

Homework 2. Due April 18, 2011

Q1. Let X be an irreducible countable state Markov chain, and F a finite subset. Let $\tau = \inf\{n \geq 0 : X_n \notin F\}$. Show that $\tau < \infty$ almost surely, and furthermore, $\mathbb{P}(\tau > n)$ decays exponentially fast in n .

Q2. Let X be an irreducible recurrent countable state Markov chain with state space S . Let $\tau_x := \inf\{n \geq 1 : X_n = x\}$ for any $x \in S$, and let $\mathbb{E}_x[\cdot]$ denote expectation with respect to a Markov chain starting from x . Prove that if X is recurrent, then for any $x \neq y \in S$, we have

$$\mathbb{E}_x \left[\sum_{i=1}^{\tau_x} 1_{\{X_i=y\}} \right] \mathbb{E}_y \left[\sum_{i=1}^{\tau_y} 1_{\{X_i=x\}} \right] = 1.$$

Q3. For some $r > 0$ and $a \in \{\pm 1\}$, let X be a random walk on \mathbb{Z} with transition probabilities

$$\begin{aligned} \Pi(x, x+1) &= \frac{1}{2}(1 + ax^{-r}) = 1 - \Pi(x, x-1) && \text{for } x \geq 2, \\ \Pi(x, x-1) &= \frac{1}{2}(1 + a|x|^{-r}) = 1 - \Pi(x, x+1) && \text{for } x \leq -2, \\ \Pi(x, x \pm 1) &= \frac{1}{2} && \text{for } x = 0, \pm 1. \end{aligned}$$

- (i) By constructing suitable non-negative super-harmonic functions and super-martingales, determine for which values of $a \in \{\pm 1\}$ and $r > 0$, is the random walk recurrent, respectively, transient.
- (ii) By constructing a reversible measure, determine for which values of $a \in \{\pm 1\}$ and $r > 0$, is the random walk positive recurrent.

Q4. A knight jumps aimlessly on an 5×5 empty chessboard. At each step, it chooses an admissible position with equal probability and jumps there.

- (i) Find the stationary distribution for this Markov chain. Is it reversible?
- (ii) If the knight starts from the center, what is the probability that it will hit one of the corners before returning to the center?
- (iii) If the knight starts from a corner, what is the expected time for the knight to reach the center?

Q5. Let X be an irreducible Markov chain with a finite state space S and transition matrix Π . We can use Π to construct a continuous time Markov process $(Y_t)_{t \geq 0}$ on S via the following approximation scheme. For each $\delta \in (0, 1)$, let $\Pi^{(\delta)} := \delta\Pi + (1 - \delta)I$, and let $(X_n^{(\delta)})_{n \geq 0}$ denote the Markov chain with transition matrix $\Pi^{(\delta)}$. Next we extend the definition of $X^{(\delta)}$ to all times $t \geq 0$ by defining $X_t^{(\delta)} := X_{\lfloor t \rfloor}^{(\delta)}$. We now speed up time by a factor of $\lambda\delta^{-1}$ for some $\lambda > 0$ and define a new process $Y_t^{(\delta)} := X_{\lambda\delta^{-1}t}^{(\delta)}$. The claim is that if $X_0^{(\delta)}$ has the same distribution for all $\delta > 0$, then as $\delta \downarrow 0$, the sequence of processes $(Y_t^{(\delta)})_{t \geq 0}$ converges in distribution to a limiting process $(Y_t)_{t \geq 0}$.

- (i) Give a direction construction of the limiting process Y , and prove that it is Markov.
- (ii) The Markov process Y defines a semigroup $S_t f(x) := \mathbb{E}_x[f(Y_t)]$, for any $t \geq 0$ and $f : S \rightarrow \mathbb{R}$. Prove that $(S_t)_{t \geq 0}$ is indeed a semigroup, i.e., $S_{t+s} = S_t S_s = S_s S_t$. Determine the generator of the semigroup, defined by $Lf := \lim_{t \downarrow 0} \frac{S_t f - f}{t}$, where the convergence is in $\|\cdot\|_\infty$ norm.

Q6. Let X be a nearest-neighbor random walk on \mathbb{Z} . Use coupling to prove that if $h : \mathbb{Z} \rightarrow \mathbb{R}$ is a bounded harmonic function for X , then h must be a constant function. Now relax the nearest-neighbor assumption. Finally extend this result to all irreducible random walks on \mathbb{Z}^d for $d \geq 1$ by coupling one coordinate at a time.

Q7. Let $(X_n)_{n \in \mathbb{Z}}$ be a stochastic process with $X_n \in \{0, 1\}$ a.s. for all $n \in \mathbb{Z}$. Suppose that X is ergodic with respect to the shift map $(\theta X)_n := X_{n+1}$, $n \in \mathbb{Z}$, and X is a.s. not identically 0.

- (i) Show that $\mathbb{P}(X_0 = 1) > 0$. If μ denotes the conditional distribution of $\tau^+ := \inf\{n \geq 1 : X_n = 1\}$ conditional on $X_0 = 1$, then use the ergodic theorem to show that μ has finite mean. Furthermore, show that conditional on $X_0 = 1$, $\tau^- := \sup\{n \leq -1 : X_n = 1\}$ has the same distribution as $-\tau^+$. However, conditional on $X_0 = 1$, τ^+ and τ^- need not be independent in general.
- (ii) Let ν denote the distribution of $\inf\{n \geq 1 : X_n = 1\} - \sup\{n \leq 0 : X_n = 1\}$. Determine the relation between ν and μ and show that ν need not have finite first moment in general.