Lecture 4

1 Strong Law of Large Numbers

We will give two proofs for the SLLN, assuming only integrability of the i.i.d. random variables. The first approach is due to Etemadi, which reduces the problem first to the case of i.i.d. non-negative random variables. The second approach reduces the problem to the question of almost sure convergence of an infinite series of independent, but not necessarily identically distributed random variables. The second approach will also allow us to treat non-integrable random variables, and to consider S_n/n^α with $\alpha \neq 1$.

Theorem 1.1 [Strong Law of Large Numbers] *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Then*

$$\lim_{n\to\infty} \frac{S_n}{n} := \lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{n} = \mathbb{E}[X_1] \qquad \text{almost surely.} \tag{1.1}$$

Proof. Let $X_i^+ := \max\{X_i, 0\}$ and $X_i^- = \max\{-X_i, 0\}$. Then $\mathbb{E}[X_i^+] < \infty$, $\mathbb{E}[X_i -] < \infty$, and $X_i = X_i^+ - X_i^-$. The SLLN for S_n/n will follow once we prove the SLLN for $\sum_{i=1}^n X_i^+/n$ and $\sum_{i=1}^n X_i^-/n$. Therefore to prove (1.1), we can assume without loss of generality that $X_i \geq 0$. The advantage of having $X_i \geq 0$ is that S_n is monotone in n, which allows us to first prove the SLLN along a sparse subsequence of $n_k \uparrow \infty$, and then use monotonicity to control the indices between the consecutive n_k's.

As in our proof of the WLLN, we will truncate $(X_i)_{1\leq i\leq n}$, except we now choose a different truncation for each X_i. More precisely, for each $i \in \mathbb{N}$, let $Y_i = X_i 1_{\{X_i \leq i\}}$. Note that

$$\sum_{i=1}^\infty \mathbb{P}(Y_i \neq X_i) = \sum_{i=1}^\infty \mathbb{P}(X_i > i) = \sum_{i=1}^\infty \mathbb{E}[1_{\{X_1 > i\}}] = \mathbb{E}\Big[\sum_{i=1}^\infty 1_{\{i < X_1\}}\Big] \leq \mathbb{E}[X_1] < \infty.$$

Therefore by Borel-Cantelli, a.s. the events $\{X_i \neq Y_i\}_{i\in\mathbb{N}}$ can only occur finitely many times. In particular, almost surely

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{n} = \lim_{n\to\infty} \frac{\sum_{i=1}^n Y_i}{n}. \tag{1.2}$$

To prove that the second limit equals $\mathbb{E}[X_1]$ a.s., we claim that it suffices to prove that for any $\rho > 1$, with $n_k = \lceil \rho^k \rceil$, we have

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{n_k} Y_i}{n_k} = \mathbb{E}[X_1] \qquad \text{a.s.} \tag{1.3}$$

Indeed, given $\rho > 1$, for any $n \in \mathbb{N}$, let $k := k(n)$ be the unique integer with $n_{k-1} < n \leq n_k$. Then the monotonicity of $\sum_{i=1}^n Y_i$ in n and (1.3) imply that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n Y_i}{n} \leq \limsup_{k\to\infty} \frac{\sum_{i=1}^{n_k} Y_i}{n_{k-1}} = \lim_{k\to\infty} \frac{\sum_{i=1}^{n_k} Y_i}{n_k} \frac{n_k}{n_{k-1}} = \rho\mathbb{E}[X_1],$$
$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n Y_i}{n} \geq \liminf_{k\to\infty} \frac{\sum_{i=1}^{n_{k-1}} Y_i}{n_k} = \lim_{k\to\infty} \frac{\sum_{i=1}^{n_{k-1}} Y_i}{n_{k-1}} \frac{n_{k-1}}{n_k} = \rho^{-1}\mathbb{E}[X_1].$$

Since $\rho > 1$ can be chosen arbitrarily close to 1, it follows that $\lim_{n\to\infty} \sum_{i=1}^n Y_i/n = \mathbb{E}[X_1]$.

It only remains to verify (1.3). Note that by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}[Y_i]}{n} = \lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}[X_1 1_{\{X_1\le i\}}]}{n} = \lim_{n\to\infty} \mathbb{E}\Big[X_1 \frac{\sum_{i=1}^n 1_{\{X_1\le i\}}}{n}\Big] = \mathbb{E}[X_1].$$

Therefore (1.3) is equivalent to

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{n_k} \tilde{Y}_i}{n_k} = 0, \qquad \text{where } \tilde{Y}_i = Y_i - \mathbb{E}[Y_i]. \tag{1.4}$$

For any $\epsilon > 0$, by Markov's inequality,

$$\mathbb{P}\Big(\Big|\frac{\sum_{i=1}^n \tilde{Y}_i}{n}\Big| > \epsilon\Big) \le \epsilon^{-2} n^{-2} \mathbb{E}\Big[\Big(\sum_{i=1}^n \tilde{Y}_i\Big)^2\Big] = \epsilon^{-2} n^{-2} \sum_{i=1}^n \mathbb{E}[\tilde{Y}_i^2] \le \epsilon^{-2} n^{-2} \sum_{i=1}^n \mathbb{E}[Y_i^2]$$

Therefore

$$\begin{aligned}\sum_{k=1}^\infty \mathbb{P}\Big(\Big|\frac{\sum_{i=1}^{n_k} \tilde{Y}_i}{n_k}\Big| > \epsilon\Big) &\le \epsilon^{-2} \sum_{k=1}^\infty \rho^{-2k} \sum_{i=1}^{\rho^k} \mathbb{E}[Y_i^2] \\ &= \epsilon^{-2} \sum_{i=1}^\infty \mathbb{E}[Y_i^2] \sum_{k:\, \rho^k \ge i} \rho^{-2k} \le C_\rho \epsilon^{-2} \sum_{i=1}^\infty \mathbb{E}[Y_i^2]/i^2,\end{aligned}$$

where $C_\rho = \sum_{m=0}^\infty \rho^{-2m} = \rho^2/(\rho^2-1)$. We will show that $\sum_{i=1}^\infty \mathbb{E}[Y_i^2]/i^2 < \infty$, which then implies by Borel-Cantelli that the events $\{|\frac{\sum_{i=1}^{n_k} \tilde{Y}_i}{n_k}| > \epsilon\}$ can only occur a.s. for finitely many k's for any given $\epsilon > 0$, which then implies (1.4). Note here the importance of restricting to the exponentially increasing subsequence $n_k = \lceil \rho^k \rceil$, which allows the application of Borel-Cantelli.

To conclude the proof, note that since $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z \ge l)\mathrm{d}l$ for any non-negative random variable Z, we have

$$\begin{aligned}\sum_{n=1}^\infty n^{-2}\mathbb{E}[Y_n^2] = \sum_{n=1}^\infty n^{-2}\mathbb{E}[X_1^2 1_{\{X_1\le n\}}] &= \sum_{n=1}^\infty n^{-2} \int_0^\infty \mathbb{P}(X_1^2 1_{\{X_1\le n\}} \ge l^2)\mathrm{d}(l^2) \\ &= \sum_{n=1}^\infty n^{-2} \int_0^\infty \mathbb{P}(n \ge X_1 \ge l) 2l\mathrm{d}l \\ &\le \int_0^\infty \Big(2l \sum_{n=\lceil l\rceil}^\infty n^{-2}\Big)\mathbb{P}(X_1 \ge l)\mathrm{d}l \\ &\le C\int_0^\infty \mathbb{P}(X_1 \ge l)\mathrm{d}l = C\mathbb{E}[X_1] < \infty,\end{aligned} \tag{1.5}$$

where in the first inequality, we used Fubini to interchange the integral and summation, and we used the fact that $2l\sum_{i=\lceil l\rceil}^\infty i^{-2} \le C$ for some $C > 0$ uniformly for all $l > 0$, which follows by comparing $\sum_{i=k}^\infty i^{-2}$ with the integral $\int_{k-1}^\infty x^{-2}\mathrm{d}x$. ∎

Corollary 1.2 *If $(X_i)_{i\in\mathbb{N}}$ are i.i.d. with $\mathbb{E}[X_i^+] = \infty$ and $\mathbb{E}[X_i^-] < \infty$, then $\frac{1}{n}\sum_{i=1}^n X_i \to \infty$ almost surely.*

Proof. Since $\frac{1}{n}\sum_{i=1}^n X_i = \frac{1}{n}\sum_{i=1}^n X_i^+ - \frac{1}{n}\sum_{i=1}^n X_i^-$, where the second term converges a.s. to $\mathbb{E}[X_1^-] < \infty$, it suffices to show that the first term tends to ∞ a.s. For $M > 0$, note that a.s.

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^+ \geq \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \min\{X_i^+, M\} = \mathbb{E}[\min\{X_1^+, M\}],$$

which increases to ∞ as $M \uparrow \infty$ by the monotone convergence theorem and the assumption $\mathbb{E}[X_1^+] = \infty$. Therefore $\lim_{n\to\infty} \frac{1}{n} X_i^+ = \infty$. ∎

2 Convergence of Random Series

We now give a second approach for proving the SLLN, which transforms the problem to the a.s. convergence of a single series of independent, but not necessarily identically distributed random variables. (Note that the SLLN is concerned with a triangular array $X_{n,1} + \cdots + X_{n,n}$, where $X_{n,i} = X_i/n$.) We first prove some preliminary results, including Kolmogorov's inequality and Kolmogorov's three-series theorem.

Lemma 2.1 [Kolmogorov's Inequality] *Let $(X_i)_{i\in\mathbb{N}}$ be independent random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{V}\mathrm{ar}(X_i) = \sigma_i^2$. Let $S_k := \sum_{i=1}^k X_i$, and let $M_n := \max\{|S_k| : 1 \leq k \leq n\}$. Then*

$$\mathbb{P}(M_n \geq l) \leq l^{-2}\mathbb{E}[S_n^2] = l^{-2}\sum_{i=1}^n \sigma_i^2.$$

Kolmogorov's inequality controls the fluctuation of sums of centered independent random variables by their variances. It is a special case of Doob's inequality in martingale theory.

Proof. Let $\tau := \min\{1 \leq i \leq n : |S_i| \geq l\}$. Then $\{M_n \geq l\}$ is the disjoint union of the events $\{\tau = k\}$, for $1 \leq k \leq n$. We can write

$$\begin{aligned}
\mathbb{P}(M_n \geq l) = \sum_{k=1}^n \mathbb{P}(\tau = k, |S_k| \geq l) \ &\leq \ \sum_{k=1}^n l^{-2}\mathbb{E}[1_{\{\tau=k\}} S_k^2] \\
&= \ \sum_{k=1}^n l^{-2}\mathbb{E}[1_{\{\tau=k\}}(S_n^2 - 2S_k(S_n - S_k) - (S_n - S_k)^2)] \\
&\leq \ \sum_{k=1}^n l^{-2}\mathbb{E}[1_{\{\tau=k\}} S_n^2] = l^{-2}\mathbb{E}[1_{\{M_n\geq l\}} S_n^2] \\
&\leq \ l^{-2}\mathbb{E}[S_n^2] = l^{-2}\sum_{i=1}^n \sigma_i^2,
\end{aligned}$$

where we used $(S_n - S_k)^2 \geq 0$, and the fact that $\mathbb{E}[1_{\{\tau=k\}} S_k(S_n - S_k)] = 0$ because S_k and the event $\{\tau = k\}$ depend only on $X_1, \ldots, X_k$ while $S_n - S_k = X_{k+1} + \cdots + X_n$ has mean 0 and is independent of $X_1, \ldots, X_k$. ∎

Using Kolmogorov's inequality, we can prove the following result.

Theorem 2.2 [Kolmogorov's Two Series Theorem] *Let $(Y_n)_{n\in\mathbb{N}}$ be independent with $\mathbb{E}[Y_n] = a_n$ and $\mathbb{V}\mathrm{ar}(Y_n) = \sigma_n^2$, such that $\sum_{n=1}^\infty a_n$ converges in $\mathbb{R}$ and $\sum_{n=1}^\infty \sigma_i^2 < \infty$. Then $\sum_{n=1}^\infty Y_n$ converges in $\mathbb{R}$ almost surely.*

When $a_n = 0$ for all $n \in \mathbb{N}$, this is a special case of the martingale convergence theorem.

Proof. Since $\mathbb{V}\mathrm{ar}(Y_n - a_n) = \mathbb{V}\mathrm{ar}(Y_n)$, without loss of generality, we may assume that $a_n = 0$ for all $n \in \mathbb{N}$. Denote $S_n = \sum_{i=1}^n Y_i$. We need to show that with probability 1,

$$\limsup_{n\to\infty} S_n - \liminf_{n\to\infty} S_n = 0.$$

Note that for any $m \in \mathbb{N}$,

$$\limsup_{n\to\infty} S_n - \liminf_{n\to\infty} S_n = \limsup_{n\to\infty}(S_n - S_m) - \liminf_{n\to\infty}(S_n - S_m) \leq 2\max_{k\geq 1}\Big|\sum_{i=1}^k Y_{m+i}\Big|.$$

Therefore for any $\epsilon > 0$ and any $m \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}\big(\limsup_{n\to\infty} S_n - \liminf_{n\to\infty} S_n \geq \epsilon\big) &\leq \mathbb{P}\Big(2\max_{k\geq 1}\Big|\sum_{i=1}^k Y_{m+i}\Big| \geq \epsilon\Big) \\
&= \lim_{N\to\infty} \mathbb{P}\Big(\max_{1\leq k\leq N}\Big|\sum_{i=1}^k Y_{m+i}\Big| \geq \epsilon/2\Big) \\
&\leq \limsup_{N\to\infty} 4\epsilon^{-2}\sum_{i=m+1}^{m+N}\sigma_i^2 = 4\epsilon^{-2}\sum_{i=m+1}^{\infty}\sigma_i^2,
\end{aligned}$$

where in the first equality we used the countable additivity of $\mathbb{P}$, and in the last inequality we used Kolmogorov's inequality. Since $\sum_{i=m+1}^\infty \sigma_i^2 \to 0$ as $m \to \infty$ by our assumption $\sum_{i=1}^\infty \sigma_i^2 < \infty$, it follows that the event $\{\limsup_{n\to\infty} S_n - \liminf_{n\to\infty} S_n \geq \epsilon\}$ has probability 0 for any $\epsilon > 0$, and hence $\lim_{n\to\infty} S_n$ must exist in $\mathbb{R}$ with probability 1. ∎

Although not needed later for the proof of the SLLN, we still give here a necessary and sufficient condition for the convergence of a series of independent random variables.

Theorem 2.3 [Kolmogorov's Three Series Theorem] *Let $(X_n)_{n\in\mathbb{N}}$ be independent. The random series $\sum_{n=1}^\infty X_n$ converges almost surely in $\mathbb{R}$ if and only if the following conditions are satisfied:*

(i) *For some $A > 0$, $\sum_{n=1}^\infty \mathbb{P}(|X_n| \geq A)$ converges;*

(ii) *Let $Y_n := X_n 1_{\{|X_n|\leq A\}}$, then $\sum_{n=1}^\infty \mathbb{E}[Y_n]$ converges;*

(iii) *$\sum_{n=1}^\infty \mathbb{V}\mathrm{ar}(Y_n)$ converges.*

Proof. Let us first show that conditions (i)–(iii) are sufficient. Condition (i) and Borel-Cantelli imply that almost surely, $X_n = Y_n$ for all n large, and hence $\sum_n X_n$ converges if and only if $\sum_n Y_n$ converges. The almost sure convergence of $\sum_n Y_n$ follows from conditions (ii)–(iii) and Kolmogorov's Two Series Theorem.

Conversely, let's assume that $\sum_n X_n$ converges almost surely. Then condition (i) must hold for every $A > 0$, because otherwise there exists some $A > 0$ such that almost surely $\{|X_n| \geq A\}$ occurs infinitely often by Borel-Cantelli, which is incompatible with the convergence of $\sum_n X_n$. Thus (i) holds with $A = 1$, which implies that also $\sum_n Y_n = \sum_n X_n 1_{\{|X_n|\leq 1\}}$ converges. Suppose that we have verified condition (iii), then by Kolmogorov's Two Series Theorem, $\sum_n (Y_n - \mathbb{E}[Y_n])$ also converges, which together with the convergence of $\sum_n Y_n$ imply the convergence of $\sum_n \mathbb{E}[Y_n]$. Therefore we only need to verify condition (iii), assuming that $\sum_n Y_n$ converges a.s., where Y_n are independent and bounded by 1.

We claim that we can assume without loss of generality that $\mathbb{E}[Y_n] = 0$. Indeed, let $(Y'_n)_{n\in\mathbb{N}}$ be an independent copy of $(Y_n)_{n\in\mathbb{N}}$, then $Z_n := Y_n - Y'_n$ is a sequence of independent random variables bounded by 2, $\mathbb{V}\mathrm{ar}(Z_n) = 2\,\mathbb{V}\mathrm{ar}(Y_n)$, and $\sum_n Z_n = \sum_n Y_n - \sum_n Y'_n$ converges almost surely. Therefore we only need to verify condition (iii) for $(Z_n)_{n\in\mathbb{N}}$, which have mean 0. This follows from the lemma below. ∎

Lemma 2.4 *If $\sum_{n=1}^\infty Z_n$ converges for a series of independent random variables Z_n, which have mean 0, variance σ_n^2, and are uniformly bounded by $C > 0$, then $\sum_{n=1}^\infty \sigma_n^2 < \infty$.*

We remark that there is also a generalization of this result for martingales, where Z_n are replaced by the increments of a martingale sequence. To see heuristically why such a result should hold, note that if $S_n := \sum_{i=1}^n Z_i$ is contained in a finite interval $[-L, L]$ for all n, then we cannot have $\sum_{i=1}^\infty \sigma_i^2 = \infty$, because $L^2 \geq \mathbb{E}[S_n^2] = \sum_{i=1}^n \sigma_i^2$ uniformly in n. In general, $(S_n)_{n\in\mathbb{N}}$ cannot be confined to a fixed interval. Nevertheless, S_n converges a.s. to a finite limit means that we can choose L large enough, such that with probability close to 1, S_n is contained in $[-L, L]$ for all n large enough, and the argument can be adapted to work.

Proof. Let $S_n := \sum_{i=1}^n Z_i$. Fix $L > 0$ and let $\tau_L := \min\{i \geq 0 : |S_i| \geq L\}$. Note that by our assumptions, $\mathbb{P}(\{\tau_L = \infty\}) \uparrow 1$ as $L \uparrow \infty$, and $|S_{\tau_L}| \leq L + C$ if $\tau_L < \infty$. The key observation is that

$$\mathbb{E}[S_n^2] = \sum_{i=1}^n \mathbb{E}[Z_i^2] + 2 \sum_{1\leq i<j\leq n} \mathbb{E}[Z_i Z_j] = \sum_{i=1}^n \sigma_i^2,$$

because $(Z_i)_{i\in\mathbb{N}}$ are independent with mean 0, and if we replace n by $n \wedge \tau_L := \min\{n, \tau_L\}$, then

$$\mathbb{E}[S_{n\wedge\tau_L}^2] = \mathbb{E}\Big[\Big(\sum_{j=1}^n Z_j 1_{\{j\leq\tau_L\}}\Big)^2\Big] = \sum_{j=1}^n \mathbb{E}\Big[Z_j^2 1_{\{j\leq\tau_L\}}\Big] + 2 \sum_{1\leq i<j\leq n} \mathbb{E}[Z_i Z_j 1_{\{j\leq\tau_L\}}].$$

Note that the event $\{j \leq \tau_L\} = \{\tau_L \leq j-1\}^c$ is determined by $Z_1, \ldots, Z_{j-1}$ and hence independent of Z_j. Therefore

$$\begin{aligned}(L+C)^2 \;\geq\; \mathbb{E}[S_{n\wedge\tau_L}^2] \;&=\; \sum_{j=1}^n \sigma_j^2 \mathbb{P}(j \leq \tau_L) + 2 \sum_{1\leq i<j\leq n} \mathbb{E}[Z_i 1_{\{j\leq\tau_L\}}]\mathbb{E}[Z_j] \\ &\geq\; \mathbb{P}(\tau_L = \infty) \sum_{j=1}^n \sigma_j^2.\end{aligned}$$

Choose L large such that $\mathbb{P}(\tau_L = \infty) > 0$ and then let $n \uparrow \infty$, we obtain $\sum_{j=1}^\infty \sigma_j^2 < \infty$. ∎

3 Strong Law of Large Numbers Revisited

We now apply Kolmogorov's Two Series Theorem to give another proof of Theorem 1.1, the Strong Law of Large Numbers. The connection between the two (one concerns the a.s. convergence of $\frac{1}{n}\sum_{i=1}^n X_i$, the other $\sum_{i=1}^n Y_i$) is through the next lemma.

Lemma 3.1 [Kronecker's Lemma] *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and $(a_n)_{n\in\mathbb{N}}$ be another sequence with $a_n \uparrow \infty$. If $\sum_{n=1}^\infty \frac{x_n}{a_n}$ converges, then $\lim_{n\to\infty} a_n^{-1} \sum_{i=1}^n x_i = 0$.*

By Kronecker's Lemma, to prove $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i = 0$ a.s., it suffices to show that the random series $\sum_{n=1}^\infty \frac{X_n}{n}$ converges a.s. Proving that a series converges is usually easier than proving that a sequence converges to a specific limit.

Exercise 3.2 Prove Lemma 3.1 by writing $x_m = a_m(b_m - b_{m-1})$, with $b_m = \sum_{i=1}^m x_i/a_i$, and then perform *summation by parts* to the sum $\sum_{m=1}^n x_m = \sum_{m=1}^n a_m(b_m - b_{m-1})$.

Proof of Theorem 1.1 (SLLN). As shown in (1.2) and (1.4), proving $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i = \mathbb{E}[X_1]$ is equivalent to proving that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \tilde{Y}_i = 0,$$

where $\tilde{Y}_i = X_1 1_{\{|X_i|\leq i\}} - \mathbb{E}[X_1 1_{\{|X_i|\leq i\}}]$. By Kronecker's Lemma, it suffices to prove that $\sum_{n=1}^\infty \tilde{Y}_n/n$ converges a.s. Since $\mathbb{E}[\tilde{Y}_n] = 0$, by Kolmogorov's Two Series Theorem, it only remains to show that $\sum_n \mathbb{V}\mathrm{ar}(\tilde{Y}_n/n) < \infty$. Note that

$$\sum_{n=1}^\infty \mathbb{V}\mathrm{ar}\left(\frac{\tilde{Y}_n}{n}\right) \leq \sum_{n=1}^\infty \frac{1}{n^2}\mathbb{E}[Y_n^2] = \sum_{n=1}^\infty \frac{1}{n^2}\mathbb{E}[X_1^2 1_{\{|X_1|\leq n\}}],$$

which has been shown to be finite in (1.5). This completes the proof. ∎

Kolmogorov's Three Series Theorem can be used to investigate choices of $a_n \uparrow \infty$ slower than n, and yet still give $\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{a_n} = 0$ for an i.i.d. sequence $(X_i)_{i\in\mathbb{N}}$ with mean 0; or consider cases with $\mathbb{E}[|X_1|] = \infty$ where we need to choose $a_n \uparrow \infty$ faster than n. We collect here several such results, and refer the proof to Durrett [1].

Theorem 3.3 *Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. with $\mathbb{E}[X_1] = 0$ and $\mathbb{V}\mathrm{ar}(X_1) = \sigma^2 < \infty$. Let $S_n := \sum_{i=1}^n X_i$. Then for any $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\epsilon}} = 0 \qquad a.s.$$

The optimal scaling is given by the Law of Iterated Logarithm (LIL), which states that $\limsup_{n\to\infty} S_n/\sqrt{n\log\log n} = \sigma\sqrt{2}$ a.s.

Theorem 3.4 *Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[|X_1|^p] < \infty$ for some $1 < p < 2$, then $\lim_{n\to\infty} S_n/n^{1/p} = 0$ a.s.*

Exercise 3.5 *Show that if $(X_n)_{n\in\mathbb{N}}$ are i.i.d. with $S_n/n^{1/p} \to 0$ a.s. for some $p > 0$, then $\mathbb{E}[|X_1|^p] < \infty$.*

Theorem 3.6 *Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. with $\mathbb{E}[|X_1|] = \infty$. Let a_n be a sequence of positive numbers such that a_n/n is increasing. If $\sum_n \mathbb{P}(|X_1| \geq a_n) < \infty$, then $\lim_{n\to\infty} |S_n|/a_n = 0$; otherwise $\limsup_{n\to\infty} |S_n|/a_n = \infty$.*

References

[1] R. Durrett. *Probability: Theory and Examples*, Duxbury Press.