1. Consider the following Poisson problem:

$$-\nabla^2 u = f, \quad (x, y) \in \Omega = (0, 1) \times (0, 1)$$

with boundary conditions:

$$u(0, y) = g_1(y), \quad u(1, y) = g_2(y), \quad y \in [0, 1]$$

$$\frac{\partial u}{\partial y}(x, 0) = g_3(x), \quad u(x, 1) = g_4(x), \quad x \in [0, 1].$$

The functions $f(x, y), g_1(y), g_2(y), g_3(x)$ and $g_4(x)$ are determined so that the problem has the exact solution:

$$u(x, y) = \frac{a}{\pi} e^{-a(2(x-1/2)^2+(y-1/2)^2)}.$$

(a). Introduce an uniform mesh with step size $h = 1/(N + 1)$ in both $x$ and $y$ directions. Discretize the Laplacian by the five-point formula. Use the following first order approximation for the Neumann boundary condition at $y = 0$:

$$\frac{\partial u}{\partial y}(x, 0) \approx \frac{u(x_i, h) - u(x_i, 0)}{h}, \quad i = 1, 2, \ldots, N.$$ 

Give the finite difference equations you obtain. How many unknowns in your system? Write the system as a matrix equation using natural ordering for the unknowns, as discussed in class.

Solve the problem with $N = 20, 40, 80$ and $160$ for $a = 10$. Plot the solution. Calculate the maximum error for each $N$. Plot the error against $N$ in a log-log scale. What is the order of convergence?

(b). Replace the first order approximation of the Neumann boundary condition with central difference:

$$\frac{\partial u}{\partial y}(x, 0) \approx \frac{u(x_i, h) - u(x_i, -h)}{2h}, \quad i = 1, 2, \ldots, N$$

where $\{(x_i, -h)\}$ are ghost points. Give the difference equations. How many unknowns in your system? Solve the problem with $N = 20, 40, 80$ and $160$ for $a = 10$. Plot the maximum error against $N$ in a log-log scale. What is the order of convergence?
2. Consider the boundary value problem:

\[
\begin{cases}
-\varepsilon^2 u'' + u = 0, & 0 < x < 1 \\
u(0) = 1/\varepsilon, & u'(1) = 0.
\end{cases}
\]

This equation has the exact solution:

\[u(x) = \frac{\cosh \left(\frac{(1 - x)}{\varepsilon}\right)}{\varepsilon \cosh \left(\frac{1}{\varepsilon}\right)}.
\]

(a). Give the weak formulation of the problem.

(b). Solve the problem using the finite element method with piecewise linear function on a uniform grid with stepsize size \( h = 1/N \). How many basis functions are needed? Give the basis functions you are using. Derive the linear system of equations, and write it in the matrix form \( Au = b \). Give the size of the system, the entries \( a_{ij} \) and \( b_i \).

Solve the problem with \( N = 40, 80, 160, 320 \) and \( 640 \) for \( \varepsilon = 0.1 \). Plot the numerical solution. Plot the error in the \( L^2 \) and \( H^1 \) norms against \( N \) on a log-log scale. What order of convergence in these two norms do you observe?