

Homework 2. Due March 30, 2009

Q1. Let X be an irreducible countable state Markov chain, and F a finite subset. Let $\tau = \inf\{n \geq 0 : X_n \notin F\}$. Show that $\tau < \infty$ almost surely, and furthermore, $\mathbb{P}(\tau > n)$ decays exponentially fast in n .

Q2. Let X be a homogeneous Markov chain with state space $\{0, 1, \dots, N\}$. Suppose furthermore that X_n is a martingale for any initial distribution. Show that

- (i) 0 and N are absorbing states, i.e., the transition probabilities have $p(0, 0) = p(N, N) = 1$.
- (ii) If $\mathbb{P}_x(\tau_0 \wedge \tau_N < \infty) > 0$ for all x , then all states other than 0 and N are transient, and $\mathbb{P}_x(\tau_N < \tau_0) = \frac{x}{N}$.

Q3. Let X an irreducible countable state Markov chain with state space S . For $x, y \in S$, let $w_{x,y} = \mathbb{P}_x(\tau_y < \tau_x)$, where $\tau_z = \inf\{n \geq 1 : X_n = z\}$. Show that for the Markov chain starting from x , the expected number of visits to y before returning to x satisfies

$$\sum_{n=1}^{\infty} \mathbb{P}_x(X_n = y, \tau_x > n) = \frac{w_{xy}}{w_{yx}}.$$

Q4. Let X be a birth-death chain on state space $\{0, 1, \dots\}$ with transition probabilities $\Pi(i, i+1) = p_i$, $\Pi(i, i-1) = q_i$, $\Pi(i, i) = r_i$, where $\Pi(0, -1) = 0$. Assume that $p_i, q_i > 0$ so that the chain is irreducible. Find a necessary and sufficient condition for the chain to be positive recurrent.

Q5. A knight jumps aimlessly on an 8X8 empty chessboard. At each step, it chooses an admissible position with equal probability and jumps there. Find the stationary distribution for this Markov chain. Is it reversible? If the knight starts from a corner, what is the expected number of jumps it would take for the knight to return to the same corner for the first time?

Q6. Let X be a Markov chain with countable state space S , transition matrix Π , and stationary measure μ . For $x, y \in S$ with $\Pi(x, y) > 0$ and $\mu(x), \mu(y) > 0$, define $\hat{\Pi}(y, x) = \frac{\mu(x)\Pi(x, y)}{\mu(y)}$. Show that

- (i) $\hat{\Pi}$ is also a Markov chain transition kernel with stationary measure μ .
- (ii) If \hat{X} denotes a Markov chain with transition kernel $\hat{\Pi}$, show that \hat{X} is irreducible if and only if X is irreducible, and \hat{X} is transient (null/positive recurrent) if and only if X is.
- (iii) If X_0 has distribution μ so that the chain is in equilibrium, show that the time-reversed sequence $(X_n, X_{n-1}, \dots, X_0)$ is equally distributed with $(\hat{X}_0, \hat{X}_1, \dots, \hat{X}_n)$ where \hat{X}_0 has distribution μ .

If we think of $\mu(x)\Pi(x, y)$ as the mass flow from x to y , then $\hat{\Pi}$ simply defines a Markov chain which reverses the flow in equilibrium. It is thus no surprise that (iii) says that, in equilibrium, \hat{X} is simply the time reversal of X . When X is reversible, $\hat{\Pi} = \Pi$ so that (iii) reads $(X_n, \dots, X_0) \stackrel{d}{=} (X_0, \dots, X_n)$, hence *reversible*. But this is true only in equilibrium.