

Lecture 6. March 2, 2009

1 Markov chains with a countable state space

1.1 Classification of states

Definition 1.1 [Irreducible Markov chains] A Markov chain with a countable state space S is called irreducible if for all $x, y \in S$, $p^{(n)}(x, y) > 0$ for some $n \geq 0$.

Definition 1.2 [Transience, null recurrence, and positive recurrence] Let $\tau_y := \inf\{n > 0 : X_n = y\}$ be the first hitting time (after time 0) of the state $y \in S$ by the Markov chain X . Any state $x \in S$ can then be classified into the following three types:

- (i) Transient, if $\mathbb{P}_x(\tau_x < \infty) < 1$.
- (ii) Null recurrent, if $\mathbb{P}_x(\tau_x < \infty) = 1$ and $\mathbb{E}_x[\tau_x] = \infty$.
- (iii) Positive recurrent, if $\mathbb{P}_x(\tau_x < \infty) = 1$ and $\mathbb{E}_x[\tau_x] < \infty$.

Lemma 1.3 Let $\rho_{xy} = \mathbb{P}_x(\tau_y < \infty)$ for $x, y \in S$. Let $G(x, y) = \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y) = \sum_{n=0}^{\infty} p^{(n)}(x, y)$. If y is transient, then

$$G(x, y) = \begin{cases} \frac{\rho_{xy}}{1 - \rho_{yy}} & \text{if } x \neq y, \\ \frac{1}{1 - \rho_{yy}} & \text{if } x = y. \end{cases} \quad (1.1)$$

If y is recurrent, then $G(x, y) = \infty$ for all $x \in S$ with $\rho_{xy} > 0$.

Lemma 1.4 If $x \in S$ is recurrent, $y \neq x$, and $\rho_{xy} := \mathbb{P}_x(\tau_y < \infty) > 0$, then $\mathbb{P}_x(\tau_y < \tau_x) > 0$, $\rho_{yx} := \mathbb{P}_y(\tau_x < \infty) = 1$, and y is also recurrent.

Proof. If $\mathbb{P}_x(\tau_y < \tau_x) = 0$ so that the Markov chain starting from x returns to x before visiting y almost surely, then when it returns to x , it starts afresh and will not visit y before a second return to x . Iterating this reasoning, the Markov chain will visit x infinitely often before visiting y , which means it will never visit y , contradicting the assumption.

Suppose that $\rho_{yx} < 1$. Let $k = \inf\{i > 0 : p^{(k)}(x, y) > 0\}$. Then there exists a sequence $y_1, \dots, y_{k-1} \in S$, all distinct from x and y such that $p(x, y_1)p(y_1, y_2) \cdots p(y_{k-1}, y) > 0$. Then

$$\mathbb{P}_x(\tau_x = \infty) \geq p(x, y_1) \cdots p(y_{k-1}, y)(1 - \rho_{yx}) > 0,$$

which contradicts the recurrence of x . Hence $\rho_{yx} = 1$.

Since $\rho_{xy} > 0$ and $\rho_{yx} > 0$, there exist $K, L \in \mathbb{N}$ such that $p^{(K)}(y, x) > 0$ and $p^{(L)}(x, y) > 0$. Then

$$p^{(K+n+L)}(y, y) \geq p^{(K)}(y, x)p^{(n)}(x, x)p^{(L)}(x, y).$$

Summing over n then yields $G(y, y) \geq p^{(K)}(y, x)G(x, x)p^{(L)}(x, y) = \infty$. Therefore y must also be recurrent by Lemma 1.3. ■

Lemma 1.5 *For an irreducible Markov chain, all states are of the same type.*

Proof. Lemma 1.4 has shown that if x is recurrent, then so is any other $y \in S$ by the irreducibility assumption. It remains to show that if x is positive recurrent, then so is any $y \in S$. Let $p = \mathbb{P}_x(\tau_y < \tau_x)$, which is positive by Lemma 1.4. Then

$$\mathbb{E}_x[\tau_x] \geq \mathbb{P}_x(\tau_y < \tau_x) \mathbb{E}_y[\tau_x].$$

Therefore $\mathbb{E}_y[\tau_x] \leq \frac{1}{p} \mathbb{E}_x[\tau_x] < \infty$. On the other hand,

$$\begin{aligned} \mathbb{E}_x[\tau_y] &\leq \mathbb{E}[1_{\{\tau_y < \tau_x\}} \tau_x] + \mathbb{E}_x[1_{\{\tau_x < \tau_y\}} \tau_y] \\ &= \mathbb{E}[1_{\{\tau_y < \tau_x\}} \tau_x] + \mathbb{E}_x[1_{\{\tau_x < \tau_y\}} (\tau_x + \mathbb{E}_x[\tau_y])] \\ &= \mathbb{E}_x[\tau_x] + (1 - p) \mathbb{E}_x[\tau_y]. \end{aligned}$$

Therefore $\mathbb{E}_x[\tau_y] \leq \frac{1}{p} \mathbb{E}_x[\tau_x]$, and

$$\mathbb{E}_y[\tau_y] \leq \mathbb{E}_y[\tau_x] + \mathbb{E}_x[\tau_y] \leq \frac{2}{p} \mathbb{E}_x[\tau_x] < \infty,$$

which proves the positive recurrence of y . ■

Definition 1.6 [Classification of irreducible Markov chains] *An irreducible Markov chain is called transient, resp. null recurrent or positive recurrent, if all its states are transient, resp. null recurrent or positive recurrent.*

The restriction to irreducible Markov chains is partly justified by the following result.

Theorem 1.7 [Decomposition theorem] *Let $R = \{x \in S : \rho_{xx} = 1\}$ be the set of recurrent states of a Markov chain. Then R can be decomposed as the disjoint union of R_i , each of which is irreducible and closed in the sense that if $x \in R_i$ and $\rho_{xy} > 0$, then $y \in R_i$. For each i , the states in R_i are either all null recurrent or all positive recurrent.*

Proof. If $x \in R$ and $\rho_{xy} > 0$, then by Lemma 1.4, $y \in R$ and $\rho_{yx} = 1$. If $\rho_{xy} > 0$ and $\rho_{yz} < 0$, then also $\rho_{xz} \geq \rho_{xy} \rho_{yz} > 0$. Therefore the relation $x \sim y$ if $\rho_{xy} > 0$ defines an equivalence relation on R . The equivalence classes R_i are what we seek. Restricted to each R_i , we have an irreducible Markov chain. Hence by Lemma 1.5, all states in R_i are of the same type. ■

Example 1.8 *The symmetric simple random walk on \mathbb{Z} is recurrent, while the asymmetric simple random walk on \mathbb{Z} is transient.*

1.2 Recurrence/transience, harmonic functions and martingales

Lemma 1.3 shows that the Green function of an irreducible Markov chain is finite if and only if the chain is transient. This provides one way of determining the transience or recurrence of a Markov chain. As an example, we consider the simple random walk on \mathbb{Z}^d defined by $X_0 = x \in \mathbb{Z}^d$ and $X_n = X_0 + \sum_{i=1}^n \xi_i$, where ξ_i are i.i.d. with $\mathbb{P}(\xi_1 = e) = \frac{1}{2d}$ for each of the $2d$ unit vectors $e \in \mathbb{Z}^d$.

Theorem 1.9 [Recurrence and transience of the simple random walk] *The simple random walk is recurrent in dimensions $d = 1, 2$, and transient in dimensions $d \geq 3$.*

Proof. We will compute the Green function for the simple random walk X_n on \mathbb{Z}^d using Fourier transform. Clearly X_n is an irreducible Markov chain on \mathbb{Z}^d . Since all states are of the same type, we may assume $X_0 = 0$, the origin. Using translation invariance, let $p_n(x, y) := p_n(y - x)$ denote the n -step transition probability of the random walk. Then for $x \in \mathbb{Z}^d$,

$$p_n(x) = \sum_{x_1, \dots, x_{n-1} \in \mathbb{Z}^d} p_1(x_1)p_1(x_2 - x_1) \cdots p_1(x - x_{n-1}) = p_1^{*n}(x),$$

where p_1^{*n} denotes the n -fold convolution of p_1 . For any $k \in [-\pi, \pi]^d$, p_n has Fourier transform

$$\hat{p}_n(k) = \sum_{x \in \mathbb{Z}^d} e^{i\langle k, x \rangle} p_n(x) = \phi^n(k),$$

where $\phi(k) = \frac{1}{d} \sum_{i=1}^d \cos k_i$ is the Fourier transform of p_1 . By Fourier inversion,

$$p_n(0) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \phi^n(k) dk,$$

and hence

$$G(0, 0) = \sum_{n=0}^{\infty} p_n(0) = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \phi^n(k) dk = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{1 - \phi(k)} dk.$$

Note that $\phi(k) = 1$ if and only if $k = 0$, and $\phi(k) = 1 - \frac{|k|^2}{2d} + o(|k|^2)$ where $|k|$ denotes the Euclidean norm. For $|k| > \epsilon$, $\frac{1}{|1 - \phi(k)|}$ is uniformly bounded. Therefore to determine whether $G(0, 0)$ is finite, we can replace $\phi(k)$ by its Taylor expansion at 0, and consider

$$\frac{1}{(2\pi)^d} \int_{|k| \leq \epsilon} \frac{2d}{|k|^2} dk = \frac{2d}{(2\pi)^d} \int_0^\epsilon r^{d-3} dr,$$

which is finite if and only if $d \geq 3$. Therefore $G(0, 0) < \infty$ and the simple random walk is transient if and only if $d \geq 3$. \blacksquare

In general, it is not possible to compute explicitly the Green function of a Markov chain. An alternative approach is by finding suitable non-negative (super) martingales.

Definition 1.10 [Harmonic functions for a Markov chain] *Let X be an irreducible Markov chain with countable state space S and one-step transition probability matrix $\Pi(x, y)$. A function $f : S \rightarrow \mathbb{R}$ is said to be harmonic for X at $x \in S$ if*

$$(\Pi f)(x) = \sum_{y \in S} \Pi(x, y) f(y) = f(x), \tag{1.2}$$

and sub-harmonic, resp. super-harmonic, at x if $(\Pi f)(x) > f(x)$, resp. $(\Pi f)(x) < f(x)$. If f is (sub/super)-harmonic at all $x \in S$, then f is said to be (sub/super)-harmonic for X .

Lemma 1.11 *If f is a harmonic function for an irreducible Markov chain X , then $f(X_n)$ is a martingale. If f is sub/super-harmonic, then $f(X_n)$ is a sub/super-martingale.*

Proof. By the Markov property,

$$\mathbb{E}[f(X_{n+1})|(X_i)_{0 \leq i \leq n}] = \mathbb{E}[f(X_{n+1})|X_n] = \sum_{y \in S} \Pi(X_n, y) f(y) = f(X_n).$$

Therefore $f(X_n)$ is a martingale. The case when f is sub/super-harmonic is similar. \blacksquare

Theorem 1.12 [Sufficient condition for recurrence] *Let X be an irreducible Markov chain with countable state space S . If there exists $\phi : S \rightarrow [0, \infty)$ such that ϕ is super-harmonic for X at all $x \in S \setminus F$ with $F \subset$ finite, and if $\phi(x) \rightarrow \infty$ as $x \rightarrow \infty$ in the sense that $\{x \in S : \phi(x) \leq M\}$ is finite for all $M > 0$, then X is recurrent.*

Proof. Let the Markov chain start from $x_0 \in S \setminus F$. Let $\tau_F = \inf\{n > 0 : X_n \in F\}$, and let $\tau_M = \inf\{n > 0 : \phi(X_n) > M\}$ for any $M > \phi(x_0)$. By our assumption on ϕ , $\phi(X_{n \wedge \tau_F \wedge \tau_M})$ is a super-martingale. Therefore

$$\phi(x_0) \geq \mathbb{E}[\phi(X_{n \wedge \tau_F \wedge \tau_M})] \geq M \mathbb{P}(\tau_M < n \wedge \tau_F). \quad (1.3)$$

Note that because $\{x : \phi(x) \leq M\}$ is a finite set, by the irreducibility of the Markov chain, $\tau_M < \infty$ almost surely. Sending $n \rightarrow \infty$ in (1.3) then yields $\mathbb{P}(\tau_M < \tau_F) \leq \phi(x_0)/M$, and hence

$$\mathbb{P}(\tau_F < \tau_M < \infty) \geq 1 - \frac{\phi(x_0)}{M}.$$

Sending $M \rightarrow \infty$ then gives $\mathbb{P}(\tau_F < \infty) = 1$. Therefore the set F is recurrent in the sense that X will return to F infinitely often almost surely. In particular, $\sum_{y \in F} G(x_0, y) = \infty$. Since F is a finite set, $G(x_0, y) = \infty$ for some $y \in F$, which implies that X is recurrent. \blacksquare

Theorem 1.13 [Necessary and sufficient condition for transience] *Let X be an irreducible Markov chain with countable state space S . A necessary and sufficient condition for X to be transient is the existence of a non-constant, non-negative super-harmonic function ϕ .*

Proof. Suppose that X is recurrent and ϕ is a non-negative non-constant super-harmonic function. We will derive a contradiction. Let $x, y \in S$ with $\phi(x) < \phi(y)$. Let the Markov chain start at x , and let $\tau_y = \inf\{n > 0 : X_n = y\}$. Then τ_y is a stopping time, and $\phi(X_{n \wedge \tau_y})$ is a super-martingale. In particular,

$$\phi(x) \geq \phi(X_{n \wedge \tau_y}) \geq \phi(y) \mathbb{P}(\tau_y \leq n).$$

By recurrence, $\tau_y < \infty$ almost surely. Therefore sending $n \rightarrow \infty$ leads to $\phi(x) \geq \phi(y)$, which is a contradiction.

For the converse, we claim that if X is transient, then there exists a non-negative non-constant super-harmonic function. Fix an $x_0 \in S$, and let P_x denote probability for the Markov chain starting at $x \in S$. We claim that $\phi(x) = \mathbb{P}_x(\tau_{x_0} < \infty)$ is a super-harmonic function, where $\tau_{x_0} = \inf\{n \geq 0 : X_n = x_0\}$. Note that $\phi \in [0, 1]$, $\phi(x_0) = 1$, and by transience, there exists $y \in S$ with $\phi(y) < 1$. By examining the Markov chain starting at x after one step, we note that

$$\mathbb{P}_x(\tau_{x_0} < \infty) = \begin{cases} 1, & \text{if } x = x_0, \\ \sum_{y \in S} \Pi(x, y) \mathbb{P}_y(\tau_{x_0} < \infty), & \text{if } x \neq x_0. \end{cases}$$

Therefore ϕ is harmonic at $x \neq x_0$ and super-harmonic at x_0 , and hence ϕ is a non-negative non-constant super-harmonic function. \blacksquare

We now illustrate Theorems 1.12 and 1.13 by an example.

Example 1.14 [Birth-death chains] Let X be a homogeneous Markov chain on $\{0, 1, 2, \dots\}$ with one-step transition probabilities $p(i, i+1) = p_i$, $p(i, i-1) = q_i$ and $p(i, i) = r_i$ with $q_0 = 0$. This is called the birth-death chain. We assume that $p_i > 0$ for $i \geq 0$ and $q_i > 0$ for $i \geq 1$ so that the chain is irreducible. Let us find a non-negative ϕ which is harmonic for the birth-death chain at all $x \in \mathbb{N}$. W.l.o.g., we may assume $\phi(0) = 0$ and $\phi(1) = 1$. For each $x \in \mathbb{N}$, we have

$$\phi(x) = p_x \phi(x+1) + r_x \phi(x) + q_x \phi(x-1),$$

which sets up the recursive relation

$$\phi(x+1) - \phi(x) = \frac{q_x}{p_x} (\phi(x) - \phi(x-1)).$$

Therefore

$$\phi(x) - \phi(x-1) = \prod_{i=1}^{x-1} \frac{q_i}{p_i},$$

and

$$\phi(x) = \sum_{i=1}^x \prod_{j=1}^{i-1} \frac{q_j}{p_j}.$$

Clearly ϕ is harmonic for X at all $x \in \mathbb{N}$ and is non-negative and monotonically increasing. We claim that the chain is recurrent if and only if

$$\phi(\infty) := \lim_{x \rightarrow \infty} \phi(x) = \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} \frac{q_j}{p_j} = \infty.$$

Indeed, if $\phi(\infty) = \infty$, then Theorem 1.12 applies with $F = \{0\}$. If $\phi(\infty) < \infty$, then $\tilde{\phi}(x) = \phi(\infty) - \phi(x)$ defines a non-negative non-constant super-harmonic function for X , so that Theorem 1.13 applies.

More directly, we note that $\phi(X_{n \wedge \tau_0 \wedge \tau_N})$ is a martingale for any $N \in \mathbb{N}$, which by the optional stopping theorem implies that if the chain starts at $X_0 = 1$, then

$$\phi(1) = \mathbb{P}_1(\tau_0 < \tau_N) \phi(0) + \mathbb{P}_1(\tau_N < \tau_0) \phi(N),$$

and hence

$$\mathbb{P}_1(\tau_N < \tau_0) = \frac{1}{\phi(N)}.$$

Sending $N \rightarrow \infty$, we conclude that 0 is a recurrent state if and only if $\phi(\infty) = \infty$.

A special case of the birth-death chain is an asymmetric simple random walk where $p_i = 1 - q_i = p$ for all $i \geq 1$. Then

$$\phi(x) = \sum_{i=1}^x \left(\frac{1-p}{p} \right)^{i-1}.$$

Therefore $\phi(\infty) = \infty$ if and only if $p \leq \frac{1}{2}$. If $p > \frac{1}{2}$, then

$$\mathbb{P}_1(\tau_0 = \infty) = \frac{1}{\phi(\infty)} = \frac{2p-1}{p}.$$

Lastly, we point out some special martingales for Markov chains. The first concerns the so-called the Dirichlet problem. Assume that X is an irreducible recurrent Markov chain with transition matrix Π , and let A be a given set. We are interested in finding bounded solutions of the Dirichlet boundary value problem

$$\begin{aligned} (\Pi - I)f(x) &= 0 && \text{for } x \notin A, \\ f(x) &= g(x) && \text{for } x \in A, \end{aligned} \tag{1.4}$$

where g is a given bounded function on A . Observe that f is harmonic for X at $x \notin A$. Therefore $f(X_{n \wedge \tau_A})$ is a bounded martingale, where $\tau_A = \inf\{n \geq 0 : X_n \in A\}$ is finite a.s. by the recurrence of X . By the optional stopping theorem,

$$f(x) = \mathbb{E}_x[f(X_{\tau_A})]. \tag{1.5}$$

Therefore (1.4) has a unique bounded solution, which is given by (1.5). If we let $g(x) = 1_{\{x \in B\}}$, $x \in A$, for some $B \subset A$, then the solution to (1.4) is

$$f(x) = \mathbb{P}_x(X_{\tau_A} \in B),$$

the probability that the Markov chain starting from x enters the set A by entering the subset B . The collection $\mathbb{P}_x(X_{\tau_A} \in B)$ for $B \subset A$ defines a probability measure on A , called the *harmonic measure*. Thus computing the hitting probabilities and harmonic measures for a Markov chain reduces to solving a Dirichlet problem with suitable boundary conditions.

When X is transient, bounded solutions to (1.4) is no longer unique, as can be seen from the fact that when $g = 1$ on A , both $f \equiv 1$ and $f(x) = \mathbb{P}_x(\tau_A < \infty)$ are bounded solutions to (1.4). But the latter can be shown to be the minimal non-negative solution.

A general way of generating martingales for Markov chains is to take any bounded function f on the state space S , and extract a martingale from $f(X_n)$ by subtracting a suitable predictable sequence. More precisely, let

$$\Delta_n = \mathbb{E}[f(X_n) - f(X_{n-1}) | X_{n-1}] = (\Pi - I)f(X_{n-1}).$$

Then

$$M_n = f(X_n) - f(X_0) - \sum_{i=1}^n \Delta_n = f(X_n) - f(X_0) - \sum_{i=1}^n (\Pi - I)f(X_{i-1}) \tag{1.6}$$

is a martingale. If we want to compute the expected hitting time of a set A , i.e., $\mathbb{E}_x[\tau_A]$, then it suffices to solve the equation

$$\begin{aligned} (\Pi - I)f(x) &= -1 && \text{for } x \notin A, \\ f(x) &= 0 && \text{for } x \in A. \end{aligned} \tag{1.7}$$

This is sometimes called Dynkin's equation. Indeed, if f is a bounded solution of (1.7), then by (1.6), $f(X_{n \wedge \tau_A}) - f(X_0) + n \wedge \tau_A$ is a martingale. If the Markov chain is recurrent, then it is easy to see that

$$\mathbb{E}_x[\tau_A] = f(x).$$

More generally, computing $\mathbb{E}_x[\sum_{i=0}^{\tau_A} g(X_i)]$ reduces to solving (1.7) with $(\Pi - I)f = g$ on A^c . Of course one has to be careful with (uniform) integrability issues. Analogues of (1.6) and (1.7) also exist for continuous time Markov processes, with $(\Pi - I)$ replaced by the generator of the Markov process.