1. Consider the \( \theta \)-scheme for the heat equation \( u_t = u_{xx} \):

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = \theta \left( \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \right) + (1 - \theta) \left( \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right).
\]

(a). Show that the scheme is unconditionally stable for \( \theta \geq 1/2 \). What is the stability condition for \( \theta = 0 \)?

(b). Consider the heat equation \( u_t = u_{xx} \) with the boundary condition

\[
u(0, t) = 0, \quad u(1, t) = \frac{1}{2} \left( 1 + e^{-5t} \right) \]

and the initial condition

\[
u(x, 0) = \sin^4(\pi x) + x.
\]

Solve the problem using the \( \theta \)-method with \( \theta = 0 \) and \( \theta = 1/2 \) respectively. Use \( N = 40 \) points in space. Specify the Courant number \( \mu \) you used and show a graph of your solution at times \( t = 0, 0.01, 0.04, 0.1 \) and 0.5.

2. Analyze the Lax-Friedrichs scheme for \( u_t + cu_x = 0 \):

\[
u_j^{n+1} = \frac{1}{2} (1 - \mu) u_{j+1}^n + \frac{1}{2} (1 + \mu) u_{j-1}^n,
\]

where \( \mu = c \frac{\Delta t}{\Delta x} \).

(a). What is the order of accuracy in space and time? (Carry out the Taylor series expansions.)

(b). Analyze the stability of the scheme.

3. Consider the advection equation \( u_t + u_x = 0 \) for \( 0 \leq x \leq 1 \) and \( 0 < t < 2 \) with periodic boundary condition and the initial condition \( u(x, 0) = \sin(6\pi x) \).

Solve the problem using the upwind scheme and the Lax-Wendroff scheme respectively. For each scheme, try different values of \( \mu = \Delta t/\Delta x \) and different values of \( \Delta t \) and \( \Delta x \) with \( \mu \) fixed. Plot your results, and report on what you see in terms of convergence and other interesting phenomena.