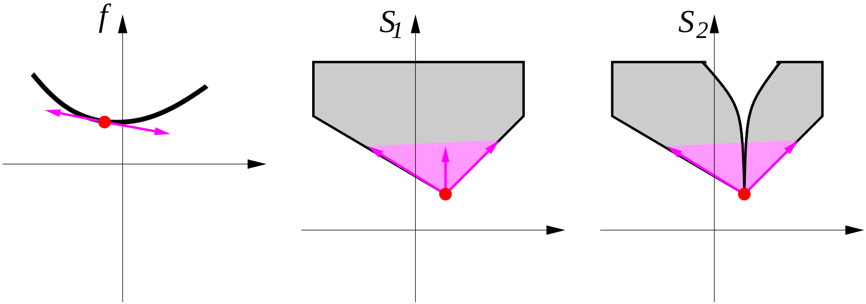
Tangent and normal cones

- Nonsmoothness of sets
  - Tangent cones and normal cones
  - Limiting normal cone $N_C(x)$ is the set of all limits of nearby normals.



Problem with derivatives from tangent cones

- For $f : X \to Y$, can define derivatives via tangent cones of $\mathrm{gph}(f) \subset X \times Y$



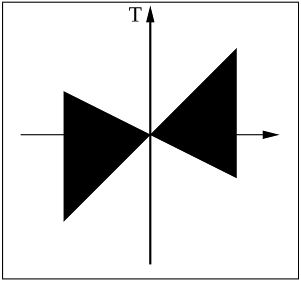
- But graphical derivatives of set-valued maps not useful
  - Right limit of $S_2(x)$ can vary a lot

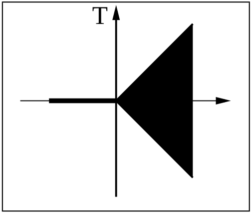
Processes

- $H : X \rightrightarrows Y$ is a process if

$$0 \in H(0) \text{ and } \alpha H(x) = H(\alpha x) \text{ for all } \alpha > 0, x \in X.$$

Examples: Linear maps are processes,
$H(x) := \kappa \|x\| \mathbb{B}$ is a process,
where $\mathbb{B} = \{x \mid \|x\| \leq 1\}$.





Differentiability of Single-valued Maps

For $f : X \to Y$,

- f is strictly differentiable at $\bar{x}$ if
  there is linear $A : X \to Y$ s.t. $\forall\ \delta > 0$, $\exists$ nh'd U_δ s.t.

$$f(x) \in f(x') + A(x - x') + \delta\|x - x'\|\mathbb{B} \text{ for all } x, x' \in U_\delta.$$

For $f : X \to Y$ and $H : X \rightrightarrows Y$ a process,

- f is strictly H-differentiable at $\bar{x}$ if $\forall\ \delta > 0$, $\exists$ nh'd U_δ s.t.

$$f(x) \in f(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} \text{ for all } x, x' \in U_\delta.$$

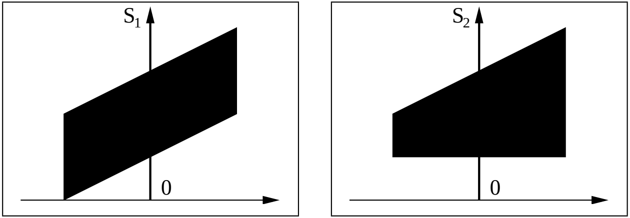
- (First studied in (Ioffe '81), (Thibault '82))

Strict H-differentiability (Pang '09)

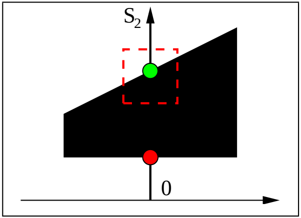
Definition

For $S : X \rightrightarrows Y$ and $H : X \rightrightarrows Y$ a process,
S is **strictly H-differentiable at $\bar{x}$** if
$\forall\ \delta > 0$, $\exists$ n'hd U_δ of $\bar{x}$ s.t.

$$S(x) \subset S(x') + \underbrace{H(x - x') + \delta\|x - x'\|\mathbb{B}}_{(H+\delta)(x-x')} \text{ for all } x, x' \in U_\delta.$$



Generalized Differentiability of Set-valued Maps (Pang '09) based on (Aubin, 84)



Definition

$S : X \rightrightarrows Y$ is pseudo strictly H-differentiable at $(\bar{x}, \bar{y})$ (if $\bar{y} \in S(\bar{x})$) if $\forall\ \delta > 0$, $\exists$ n'hds U_δ of $\bar{x}$ and V_δ of $\bar{y}$ s.t.

$$S(x) \cap V_\delta \subset S(x') + (H + \delta)(x - x') \text{ for all } x, x' \in U_\delta$$

Particular case: $T(w) := \kappa |w| \mathbb{B}$ is Aubin property (Aubin '84).

Aubin and Mordukhovich criterions

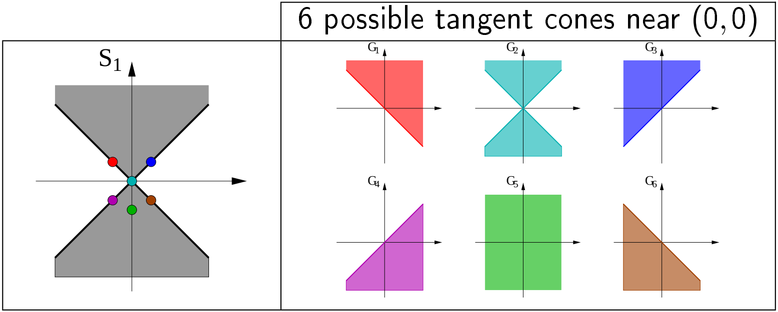
- Recall S is pseudo strictly H-differentiable at $(\bar{x}, \bar{y})$ if $\forall\ \delta > 0$, $\exists$ n'hds U_δ of $\bar{x}$ and V_δ of $\bar{y}$ s.t.

$$S(x) \cap V_\delta \subset S(x') + (H + \delta)(x - x') \text{ for all } x, x' \in U_\delta$$

  - Aubin ppty (1980s): $H(w) := \kappa \|w\| \mathbb{B}$.
  - Infimum of all κ is $\operatorname{lip} S(\bar{x} \mid \bar{y})$.
- Aubin criterion (Aubin '80s, Dontchev-Quincampoix-Zlateva 2006)
  - Uses tangent cones of graphs to find $\operatorname{lip} S(\bar{x} \mid \bar{y})$.
- Mordukhovich criterion (Mordukhovich 1993)
  - Uses normal cones of graphs to find $\operatorname{lip} S(\bar{x} \mid \bar{y})$.

Characterizing generalized derivatives of set-valued maps

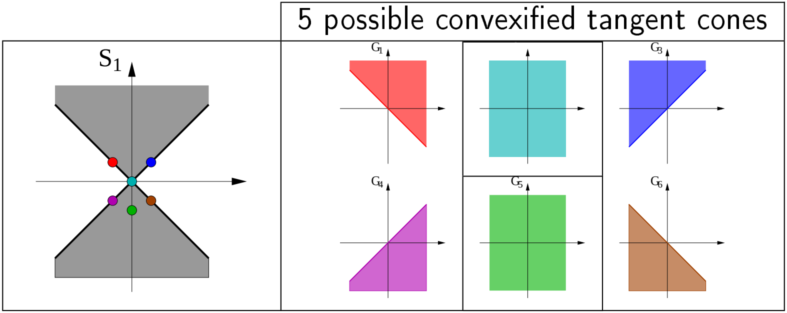
A polyhedral example



- 6 types of tangent cones at points near $(0,0)$.
- Each defines a positively homogeneous map G_i, $1 \leq i \leq 6$.
- Will see: For $H : \mathbb{R} \rightrightarrows \mathbb{R}$ convex valued,
  S is pseudo strictly H-differentiable at $(0,0)$ iff

$$G_i(p) \cap [-H(-p)] \neq \emptyset \text{ for all } p \in \mathbb{R} \setminus \{0\}, 1 \leq i \leq 6.$$

Convexified tangent cones



Finite dimensional characterization v2

Theorem

Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$*, and*
let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *be a convex-valued process.*
Then S *is pseudo strictly* H*-differentiable at* $(\bar{x}, \bar{y})$ *iff*

$$G(p) \cap [-H(-p)] \neq \emptyset \text{ for all } p \in \mathbb{R}^n \backslash \{0\} \text{ and } G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \text{ s.t.}$$

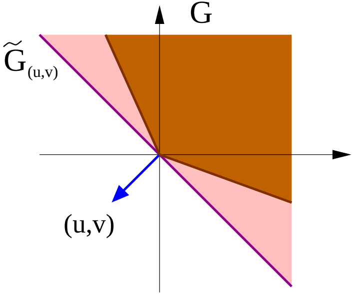
$$\mathrm{gph}(G) = \lim_{j \to \infty} T_{\mathrm{gph}(S)}(x_i, y_i)$$

$$\text{for some } (x_i, y_i) \xrightarrow[\mathrm{gph}(S)]{} (\bar{x}, \bar{y}).$$

- See paper for v1.
- See last page for example.
- Can use $\mathrm{gph}(G) = \mathrm{conv} \lim_{j \to \infty} T_{\mathrm{gph}(S)}(x_i, y_i)$ too.

Lemma on convex processes

For $(u, v) \in N_{\mathrm{gph}(G)}$,
define $\tilde{G}_{(u,v)} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ by
$\mathrm{gph}(\tilde{G}_{(u,v)}) = [\mathbb{R}_+(u, v)]^0$.



Lemma

Suppose $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a convex process, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a graph-convex process. Then

$$G(p) \cap [-H(-p)] \neq \emptyset \; \forall p \in \mathbb{R}^n \setminus \{0\}$$

iff $\tilde{G}_{(u,v)}(p) \cap [-H(-p)] \neq \emptyset \; \forall p \in \mathbb{R}^n \setminus \{0\}$ *and* $(u, v) \in N_{\mathrm{gph}(G)}$.

Finite dimensional characterization v3

Theorem

Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$,
$H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *be a convex-valued process.*
Then S *is pseudo strictly* H*-differentiable at* $(\bar{x}, \bar{y})$ *iff*

$$\tilde{G}_{(u,v)}(p) \cap [-H(-p)] \neq \emptyset \qquad \text{for all } p \in \mathbb{R}^n \backslash \{0\} \text{ and } (u,v) \in N_{\mathrm{gph}(S)}(\bar{x}, \bar{y}),$$

or equivalently, for all $p \in \mathbb{R}^n \backslash \{0\}$ *and* $u \in \mathbb{R}^m$,

$$\min_{y \in H(p)} \langle u, y \rangle \leq \min_{v \in D^* S(\bar{x}|\bar{y})(u)} \langle v, p \rangle,$$

$$\text{or} \quad \min_{y \in H(p)} \langle u, y \rangle \leq \min_{v \in \mathrm{cl\,co}\, D^* S(\bar{x}|\bar{y})(u)} \langle v, p \rangle.$$

In other words, $D^* S(\bar{x} \mid \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ determines the possible generalized derivatives $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$.

Relationship between $D^*S(\bar{x} \mid \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$

For $D : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, define $\mathscr{H}(D)$ by

$$\begin{aligned}\mathscr{H}(D) \;:=\; \{H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m : \;& H \text{ convex-compact-valued process} \\ & \|H\|^+ < \infty, \text{ and for all } p \in \mathbb{R}^n \backslash \{0\} \text{ and } u \in \mathbb{R}^m, \\ & \min_{y \in H(p)} \langle u, y \rangle \le \min_{v \in \operatorname{cl}\operatorname{co} D(u)} \langle v, p \rangle \},\end{aligned}$$

i.e., $\mathscr{H}(D^*S(\bar{x} \mid \bar{y}))$ is set of all gen. derv. $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$.

Theorem

For $D_i : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$,
$\mathscr{H}(D_1) = \mathscr{H}(D_2)$ implies $\operatorname{cl}\operatorname{co} D_1 = \operatorname{cl}\operatorname{co} D_2$.
(i.e., can figure out $D^*S(\bar{x} \mid \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ from all valid $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$.)