

## Lecture 6. Feb 28, 2011

### 1 Classification of states

We have shown that all states of an irreducible countable state Markov chain must be of the same type. This gives rise to the following classification.

**Definition 1.1 [Classification of irreducible countable state Markov chains]** *An irreducible Markov chain is called transient, resp. null recurrent or positive recurrent, if all its states are transient, resp. null recurrent or positive recurrent.*

Recall that  $\tau_y := \inf\{n \geq 1 : X_n = y\}$  and  $\rho_{xy} = \mathbb{P}_x(\tau_y < \infty)$ . The restriction to irreducible Markov chains is partly justified by the following result.

**Theorem 1.2 [Decomposition theorem]** *Let  $R = \{x \in S : \rho_{xx} = 1\}$  be the set of recurrent states of a countable state Markov chain. Then  $R$  can be decomposed as the disjoint union of  $R_i$ , each of which is irreducible and closed in the sense that if  $x \in R_i$  and  $\rho_{xy} > 0$ , then  $y \in R_i$ . For each  $i$ , the states in  $R_i$  are either all null recurrent or all positive recurrent.*

**Proof.** If  $x \in R$  and  $\rho_{xy} > 0$ , then we have shown that  $y \in R$  and  $\rho_{yx} = 1$ . If  $\rho_{xy} > 0$  and  $\rho_{yz} > 0$ , then also  $\rho_{xz} \geq \rho_{xy}\rho_{yz} > 0$ . Therefore the relation  $x \sim y$  for  $x$  and  $y$  with  $\rho_{xy} > 0$  defines an equivalence relation on  $R$ . The equivalence classes  $\{R_i\}$  are what we seek. Restricted to each  $R_i$ , we have an irreducible Markov chain whose states are all of the same type. ■

**Example 1.3** *The symmetric simple random walk on  $\mathbb{Z}$  is recurrent, while the asymmetric simple random walk on  $\mathbb{Z}$  is transient.*

### 2 Recurrence/transience, harmonic functions and martingales

We have shown that the Green's function,  $G(x, y) = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y)$ , of an irreducible countable state Markov chain  $(X_n)_{n \geq 0}$  is finite if and only if the chain is transient. This provides one way of determining the transience or recurrence of an irreducible Markov chain. As an example, we consider the simple symmetric random walk on  $\mathbb{Z}^d$  defined by  $X_0 = x \in \mathbb{Z}^d$  and  $X_n = X_0 + \sum_{i=1}^n \xi_i$ , where  $\xi_i$  are i.i.d. with  $\mathbb{P}(\xi_1 = e) = \frac{1}{2d}$  for each of the  $2d$  unit vectors  $e \in \mathbb{Z}^d$ .

**Theorem 2.1 [Simple symmetric random walk on  $\mathbb{Z}^d$ ]** *The simple symmetric random walk on  $\mathbb{Z}^d$  is recurrent in dimensions  $d = 1, 2$ , and transient in dimensions  $d \geq 3$ .*

**Proof.** We will compute the Green's function for the simple symmetric random walk  $(X_n)_{n \geq 0}$  on  $\mathbb{Z}^d$  using Fourier transform. Clearly  $X$  is an irreducible Markov chain on  $\mathbb{Z}^d$ . Since all states are of the same type, we may assume  $X_0 = 0$ , the origin. By translation invariance,

we can let  $p_n(x, y) := p_n(y - x)$  denote the  $n$ -step transition probability of the random walk. Then for  $x \in \mathbb{Z}^d$ ,

$$p_n(x) = \sum_{x_1, \dots, x_{n-1} \in \mathbb{Z}^d} p_1(x_1)p_1(x_2 - x_1) \cdots p_1(x - x_{n-1}) = p_1^{*n}(x),$$

where  $p_1^{*n}$  denotes the  $n$ -fold convolution of  $p_1$ . For any  $k \in [-\pi, \pi]^d$ ,  $p_n$  has Fourier transform

$$\hat{p}_n(k) = \sum_{x \in \mathbb{Z}^d} e^{i\langle k, x \rangle} p_n(x) = \phi^n(k),$$

where  $\phi(k) = \frac{1}{d} \sum_{i=1}^d \cos k_i$  is the Fourier transform of  $p_1$ . By Fourier inversion,

$$p_n(0) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \phi^n(k) dk,$$

and hence

$$G(0, 0) = \sum_{n=0}^{\infty} p_n(0) = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \phi^n(k) dk = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{1 - \phi(k)} dk.$$

Note that  $\phi(k) = 1$  if and only if  $k = 0$ , and  $\phi(k) = 1 - \frac{\|k\|^2}{2d} + o(\|k\|^2)$  where  $\|k\|$  denotes the Euclidean norm. For  $\|k\| > \epsilon$  and  $k \in [-\pi, \pi]^d$ ,  $\frac{1}{|1 - \phi(k)|}$  is uniformly bounded. Therefore to determine whether  $G(0, 0)$  is finite, we can replace  $\phi(k)$  by its Taylor expansion at 0, and consider

$$\frac{1}{(2\pi)^d} \int_{\|k\| \leq \epsilon} \frac{2d}{\|k\|^2} dk = \frac{2d}{(2\pi)^d} \int_0^\epsilon r^{d-3} dr,$$

which is finite if and only if  $d \geq 3$ . Therefore  $G(0, 0) < \infty$  and the simple symmetric random walk is transient if and only if  $d \geq 3$ . ■

**Exercise 2.2** *Prove that if the random walk increments  $\xi_i = X_i - X_{i-1}$ ,  $i \in \mathbb{N}$ , are i.i.d. with  $\mathbb{E}[\xi_i] = 0$  and  $\mathbb{E}[\|\xi_i\|^2] < \infty$ , then the walk on  $\mathbb{Z}^d$  is also recurrent in dimensions  $d = 1$  and  $2$ , and transient in dimensions  $d \geq 3$ . On the other hand, if  $\mathbb{E}[\xi_i] \neq 0$ , then the walk is always transient regardless of the dimension.*

In general, it is not possible to compute explicitly the Green's function of a Markov chain. An alternative approach is by finding suitable non-negative (super) martingales. Let us first illustrate this technique by a concrete example.

**Example 2.3 [Recurrence of a simple symmetric random walk on  $\mathbb{Z}$ ]** *Let  $X_n = X_0 + \sum_{i=1}^n \xi_i$  be a simple symmetric random walk on  $\mathbb{Z}$  where  $\xi_i$  are i.i.d. with  $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$ . To prove recurrence, it suffices to show that  $\mathbb{P}_1(\tau_0 < \infty) = 1$ , where  $\mathbb{P}_1$  denotes probability law for  $X$  with  $X_0 = 1$ , and  $\tau_0 = \inf\{n \geq 1 : X_n = 0\}$ . Note that  $X_n$  is a martingale, and  $X_{n \wedge \tau_0 \wedge \tau_L}$  is a bounded martingale for any  $L \in \mathbb{N}$ . Since  $\tau_0 \wedge \tau_L < \infty$  almost surely, by the optional stopping theorem,*

$$X_0 = 1 = \mathbb{E}_1[X_{\tau_0 \wedge \tau_L}] = L\mathbb{P}_1(\tau_L < \tau_0),$$

*and hence  $\mathbb{P}_1(\tau_0 < \tau_L) = 1 - L^{-1}$ . Let  $L \uparrow \infty$ , we deduce that  $\mathbb{P}_1(\tau_0 < \infty) = 1$ . The crucial fact we use here is that  $f(X_n)$ , where  $f(x) = x$ , is a martingale, and  $f$  is non-negative on  $[0, \infty)$  and increases to  $\infty$  at  $\infty$ . Note that the proof still goes through if  $f(X_n)$  is a super-martingale instead of a martingale.*

**Example 2.4 [Transience of an asymmetric simple random walk on  $\mathbb{Z}$ ]** Let  $X_n = X_0 + \sum_{i=1}^n \xi_i$  where  $\xi_i$  are i.i.d. with  $\mathbb{P}(\xi_i = 1) = 1 - \mathbb{P}(\xi_i = -1) = p$  for some  $p > \frac{1}{2}$ . To prove transience, it suffices to show that  $\mathbb{P}_1(\tau_0 < \infty) < 1$ . Note that  $f(X_n) := \left(\frac{1-p}{p}\right)^{X_n}$  is a martingale. For  $X_0 = 1$ ,  $\tau_0 \wedge \tau_L < \infty$  almost surely for any  $L \in \mathbb{N}$ , therefore,

$$\frac{1-p}{p} = \mathbb{E}_1 \left[ \left(\frac{1-p}{p}\right)^{X_{\tau_0 \wedge \tau_L}} \right] = \mathbb{P}_1(\tau_0 < \tau_L) + \left(\frac{1-p}{p}\right)^L \mathbb{P}_1(\tau_L < \tau_0) \geq \mathbb{P}_1(\tau_0 < \tau_L).$$

Let  $L \uparrow \infty$ , we deduce that  $\mathbb{P}_1(\tau_0 < \infty) \leq \frac{1-p}{p} < 1$ , thus proving transience. The crucial fact we use here is that  $f(X_n)$ , where  $f(x) = \left(\frac{1-p}{p}\right)^x$ , is a non-negative super-martingale, and  $f(0) > f(1)$ .

We now generalize the above proof approach to general irreducible Markov chains.

**Definition 2.5 [Harmonic functions for a Markov chain]** Let  $X$  be a time-homogeneous irreducible Markov chain with countable state space  $S$  and one-step transition probability matrix  $\Pi(x, y)$ . A function  $f : S \rightarrow \mathbb{R}$  is said to be harmonic for  $X$  at  $x \in S$  if

$$(\Pi f)(x) = \sum_{y \in S} \Pi(x, y) f(y) = f(x), \quad (2.1)$$

and sub-harmonic, resp. super-harmonic, at  $x$  if  $f(x) \leq (\Pi f)(x)$ , resp.  $f(x) \geq (\Pi f)(x)$ . If  $f$  is (sub/super)-harmonic at all  $x \in S$ , then  $f$  is said to be (sub/super)-harmonic for  $X$ .

**Lemma 2.6** If  $f$  is a harmonic function for an irreducible Markov chain  $X$ , then  $f(X_n)$  is a martingale. If  $f$  is sub/super-harmonic, then  $f(X_n)$  is a sub/super-martingale.

**Proof.** Let  $(\mathcal{F}_n)_{n \geq 0}$  be the canonical filtration generated by  $(X_n)_{n \geq 0}$ . By the Markov property,

$$\mathbb{E}[f(X_{n+1}) | \mathcal{F}_n] = \mathbb{E}[f(X_{n+1}) | X_n] = \sum_{y \in S} \Pi(X_n, y) f(y) = f(X_n).$$

Therefore  $f(X_n)$  is a martingale. The case when  $f$  is sub/super-harmonic is similar. ■

**Theorem 2.7 [Sufficient condition for recurrence]** Let  $X$  be an irreducible Markov chain with countable state space  $S$ . If there exists a finite set  $F \subset S$  and a function  $\phi : S \rightarrow [0, \infty)$  such that  $\phi$  is super-harmonic for  $X$  at all  $x \in S \setminus F$ , and  $\phi(x) \rightarrow \infty$  as  $x \rightarrow \infty$  in the sense that the level set  $\{x \in S : \phi(x) \leq M\}$  is finite for all  $M > 0$ , then  $X$  is recurrent.

**Proof.** Let the Markov chain start from  $x_0 \in S \setminus F$ . Let  $\tau_F = \inf\{n \geq 1 : X_n \in F\}$ , and let  $\tau_M = \inf\{n \geq 1 : \phi(X_n) > M\}$  for any  $M > \phi(x_0)$ . By our assumption on  $\phi$ ,  $\phi(X_{n \wedge \tau_F \wedge \tau_M})$  is a super-martingale. Therefore

$$\phi(x_0) \geq \mathbb{E}[\phi(X_{n \wedge \tau_F \wedge \tau_M})] \geq M \mathbb{P}(\tau_M < n \wedge \tau_F). \quad (2.2)$$

Note that because  $\{x : \phi(x) \leq M\}$  is a finite set, by the irreducibility of the Markov chain,  $\tau_M < \infty$  almost surely. Sending  $n \rightarrow \infty$  in (2.2) then yields  $\mathbb{P}(\tau_M < \tau_F) \leq \phi(x_0)/M$ , and hence

$$\mathbb{P}(\tau_F < \tau_M < \infty) \geq 1 - \frac{\phi(x_0)}{M}.$$

Sending  $M \rightarrow \infty$  then gives  $\mathbb{P}(\tau_F < \infty) = 1$ . Therefore the set  $F$  is recurrent in the sense that  $X$  will return to  $F$  infinitely often almost surely. In particular,  $\sum_{y \in F} G(x_0, y) = \infty$ . Since  $F$  is a finite set,  $G(x_0, y) = \infty$  for some  $y \in F$ , which implies that  $X$  is recurrent. ■

**Theorem 2.8 [Necessary and sufficient condition for transience]** *Let  $X$  be an irreducible Markov chain with countable state space  $S$ . A necessary and sufficient condition for  $X$  to be transient is the existence of a non-constant, non-negative super-harmonic function  $\phi$ .*

**Proof.** Suppose that  $X$  is recurrent and  $\phi$  is a non-negative non-constant super-harmonic function. We will derive a contradiction. Let  $x, y \in S$  with  $\phi(x) < \phi(y)$ . Let the Markov chain start at  $x$ , whose law we denote by  $\mathbb{P}_x$ . Let  $\tau_y = \inf\{n \geq 1 : X_n = y\}$ . Then  $\tau_y$  is a stopping time, and  $\phi(X_{n \wedge \tau_y})$  is a super-martingale. In particular,

$$\phi(x) \geq \phi(X_{n \wedge \tau_y}) \geq \phi(y)\mathbb{P}_x(\tau_y \leq n).$$

By recurrence,  $\tau_y < \infty$  almost surely. Therefore sending  $n \rightarrow \infty$  leads to  $\phi(x) \geq \phi(y)$ , which is a contradiction.

For the converse, we claim that if  $X$  is transient, then there exists a non-negative non-constant super-harmonic function. Fix an  $x_0 \in S$ . We claim that  $\phi(x) = \mathbb{P}_x(\tilde{\tau}_{x_0} < \infty)$  is a super-harmonic function, where  $\tilde{\tau}_{x_0} = \inf\{n \geq 0 : X_n = x_0\}$ . Note that  $\phi \in [0, 1]$ ,  $\phi(x_0) = 1$ , and by transience, there exists  $y \in S$  with  $\phi(y) < 1$ . By examining the Markov chain starting at  $x$  after one step, we note that

$$\mathbb{P}_x(\tilde{\tau}_{x_0} < \infty) = \begin{cases} 1, & \text{if } x = x_0, \\ \sum_{y \in S} \Pi(x, y)\mathbb{P}_y(\tilde{\tau}_{x_0} < \infty), & \text{if } x \neq x_0. \end{cases}$$

Therefore  $\phi$  is harmonic at  $x \neq x_0$  and super-harmonic at  $x_0$ , and hence  $\phi$  is a non-negative non-constant super-harmonic function. ■

We now illustrate Theorems 2.7 and 2.8 by an example.

**Example 2.9 [Birth-death chains]** *Let  $X$  be a time-homogeneous Markov chain on  $\{0, 1, 2, \dots\}$  with one-step transition probabilities  $p(i, i+1) = p_i$ ,  $p(i, i-1) = q_i$  and  $p(i, i) = r_i$  with  $q_0 = 0$ . This is called the birth-death chain. We assume that  $p_i > 0$  for  $i \geq 0$  and  $q_i > 0$  for  $i \geq 1$  so that the chain is irreducible. Let us find a non-negative  $\phi$  which is harmonic for the birth-death chain at all  $x \in \mathbb{N}$ . W.l.o.g., we may assume  $\phi(0) = 0$  and  $\phi(1) = 1$ . For  $\phi$  to be harmonic at each  $x \in \mathbb{N}$ , we must have*

$$\phi(x) = p_x\phi(x+1) + r_x\phi(x) + q_x\phi(x-1),$$

which sets up the recursion relation

$$\phi(x+1) - \phi(x) = \frac{q_x}{p_x}(\phi(x) - \phi(x-1)).$$

Therefore

$$\phi(x) - \phi(x-1) = \prod_{i=1}^{x-1} \frac{q_i}{p_i},$$

and

$$\phi(x) = \sum_{i=1}^x \prod_{j=1}^{i-1} \frac{q_j}{p_j}.$$

By construction,  $\phi$  is harmonic for  $X$  at all  $x \in \mathbb{N}$  and is non-negative and monotonically increasing. We claim that the chain is recurrent if and only if

$$\phi(\infty) := \lim_{x \rightarrow \infty} \phi(x) = \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} \frac{q_j}{p_j} = \infty.$$

Indeed, if  $\phi(\infty) = \infty$ , then Theorem 2.7 applies with  $F = \{0\}$ . If  $\phi(\infty) < \infty$ , then  $\tilde{\phi}(x) = \phi(\infty) - \phi(x)$  defines a non-negative non-constant super-harmonic function for  $X$ , so that Theorem 2.8 applies.

More directly, we note that  $\phi(X_{n \wedge \tau_0 \wedge \tau_N})$  is a martingale for any  $N \in \mathbb{N}$ , which by the optional stopping theorem implies that if the chain starts at  $X_0 = 1$ , then

$$\phi(1) = \mathbb{P}_1(\tau_0 < \tau_N)\phi(0) + \mathbb{P}_1(\tau_N < \tau_0)\phi(N),$$

and hence

$$\mathbb{P}_1(\tau_N < \tau_0) = \frac{1}{\phi(N)}.$$

Sending  $N \rightarrow \infty$ , we conclude that 0 is a recurrent state if and only if  $\phi(\infty) = \infty$ .

A special case of the birth-death chain is an asymmetric simple random walk where  $p_i = 1 - q_i = p$  for all  $i \geq 1$ . Then

$$\phi(x) = \sum_{i=1}^x \left(\frac{1-p}{p}\right)^{i-1}.$$

Therefore  $\phi(\infty) = \infty$  if and only if  $p \leq \frac{1}{2}$ . If  $p > \frac{1}{2}$ , then

$$\mathbb{P}_1(\tau_0 = \infty) = \frac{1}{\phi(\infty)} = \frac{2p-1}{p}.$$

### 3 Dirichlet problem and Poisson equation

We now elaborate more on the connection between Markov chains and potential theory. The first concerns the so-called the *Dirichlet problem*. Assume that  $X$  is an irreducible recurrent Markov chain with transition matrix  $\Pi$ , and let  $A$  be a given set. We are interested in finding bounded solutions of the Dirichlet boundary value problem

$$\begin{aligned} (\Pi - I)f(x) &= 0 & \text{for } x \in A, \\ f(x) &= g(x) & \text{for } x \notin A, \end{aligned} \tag{3.3}$$

where  $g$  is a given bounded function on  $A^c$ . Observe that such an  $f$  is harmonic for  $X$  at all  $x \in A$ . Therefore  $f(X_{n \wedge \tau_{A^c}})$  is a bounded martingale, where  $\tau_{A^c} = \inf\{n \geq 0 : X_n \notin A\}$  is finite a.s. by the recurrence of  $X$ . By the optional stopping theorem,

$$f(x) = \mathbb{E}_x[f(X_{\tau_{A^c}})]. \tag{3.4}$$

Therefore (3.3) has a unique bounded solution, which is given by (3.4). If we let  $g(x) = 1_{\{x \in B\}}$  for some  $B \subset A^c$ , then the solution to (3.3) is

$$f(x) = \mathbb{P}_x(X_{\tau_{A^c}} \in B),$$

which is the probability that the Markov chain starting from  $x$  enters the set  $A^c$  by entering the subset  $B$ . The collection  $\mathbb{P}_x(X_{\tau_{A^c}} = y)$  for  $y \in A^c$  defines a probability measure on  $A^c$ , called the *harmonic measure*. Thus computing the hitting probabilities for a Markov chain is equivalent to solving a Dirichlet problem with suitable boundary conditions.

When  $X$  is transient, bounded solutions to (3.3) are not unique in general, as can be seen from the fact that when  $g = 1$  on  $A^c$ , taken to be a finite set, both  $f \equiv 1$  and  $f(x) = \mathbb{P}_x(\tau_{A^c} < \infty)$  are bounded solutions to (3.3). However, the latter can be easily seen to be the minimal non-negative solution.

Another general way of generating martingales for Markov chains is to take any bounded function  $f$  on the state space  $S$ , and then extract a martingale from  $f(X_n)$  by subtracting a suitable predictable sequence. More precisely, let

$$\Delta_n = \mathbb{E}[f(X_n) - f(X_{n-1}) | X_{n-1}] = (\Pi - I)f(X_{n-1}).$$

Then

$$M_n = f(X_n) - f(X_0) - \sum_{i=1}^n \Delta_n = f(X_n) - f(X_0) - \sum_{i=1}^n (\Pi - I)f(X_{i-1}) \quad (3.5)$$

is a martingale. If we want to compute the expected hitting time of a set  $A$ , i.e.,  $\mathbb{E}_x[\tau_A]$ , then it suffices to solve the equation

$$\begin{aligned} (\Pi - I)f(x) &= -1 & \text{for } x \in A, \\ f(x) &= 0 & \text{for } x \notin A. \end{aligned} \quad (3.6)$$

This is sometimes called Dynkin's equation. Indeed, if  $f$  is a bounded solution of (3.6), then by (3.5),  $f(X_{n \wedge \tau_{A^c}}) - f(X_0) + n \wedge \tau_{A^c}$  is a martingale. If the Markov chain is recurrent, then it is easy to see that

$$\mathbb{E}_x[\tau_{A^c}] = f(x).$$

More generally, computing  $\mathbb{E}_x[\sum_{i=0}^{\tau_{A^c}-1} g(X_i)]$  reduces to solving (3.6) with  $(\Pi - I)f = -g$  on  $A$ , which is called the *Poisson equation*. Of course to establish this correspondence using martingales and the optional stopping theorem, one has to be careful with (uniform) integrability issues. Analogues of (3.5) and (3.6) also exist for continuous time Markov processes, with  $(\Pi - I)$  replaced by the generator of the Markov process, the most well known being the Laplacian  $\Delta$ , which is the generator of a Brownian motion.

We can even find probabilistic representations for solutions of equations of the following general form:

$$\begin{aligned} (\Pi - I + V)f(x) &= g(x) & \text{for } x \in A, \\ f(x) &= h(x) & \text{for } x \notin A, \end{aligned} \quad (3.7)$$

where to avoid integrability issues, let us assume that  $A$  is a bounded set,  $V : A \rightarrow \mathbb{R}$  satisfies  $\max_{x \in A} V(x) < 1$ ,  $h : A^c \rightarrow \mathbb{R}$  is a bounded function, and  $g = 0$  on  $A^c$ .

Indeed, the solution of the equation

$$\begin{aligned} (\Pi - I + V)f(x) &= 0 & \text{for } x \in A, \\ f(x) &= h(x) & \text{for } x \notin A \end{aligned} \quad (3.8)$$

admits the representation  $f(x) = h(x)$  if  $x \in A^c$  and  $f(x) = \mathbb{E}_x[h(X_{\tau_{A^c}}) \prod_{i=1}^{\tau_{A^c}} \frac{1}{1-V(X_{i-1})}]$ , using the observation that  $f(X_n) \prod_{i=1}^n \frac{1}{1-V(X_{i-1})}$  is a martingale if  $f$  solves (3.8). This is called the Feynman-Kac formula.

On the other hand, the solution of the equation

$$\begin{aligned} (\Pi - I + V)f(x) &= g(x) & \text{for } x \in A, \\ f(x) &= 0 & \text{for } x \notin A \end{aligned} \quad (3.9)$$

admits the representation  $f(x) = 0$  when  $x \in A^c$ , and  $f(x) = \mathbb{E}_x[\sum_{i=0}^{\tau_{A^c}-1} g(X_i) \prod_{j=0}^{i-1} \frac{1}{1-V(X_j)}]$  when  $x \in A$ , observing that if  $f$  solves (3.9) then

$$f(X_n) \prod_{i=0}^{n-1} \frac{1}{1-V(X_i)} - f(X_0) + \sum_{i=0}^{n-1} g(X_i) \prod_{j=0}^{i-1} \frac{1}{1-V(X_j)}$$

is a martingale. Finally note that any solution of (3.7) can be written as the sum of a solution to (3.8) and a solution (3.9).