Subdiffusivity of a random walk among a Poisson system of moving traps on $\mathbb{Z}$

Siva Athreya∗ Alexander Drewitz† Rongfeng Sun‡

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Abstract

We consider a random walk among a Poisson system of moving traps on $\mathbb{Z}$. In earlier work [DGRS12], the quenched and annealed survival probabilities of this random walk have been investigated. Here we study the path of the random walk conditioned on survival up to time $t$ in the annealed case and show that it is subdiffusive. As a by-product, we obtain an upper bound on the number of so-called thin points of a one-dimensional random walk, as well as a bound on the total volume of the holes in the random walk’s range.

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1 Introduction

Trapping problems have been studied in the statistical physics and probability literature for decades, where a particle modeled by a random walk or Brownian motion is killed when it meets one of the traps. When the traps are Poisson distributed in space and immobile, much has been understood, see e.g. the seminal works of Donsker and Varadhan [DV75, DV79] as well as the monograph by Sznitman [Szn98] and the references therein. However, when the traps are mobile, surprisingly little is known. In a previous work [DGRS12] (see also [PSSS13]), the long-time asymptotics of the annealed and quenched survival probabilities were identified in all dimensions, extending earlier work in the physics literature [MOBC03, MOBC04]. The goal of the current work is to investigate the path behavior of the one-dimensional random walk conditioned on survival up to time $t$ in the annealed setting, which is the first result of this type to our best knowledge. Note that the model of random walk among mobile traps is a natural model for many physical and biological phenomena, such as foraging predators vs prey, or diffusing T-cells vs cancer cells in the blood stream.

We now recall the model considered in [DGRS12]. Given an intensity parameter $\nu > 0$, we consider a family of i.i.d. Poisson random variables $(N_y)_{y \in \mathbb{Z}^d}$ with mean $\nu$. Given $(N_y)_{y \in \mathbb{Z}^d}$, we then start a family of independent simple symmetric random walks $(Y_j^{j,y})_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y}$ on $\mathbb{Z}^d$, each with jump rate $\rho \geq 0$, with $Y_j^{j,y} := (Y_j^{j,y}_t)_{t \geq 0}$ representing the path of the $j$-th trap starting from $y$ at time 0. We will refer to these as ‘$Y$-particles’ or ‘traps’. For $t \geq 0$ and $x \in \mathbb{Z}^d$, we denote by

$$\xi(t,x) := \sum_{y \in \mathbb{Z}^d, 1 \leq j \leq N_y} \delta_x(Y_j^{j,y}_t)$$

(1.1)

∗8th Mile Mysore Road, Indian Statistical Institute, Bangalore 560059, India. Email: athreya@ms.isibang.ac.in
†Universität zu Köln, Mathematisches Institut, Weyertal 86–90, 50931 Köln, Germany. Email: drewitz@math.uni-koeln.de
‡Department of Mathematics, National University of Singapore, S17, 10 Lower Kent Ridge Road Singapore, 119076. Email: matsr@nus.edu.sg
the number of traps at site \( x \) at time \( t \).

Let \( X := (X_t)_{t \geq 0} \) denote a simple symmetric random walk on \( \mathbb{Z}^d \) with jump rate \( \kappa \geq 0 \) (and later on a more general random walk, see Theorem 1.2) that evolves independently of the \( Y \)-particles. At each time \( t \), the \( X \) particle is killed with rate \( \gamma \xi(t, X_t) \), where \( \gamma \geq 0 \) is the interaction parameter — i.e., the killing rate is proportional to the number of traps that the \( X \) particle sees at that time instant. We denote the probability measure underlying the \( X \) and \( Y \)-particles by \( \mathbb{P} \), and if we consider expectations or probabilities with respect to only a subset of the defined random variables, we give those as a superscript, and sometimes also specify the starting configuration as a subscript, such as \( \mathbb{P}^X_{0} \).

Conditional on the realization of \( \xi \), the survival probability of \( X \) up to time \( t \) is then given by

\[
\mathbb{E}^X_{0} \left[ \exp \left\{ -\gamma \int_0^t \xi(s, X_s) \, ds \right\} \right].
\]

This quantity is also referred to as the ‘quenched survival probability’. Taking expectation with respect to \( \xi \) yields the ‘annealed survival probability’

\[
Z^\gamma_t := \mathbb{E}^\xi \left[ \mathbb{E}^X_{0} \left[ \exp \left\{ -\gamma \int_0^t \xi(s, X_s) \, ds \right\} \right] \right].
\]

Since we will mainly be interested in the behavior of \( X \), it is useful to integrate out \( \xi \) in order to obtain the annealed survival probability for a given realization of \( X \), i.e.,

\[
Z^\gamma_{t,X} := \mathbb{E}^\xi \left[ \exp \left\{ -\gamma \int_0^t \xi(s, X_s) \, ds \right\} \right].
\]

Note that the annealed survival probability \( Z^\gamma_t \) is also given by \( \mathbb{E}^X_{0} [Z^\gamma_{t,X}] \). In [DGRS12], the following asymptotics for the annealed survival probability have been derived.

**Theorem 1.1.** [DGRS12, Thm. 1.1] Assume that \( \gamma \in (0, \infty) \), \( \kappa \geq 0 \), \( \rho > 0 \) and \( \nu > 0 \), then

\[
\mathbb{E}^X_{0} [Z^\gamma_{t,X}] = \begin{cases} \exp \left\{ -\nu \sqrt{\frac{8\rho}{\pi}} (1 + o(1)) \right\}, & d = 1, \\ \exp \left\{ -\nu \pi \rho \frac{1}{m_1} (1 + o(1)) \right\}, & d = 2, \\ \exp \left\{ -\lambda_{d,\gamma,\kappa,\rho,\nu} t (1 + o(1)) \right\}, & d \geq 3, \end{cases}
\]

where \( \lambda_{d,\gamma,\kappa,\rho,\nu} \) depends on \( d, \gamma, \kappa, \rho, \nu \), and is called the annealed Lyapunov exponent.

**Remark 1.** The annealed and quenched survival probabilities introduced above are closely related to the parabolic Anderson model, namely, the solution of the lattice stochastic heat equation with a random potential \( \xi \):

\[
\begin{cases} \frac{\partial}{\partial t} u(t, x) = \kappa \Delta u(t, x) - \gamma \xi(t, x) u(t, x), & (t, x) \in [0, \infty) \times \mathbb{Z}, \\ u(0, x) = 1, & x \in \mathbb{Z}. \end{cases}
\]

See [DGRS12] for more details.

It is natural to ask how the asymptotics in Theorem 1.1 are actually achieved, both in terms of the behavior of \( X \) as well as that of \( \xi \). We consider the case \( d = 1 \) and investigate the typical behavior of \( X \) conditioned on survival. In the next section we state the model precisely and the main results of the paper.

2
1.1 Main Results

We shall consider the model considered in [DGRS12] but will allow the following generalisations:

\[ X \text{ is a continuous time random walk on } \mathbb{Z} \text{ with jump rate } \kappa > 0, \text{ and possess a jump kernel } p_X \text{ which is non-degenerate with zero mean.} \quad (1.4) \]

\[ Y\text{-particles (traps) are independent continuous time random walks on } \mathbb{Z} \text{ with jump rate } \rho > 0, \text{ whose jump kernel } p_Y \text{ is symmetric.} \quad (1.5) \]

As defined earlier, \( \xi \) is as in (1.1) and we shall assume that the interaction parameter \( \gamma \in (0, \infty] \) and the trap intensity \( \nu > 0 \).

Before stating our results, we introduce some notation. For \( t \in (0, \infty) \) and a càdlàg function \( f \in D([0, t], \mathbb{R}) \) (with \( D([0, t], \mathbb{R}) \) denoting the Skorokhod space), we define its supremum norm by

\[ \|f\|_t := \sup_{x \in [0, t]} |f(x)|. \quad (1.6) \]

1.1.1 Sub-diffusivity of \( X \)

We are interested in the (non-consistent) family of Gibbs measures

\[ P^\gamma_t(X \in \cdot) := \mathbb{E}^X_0 \left[ \mathbb{E}^{\xi} \left[ \exp \left\{ - \gamma \int_0^t \xi(s, X_s) \, ds \right\} \mathbb{1}_{X \in \cdot} \right] \right], \quad t \geq 0, \quad (1.7) \]

on the space of càdlàg paths on \( \mathbb{Z} \). We will bound typical fluctuations of \( X \) with respect to \( P^\gamma_t \).

Our primary result is the following bound on the fluctuation of \( X \) conditioned on survival up to time \( t \).

**Theorem 1.2.** Let \( X \) and \( Y \) be as (1.4) and (1.5) respectively. Assume that \( \exists \lambda_* > 0 \) such that

\[ \sum_{x \in \mathbb{Z}} e^{\lambda_* |x|} p_X(x) < \infty \quad \text{and} \quad \sum_{x \in \mathbb{Z}} e^{\lambda_* |x|} p_Y(x) < \infty. \quad (1.8) \]

Then there exists \( \alpha > 0 \) such that for all \( \epsilon > 0 \),

\[ P^\gamma_t \left( \|X\|_t \in (\alpha t^{\frac{1}{3}}, t^{\frac{11}{24}} + \epsilon) \right) \to 1. \quad (1.9) \]

**Remark 2.** Since \( \frac{11}{24} < \frac{1}{2} \), the above result shows that \( X \) is sub-diffusive under \( P^\gamma_t \). We believe that \( X \) in fact fluctuates on the scale of \( t^{1/3} \) (modulo lower order corrections). Interestingly, this would coincide with the fluctuation known for the case of immobile traps in dimension one (see e.g. [S90, S03]), and we conjecture that even the rescaled path converges to the same limit. We also note that this should happen even though the annealed survival probability decays at a different rate when the traps are mobile. However, the mobile trap case presents fundamental difficulties that are not present in the immobile case.

1.1.2 Thin points of \( X \)

As by-product of our analysis, we obtain bounds on the number of thin-points of a one-dimensional random walk which is of independent interest. Let \( X \) be as in (1.4) and

\[ L_t(x) := L^X_t(x) := \int_0^t \delta_x(X_s) \, ds \quad (1.10) \]
denote the local time of the random walk \( X \) at \( x \) up to time \( t \). Typically, for \( x \in \mathbb{Z} \) in the bulk of the range of \( X \), the local time \( L^X_t(x) \) will be of order \( \sqrt{t} \). We are interested in thin points. More precisely, for \( M > 0 \), a point \( x \) in the range of \( X \) is called ‘\( M \)-thin at time \( t \)’ if \( L^X_t(x) \in (0, M] \), and we denote by

\[
T_{t,M} := \{ x \in \mathbb{Z} : L^X_t(x) \in (0, M] \}
\]

the set of \( M \)-thin points at time \( t \).

For \( \gamma > 0 \), we introduce the local time functional

\[
F^\gamma_t(X) := \sum_{x \in \mathbb{Z}} e^{-\gamma L^X_t(x)} 1_{L^X_t(x) > 0}.
\]

For \( X \) with mean zero and finite variance for its increments, Proposition 2.1 below implies the existence of constants \( c(\gamma), C(\gamma) \in (0, \infty) \) such that for all \( t \in (0, \infty) \),

\[
\mathbb{E}^X_0 \left[ \exp \left\{ \frac{c(\gamma)}{1 \vee \ln t} F^\gamma_t(X) \right\} \right] \leq C(\gamma).
\]

Since \( F^\gamma_t(X) \geq e^{-\gamma M} |T_{t,M}| \), we immediately obtain the following result

**Theorem 1.3.** Let \( \gamma \in (0, \infty) \), and let \( X \) be as in (1.4). Assume that

\[
\sum_{x \in \mathbb{Z}} x^2 p_X(x) < \infty.
\]

Then, for any positive \( M \),

\[
\mathbb{P}( |T_{t,M}| \geq a ) \leq C(\gamma)e^{-\frac{c(\gamma)}{1 \vee \ln t} a} \quad \text{for all } a > 0.
\]

**Remark 3.** Thin points of Brownian motion in dimension \( d \geq 2 \) have been studied in [DPRZ00] using Lévy’s modulus of continuity. Dimension 1 is different and could be analyzed by using the Ray-Knight theorem. When \( X \) is a simple random walk on \( \mathbb{Z} \), there is still a Ray-Knight theorem to aid our analysis. But for a general random walk \( X \) as in Theorem 1.3, this approach fails.

1.1.3 Holes in the range of \( X \)

For a simple random walk \( X \) on \( \mathbb{Z} \), its range equals the interval \( [\inf_{0 \leq s \leq t} X_s, \sup_{0 \leq s \leq t} X_s] \), which is no longer true for non-simple random walks. However, for \( X \) as in (1.4), we can control the difference

\[
G_t(X) := \left( \sup_{0 \leq s \leq t} X_s - \inf_{0 \leq s \leq t} X_s \right) - |\text{Range}_{s \in [0,t]}(X_s)| = \sum_{\inf_{s \in [0,t]} X_s < x < \sup_{s \in [0,t]} X_s} 1_{L^X_t(x) = 0}.
\]

This is the total volume of the holes in the range of \( X \) by time \( t \), which will appear in the proof of Theorem 1.2 for non-simple random walks.

**Theorem 1.4.** Let \( X \) be as in (1.4). Assume that \( \exists \lambda_s > 0 \) such that

\[
\sum_{x \in \mathbb{Z}} e^{\lambda_s |x|} p_X(x) < \infty.
\]

Then there exist \( c, C > 0 \) such that for \( \lambda_t := \frac{c}{1 \vee \ln t} \), we have

\[
\mathbb{E}^X_0 \left[ \exp \{ \lambda_t G_t(X) \} \right] \leq C \quad \text{for all } t \in (0, \infty).
\]

As a consequence of (1.17), we have

\[
\mathbb{E}^X_0 [G_t(X)] = \int_0^\infty \mathbb{P}^X_0 (G_t(X) \geq m) \, dm \leq \int_0^\infty C e^{-\frac{m}{1 \ln m}} \, dm \leq C \ln t.
\]
Remark 4. We note that (1.17) cannot hold if \( \sum_{|x| \ge L} p_X(x) \) has power law decay. This is easily seen by considering the strategy that the random walk makes a single jump from 0 to a position \( x \ge t \), and then never falls below \( x \) before time \( t \). The probability of this strategy decays polynomially in \( t \), while the gain \( e^{\lambda G_t(X)} \) is more than the stretched exponential.

Throughout the paper, \( c \) and \( C \) will denote generic constants, whose values may change from line to line. Indexed constants such as \( c_1 \) and \( C_2 \) will denote values that will be fixed from their first occurrence onwards. In order to emphasise dependence of a constant on a parameter, we will write \( C(p) \) for instance.

**Layout:** The rest of the paper is organised as follows. In Section 2 we prove Theorem 1.3, in Section 3 we prove Theorem 1.4. We conclude the paper with Section 4 where we prove Theorem 1.2.

## 2 Proof of Theorem 1.3

Recall from (1.12) that \( F_t^\gamma(X) := \sum_{x \in \mathbb{Z}} e^{-\gamma L_t^X(x)} \mathbbm{1}_{L_t^X(x) > 0} \). As remarked before the statement of Theorem 1.3, it suffices to establish the following result.

**Proposition 2.1.** Let \( X \) be a random walk satisfying the assumptions of Theorem 1.3. Then for each \( \gamma > 0 \), there exist constants \( c(\gamma), C(\gamma) \in (0, \infty) \), such that for \( \lambda_t := \frac{c(\gamma)}{1 + t} \), we have

\[
E_0^X \left[ \exp \left\{ \lambda_t F_t^\gamma(X) \right\} \right] \le C(\gamma) \quad \text{for all } t \in (0, \infty).
\]

(2.1)

We will prove Proposition 2.1 by approximating \( X \) by a sequence of discrete time random walks. More precisely, for any \( 0 < q < \frac{1}{\kappa} \), where \( \kappa \) is the jump rate of \( X \), let \( X^q \) denote the discrete time random walk with transition probability

\[
P_0^{X^q}(X^q(1) = 0) = 1 - \kappa q, \quad \text{and} \quad P_0^{X^q}(X^q(1) = x) = \kappa q p_X(x) \quad \forall x \in \mathbb{Z},
\]

(2.2)

where \( p_X(\cdot) \) is the jump probability kernel of \( X \). Let \( X^q(s) := X^q([s]) \) for all \( s \ge 0 \). It is then a standard fact that the sequence of discrete time random walks \( (X^q(s/q))_{s \ge 0} \) converges in distribution to \( (X_s)_{s \ge 0} \) as \( q \downarrow 0 \). Proposition 2.1 will then follow from its analogue for \( (X^q(s/q))_{s \ge 0} \), together with the following lemma.

**Lemma 2.2.** Let \( X \) be an arbitrary continuous time random walk on \( \mathbb{Z} \), and let \( X^q(\cdot/q) \) be its discrete time approximation defined above. Then for any \( \lambda, t \in [0, \infty) \),

\[
\lim_{n \to \infty} E_0^{X^q} \left[ \exp \left\{ \lambda F_t^\gamma(X^q(\cdot/n)) \right\} \right] = E_0^X \left[ \exp \left\{ \lambda F_t^\gamma(X) \right\} \right].
\]

(2.3)

**Proof.** By coupling the successive non-trivial jumps of \( X^q \) with those of \( X \), it is easily seen that the local time process \( (L_t^X(\cdot/n))_{x \in \mathbb{Z}} \) converges in distribution to \( (L_t^X(x))_{x \in \mathbb{Z}} \), and hence \( F_t^\gamma(X^q(\cdot/n)) \) also converges in distribution to \( F_t^\gamma(X) \) as \( n \to \infty \). Therefore to establish (2.3), it suffices to show that \( (\exp\{\lambda F_t^\gamma(X^q(\cdot/n))\})_{n \in \mathbb{N}} \) are uniformly integrable.

Note that

\[
F_t^\gamma(X^q(\cdot/n)) \le |\text{Range}_{s \in [0,nt]}(X^q(\cdot/n))| \le |\text{Range}_{s \in [0,nt]}(X^q(s))|
\]

(2.4)

is bounded by the number of non-trivial jumps of \( X^{1/n} \) before time \( nt \), which is a binomial random variable \( \text{Bin}(nt, \kappa/n) \). Since the exponential moment generating function of the sequence of \( \text{Bin}(nt, \kappa/n) \) random variables converges to that of a Poisson random variable with mean \( \kappa t \), the uniform integrability of \( (\exp\{\lambda F_t^\gamma(X^q(\cdot/n))\})_{n \in \mathbb{N}} \) then follows.

\[\square\]

We will also need the following result.
Lemma 2.3. Let $X$ be a continuous time random walk on $\mathbb{Z}$ with jump rate $\kappa > 0$, whose jump kernel has mean zero and variance $\sigma^2 \in (0, \infty)$. Let $L_t^X(0)$ be its local time at 0 by time $t$, and $\tau_0$ the first hitting time of 0. Then there exists $C > 0$ such that

$$E_0^X[e^{-\gamma L_t^X(0)}] \sim \frac{\sigma}{\gamma} \sqrt{\frac{2\kappa}{\pi t}} \text{ as } t \to \infty \quad \text{and} \quad P_x^X(\tau_0 \geq t) \leq 1 + \frac{C|z|}{\sqrt{t}} \forall \ t > 0, \ z \in \mathbb{Z}. \quad (2.5)$$

Furthermore, if $X^{1/2}(n)$ denote the random walks that approximate $X$ as in Lemma 2.2, then there exists $C' > 0$ such that for any $T > 0$,

$$E_0^X[e^{-\gamma L_t^{X^{1/2}(n)}}] \leq 1 + \frac{C'}{\sqrt{t}} \quad \text{and} \quad P_x^{X^{1/2}}(\tau_0 \geq nt) \leq 1 + \frac{C'|z|}{\sqrt{t}} \quad (2.6)$$

uniformly in $t \in [0, T]$, $z \in \mathbb{Z}\setminus\{0\}$, and $n$ sufficiently large.

Proof. When $X$ is a continuous time simple symmetric random walk, the first part of (2.5) was proved in [DGRS12, Section 2.2] using the local central limit theorem and Karamata’s Tauberian theorem. The same proof can also be applied to general $X$ with mean zero and finite variance. The second part of (2.5) follows from Theorem 5.1.7 of [LL10].

By (2.5),

$$E_0^X[e^{-\gamma L_t^X(0)}] \leq 1 + \frac{C}{\sqrt{t}}$$

for some $C$ uniformly in $t > 0$. By the same reasoning as in the proof of Lemma 2.2, $E_0^X[e^{-\gamma L_t^{X^{1/2}(n)}}]$ is a family of decreasing continuous functions in $t$ that converge pointwise to the continuous function $E_0^X[e^{-\gamma L_t^X(0)}]$ as $n \to \infty$. Therefore this convergence must be uniform on $[0, T]$, which implies the first part of (2.6). The second part of (2.6) follows by the same argument.

Proof of Proposition 2.1. We may restrict our attention to $t \geq t_0$ for some large $t_0$, since otherwise (2.1) is easily shown if we bound $F_t^X(X)$ by the number of jumps of $X$ before time $t$.

Due to Lemma 2.2, it then suffices to show that for some $C(\gamma) < \infty$ and for all $t \geq t_0$,

$$\lim_{n \to \infty} E_0^X \left[ \exp \left\{ \lambda t F_t^X(X^{1/2}(\cdot)) \right\} \right] \leq C(\gamma). \quad (2.7)$$

Denote $L_{nt}(\cdot) := L_{nt}^{X^{1/2}(\cdot)}$ for simplicity, and for $x \in \mathbb{Z}$, let $\tau_x$ denote the first time $X^{1/2}$ visits $x$. By Taylor expansion and the definition of $F_t^X$, we have

$$E_0^X \left[ \exp \left\{ \lambda t F_t^X(X^{1/2}(\cdot)) \right\} \right] = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{x_1, \ldots, x_k \in \mathbb{Z}} E_0^X \left[ \prod_{i=1}^{k} e^{-\gamma L_{nt}(x_i)} 1_{L_{nt}(x_i) > 0} \right]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{0 \leq s_1, \ldots, s_k \leq nt} E_0^X \left[ \prod_{i=1}^{k} e^{-\gamma L_{nt}(x_i)} 1_{\tau_{s_i} = s_i} \right]$$

$$\leq 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{m=1}^{k} \frac{m!}{(k-m)!} \sum_{0 \leq t_1 < \cdots < t_m \leq nt} E_0^X \left[ \prod_{i=1}^{m} e^{-\gamma L_{nt}(y_i)} 1_{\tau_{y_i} = t_i} \right]$$

$$= 1 + \sum_{m=1}^{\infty} e^{\lambda m} \lambda^m \sum_{0 \leq s_1 < s_2 < \cdots < s_m \leq nt} E_0^X \left[ \prod_{i=1}^{m} e^{-\gamma L_{nt}(x_i)} 1_{\tau_{x_i} = s_i} \right], \quad (2.8)$$

where in the inequality, we took advantage of the fact that for any $0 \leq t_1 < t_2 < \cdots < t_m \leq nt$ with $1 \leq m \leq k$, the number of ways of choosing $s_1, \ldots, s_k$ from $\{t_1, \ldots, t_m\}$ so that each $t_i$ is chosen at least once is given by $m! S(k, m) \leq \frac{1}{2} \frac{k!m^{k-m}}{(k-m)!}$, where $S(k, m)$ is called a Stirling number.
of the second kind [RD69, Theorem 3]. We also used that when \( \tau_{x_i} = s_i = \tau_{x_j} = s_j \), we must have \( x_i = x_j \).

Using \( L_{nt}^{(n)}(x_i) \geq L_{st}^{(n)}(x_i) \) for \( 1 \leq i \leq m - 1 \), and applying the strong Markov property at time \( \tau_{x_k} = s_k \), we can bound the expectation in (2.8) by

\[
\mathbb{E}^{X^\frac{1}{n}}_0 \left[ \prod_{i=1}^{m-1} e^{-\frac{2}{n} L_{nt}^{(n)}(x_i)} \prod_{i=1}^{m} 1_{\tau_{x_i} = s_i} \right] \leq \mathbb{E}^{X^\frac{1}{n}}_0 \left[ \prod_{i=1}^{m-1} e^{-\frac{2}{n} L_{st}^{(n)}(x_i)} \prod_{i=1}^{m} 1_{\tau_{x_i} = s_i} \right] \cdot C \phi \left( t - \frac{s_m}{n} \right),
\]

(2.9)

where \( \phi(u) := 1 \land \frac{1}{\sqrt{u}} \) and we applied (2.6) to obtain the inequality.

We now bound the expectation in (2.9), summed over \( x_m \in \mathbb{Z} \). Let \( r := \lfloor \frac{s_{m-1} + s_m}{2} \rfloor \). Using \( L_{st}^{(n)}(x_i) \geq L_{r}^{(n)}(x_i) \) for \( 1 \leq i \leq m - 1 \) and applying the Markov property at time \( r \) gives

\[
\sum_{x_m \in \mathbb{Z}} \mathbb{E}^{X^\frac{1}{n}}_0 \left[ \prod_{i=1}^{m-1} e^{-\frac{2}{n} L_{st}^{(n)}(x_i)} \prod_{i=1}^{m} 1_{\tau_{x_i} = s_i} \right] = \sum_{x_m \in \mathbb{Z}} \mathbb{E}^{X^\frac{1}{n}}_0 \left[ \prod_{i=1}^{m-1} e^{-\frac{2}{n} L_{st}^{(n)}(x_i)} \prod_{i=1}^{m} 1_{\tau_{x_i} = s_i} \cdot 1_{X^\frac{1}{n} (r) = y} \right] \cdot \sum_{x_m \in \mathbb{Z}} \mathbb{E}^{X^\frac{1}{n}}_y \left( \tau_{x_m} = s_m - r \right).
\]

(2.10)

If \( \tilde{X}^\frac{1}{n} \) denotes the time-reversal of \( X^\frac{1}{n} \), which has the same increment distribution as \( -X^\frac{1}{n} \), then by time reversal and translation invariance, we have

\[
\sum_{x_m \in \mathbb{Z}} \mathbb{E}^{X^\frac{1}{n}}_y \left( \tau_{x_m} = s_m - r \right) = \mathbb{E}^{X^\frac{1}{n}}_0 \left( \tilde{X}^\frac{1}{n} (1) \neq 0, \tilde{\tau}_0 > s_m - r \right) = \frac{\kappa}{n} \sum_{z \in \mathbb{Z}} p_X (z) \mathbb{E}^{X^\frac{1}{n}}_z \left( \tau_0 \geq s_m - r \right)
\]

\[
\leq \frac{\kappa}{n} \sum_{z \in \mathbb{Z}} |z| p_X (z) \left( 1 \land \frac{C'}{\sqrt{m} - \frac{r}{n}} \right) \leq \frac{C}{n} \phi \left( \frac{s_m - s_{m-1}}{n} \right),
\]

(2.11)

where \( \tilde{\tau}_0 := \min \{ i \geq 1 : \tilde{X}^\frac{1}{n} (i) = 0 \} \), and we applied (2.6) in the first inequality. Note that this bound no longer depends on \( y \).

Substituting the bound of (2.11) into (2.10), and then successively into (2.9) and (2.8), we obtain

\[
\sum_{x_1, \ldots, x_m \in \mathbb{Z}} \mathbb{E}^{X^\frac{1}{n}}_0 \left[ \prod_{i=1}^{m} e^{-\frac{2}{n} L_{nt}^{(n)}(x_i)} 1_{\tau_{x_i} = s_i} \right] \leq \frac{C^2}{n} \phi \left( t - \frac{s_m}{n} \right) \phi \left( \frac{s_m - s_{m-1}}{n} \right) \sum_{x_1, \ldots, x_{m-1} \in \mathbb{Z}} \mathbb{E}^{X^\frac{1}{n}}_0 \left[ \prod_{i=1}^{m-1} e^{-\frac{2}{n} L_{nt}^{(n)}(x_i)} 1_{\tau_{x_i} = s_i} \right].
\]

(2.12)

We can now iterate this bound to obtain

\[
\sum_{x_1, \ldots, x_m \in \mathbb{Z}} \mathbb{E}^{X^\frac{1}{n}}_0 \left[ \prod_{i=1}^{m} e^{-\frac{2}{n} L_{nt}^{(n)}(x_i)} 1_{\tau_{x_i} = s_i} \right] \leq \frac{C^m}{n^m} \phi \left( \frac{s_1}{n} \right) \prod_{i=2}^{m} \phi \left( \frac{s_i - s_{i-1}}{n} \right) \cdot \phi \left( t - \frac{s_m}{n} \right),
\]

(2.13)

where \( \phi(u) = 1 \land \frac{1}{\sqrt{u}} \). Therefore the inner summand in (2.8) can be bounded by

\[
\sum_{0 \leq s_1 < s_2 < \ldots < s_m \leq nt} \frac{C^m}{n^m} \phi \left( \frac{s_1}{n} \right) \prod_{i=2}^{m} \phi \left( \frac{s_i - s_{i-1}}{n} \right) \cdot \phi \left( t - \frac{s_m}{n} \right) \leq C^m \int_0^{t_1} \cdots \int_0^{t_m} \phi(t_1) \phi(t - t_m) \prod_{i=2}^{m} \phi'(t_i - t_{i-1}) dt_1 \cdots dt_m.
\]
Note that given $t_{j-1} < t_{j+1},$

$$\int_{t_{j-1}}^{t_{j+1}} \phi^2(t_j - t_{j-1}) \phi^2(t_{j+1} - t_j) dt_j = \int_{t_{j-1}}^{t_{j+1}} \left( 1 \wedge \frac{1}{t_j - t_{j-1}} \right) \left( 1 \wedge \frac{1}{t_{j+1} - t_j} \right) dt_j \leq 4(\ln t) \left( 1 \wedge \frac{1}{t_{j+1} - t_j} \right) = 4(\ln t) \phi^2(t_{j+1} - t_j), \quad (2.15)$$

where the bound clearly holds when $t_{j+1} - t_{j-1} \leq 1$. When $t_{j+1} - t_{j-1} > 1$, the inequality is obtained by dividing the interval of integration into $[t_{j-1}, (t_{j+1} - t_{j-1})/2]$ and $[(t_{j+1} - t_{j})/2, t_{j+1}]$, where in the first case we use the bound $\frac{1}{t_{j+1} - t_j} \leq \frac{2}{t_{j+1} - t_{j-1}}$, and in the second case we use the bound $\frac{1}{t_{j-1} - t_{j-1}} \leq \frac{2}{t_{j+1} - t_{j-1}}$.

Applying (2.15) repeatedly to (2.14) to integrate out $t_2, \ldots, t_{m-1}$, we can bound the right-hand side of (2.14) from above by

$$C^m(4 \ln t)^{m-2} \int_{0 < t_1 < t_m < t} \left( 1 \wedge \frac{1}{\sqrt{t_1}} \right) \left( 1 \wedge \frac{1}{\sqrt{t_m - t_1}} \right) \left( 1 \wedge \frac{1}{\sqrt{t - t_m}} \right) dt_1 dt_m \leq \tilde{C}^m(\ln t)^{m-1}, \quad (2.16)$$

where the integral is bounded by considering the three cases: $t_1 \geq t/3$, $t_m - t_1 \geq t/3$, or $t - t_m \geq t/3$. Substituting this bound for (2.14) back into (2.8) then gives

$$\mathbb{E}_0^X \left[ \exp \left\{ \lambda t F_0^\gamma (X \frac{1}{n} (\cdot)) \right\} \right] \leq 1 + \sum_{m=1}^{\infty} e^{m \lambda \gamma} \tilde{C}^m \left( \ln t \right)^{m-1} \leq \frac{1}{1 - e^{\gamma} \tilde{C} \lambda \gamma} =: C(\gamma) < \infty$$

uniformly in $n$ if $c(\gamma)$ is chosen small enough such that $e^{\gamma} c(\gamma) < 1/\tilde{C}$. This finishes the proof. \hfill \Box

3 Proof of Theorem 1.4

The proof follows the same line of argument as that of Proposition 2.1, except for some complications. We first approximate $X$ by the family of discrete time random walks $X^\frac{1}{n}(\cdot)$, $n \in \mathbb{N}$. Recall from (1.15) that

$$G_t(X) := \left( \sup_{0 \leq s \leq t} X_s - \inf_{0 \leq s \leq t} X_s \right) - |\text{Range}_{x \in [0, t]}(X_s)| = \sum_{\inf_{s \in [0, t]} X_s < x < \sup_{s \in [0, t]} X_s} 1_{L^X_s(x) = 0}. \quad (3.1)$$

The following is an analogue of Lemma 2.2.

**Lemma 3.1.** Let $X$ be a continuous time random walk on $\mathbb{Z}$, whose jump kernel $p_X(x)$ satisfies $\sum_{x \in \mathbb{Z}} p_X(x) e^{\lambda |x|} < \infty$ for some $\lambda_* > 0$. Let $X^\frac{1}{n}(\cdot)$ be the discrete time approximation of $X$ defined as in (2.2). Then for any $\lambda < \lambda_*$ and $t \in [0, \infty)$, we have

$$\lim_{n \to \infty} \mathbb{E}_0^X \left[ \exp \left\{ \lambda G_t(X^\frac{1}{n}(\cdot)) \right\} \right] = \mathbb{E}_0^X \left[ \exp \left\{ \lambda G_t(X) \right\} \right]. \quad (3.2)$$

**Proof.** Clearly $G_t(X^\frac{1}{n}(\cdot))$ converges in distribution to $G_t(X)$ as $n \to \infty$. It remains to show the uniform integrability of $(\exp \left\{ \lambda G_t(X^\frac{1}{n}(\cdot)) \right\})_{n \in \mathbb{N}}$. Similarly, as in (2.4) we have,

$$G_t(X^\frac{1}{n}(\cdot)) \leq \sup_{0 \leq i \leq nt} X^\frac{1}{n}(i) - \inf_{0 \leq i \leq nt} X^\frac{1}{n}(i)$$

is bounded by the sum of the sizes of the jumps of $X^\frac{1}{n}$ before time $nt$, which is a compound binomial random variable with binomial parameters $(nt, \kappa/n)$ and summand distribution $\tilde{p}_X(x) = p_X(x) 1_{x \geq 0} + p_X(-x) 1_{x < 0}$. As $n \to \infty$, this converges to a compound Poisson random variable with Poisson parameter $\kappa t$ and summand distribution $\tilde{p}_X$. Since we assume $\sum_{x \in \mathbb{Z}} e^{\lambda_* |x|} p_X(x) < \infty$ for some $\lambda_* > 0$, it is then easily seen that $(\exp \left\{ \lambda G_t(X^\frac{1}{n}(\cdot)) \right\})_{n \in \mathbb{N}}$ is uniformly integrable for $\lambda < \lambda_*$. \hfill \Box
Proof of Theorem 1.4. As in the proof of Proposition 2.1, it suffices to show that for some $C < \infty$ and for all $t$ sufficiently large,

$$
\lim_{n \to \infty} \mathbb{E}_0^{X^\frac{1}{2}} \left[ \exp \left\{ \lambda_t \gamma_t(X^\frac{1}{2}(\cdot,n)) \right\} \right] \leq C. \tag{3.3}
$$

Given $X^\frac{1}{2}(0) = 0$, for $x \in \mathbb{Z}$, define

$$
\tau_x := \begin{cases} 
\min \{ i \geq 0 : X^\frac{1}{2}(i) \geq x \} & \text{if } x \geq 0, \\
\min \{ i \geq 0 : X^\frac{1}{2}(i) \leq x \} & \text{if } x \leq 0.
\end{cases}
$$

Using (3.1), as in (2.8), we can expand

$$
\mathbb{E}_0^{X^\frac{1}{2}} \left[ \exp \left\{ \lambda_t \gamma_t(X^\frac{1}{2}(\cdot,n)) \right\} \right] = 1 + \sum_{m=1}^\infty \lambda_m^k \sum_{m=1}^\infty \sum_{m=1}^k \mathbb{E}_0^{X^\frac{1}{2}} \left[ \prod_{i=1}^k \mathbb{I}_{t_{\lambda_i} \leq \tau_{x_i}} \right] = 1 + \sum_{m=1}^\infty \lambda_m^k \sum_{m=1}^k \mathbb{E}_0^{X^\frac{1}{2}} \left[ \prod_{i=1}^m \mathbb{I}_{t_{\lambda_i} = \tau_{x_i} > nt} \right], \tag{3.4}
$$

where in the third line, we summed over all ordered non-empty disjoint sets $(I_1, \ldots, I_m)$ which partition $\{1, \ldots, k\}$. Note that when $\tau_{x_j} = t_m$ for all $j \in I_m$, $x_j$ must be strictly between $y_m$ and $z_m$ for all $j \in I_m$. By the Markov property at time $t_m$ and by Lemma 2.3, we can bound

$$
\mathbb{E}_0^{X^\frac{1}{2}} \left[ \prod_{i=1}^m \mathbb{I}_{X^\frac{1}{2}(t_{\lambda_i-1}) = y_i, X^\frac{1}{2}(t_{\lambda_i}) = z_i} \mathbb{I}_{\tau_{x_j} = t_m, \tau_{x_j} > nt} \right] \leq \mathbb{E}_0^{X^\frac{1}{2}} \left[ \prod_{i=1}^{m-1} \mathbb{I}_{X^\frac{1}{2}(t_{\lambda_i-1}) = y_i, X^\frac{1}{2}(t_{\lambda_i}) = z_i} \mathbb{I}_{\tau_{x_j} = t_m, \tau_{x_j} > nt} \right] \left( \prod_{j \in I_m} \mathbb{I}_{\tau_{x_j} > t_m} \right) \mathbb{E}_0^{X^\frac{1}{2}} \left[ \prod_{j \in I_m} \mathbb{I}_{x_j \in (y_m \cup z_m, y_m \cup z_m)} \right] \leq \mathbb{E}_0^{X^\frac{1}{2}} \left[ \prod_{i=1}^{m-1} \mathbb{I}_{X^\frac{1}{2}(t_{\lambda_i-1}) = y_i, X^\frac{1}{2}(t_{\lambda_i}) = z_i} \mathbb{I}_{\tau_{x_j} = t_m, \tau_{x_j} > nt} \right] \left( \prod_{j \in I_m} \mathbb{I}_{\tau_{x_j} > t_m} \right) \mathbb{E}_0^{X^\frac{1}{2}} \left[ \prod_{j \in I_m} \mathbb{I}_{x_j \in (y_m \cup z_m, y_m \cup z_m)} \right] \times \frac{\Phi}{n} \mathbb{P}_X(z_m - y_m) \mathbb{P}_m(x_m - y_m) \mathbb{P}_m(y_m - z_m) \mathbb{P}_m(z_m - y_m) C |z_m - y_m| \Phi \left( \frac{t_m - \frac{t_m}{n}}{n} \right) \prod_{j \in I_m} \mathbb{I}_{x_j \in (y_m \cup z_m, y_m \cup z_m)}, \tag{3.5}
$$

where as before $\phi(u) = 1 \wedge \frac{1}{\sqrt{u}}$, and this is the analogue of (2.9) in the proof of Proposition 2.1.

Let $r := \left[ \frac{t_m - \frac{t_m}{n}}{2} \right]$. Applying the Markov property at time $r$ and summing the above bound
over \( y_m, z_m \) and \((x_j)_{j \in I_m}\) then gives
\[
C^{-K} n \left( \frac{t_m - t_m - 1}{2} \right) \sum_{v, w, u \in \mathbb{Z}} p_X(v) \left| v \right| |I_m| + 2
\times E_0^{X_0^+} \left[ \prod_{i=1}^{m} \left( 1_{X_0^+(t_{i-1}) = y_i, X_0^+(t_i) = z_i} \prod_{j \in I_i} 1_{\tau_j = t_i, \tau_j > r} \right) \right] = \tilde{C}^{m} (\ln t)^{m-1} \prod_{i=1}^{m} M(|I_i| + 2),
\]
where \( M(\alpha) := \sum_{x \in \mathbb{Z}} |x|^\alpha p_X(x) \). Substituting this bound into (3.4) then gives (uniformly in \( n \))
\[
E_0^{X_0^+} \left[ \exp \{ \lambda_G t (X_0^+ - n) \} \right] \leq 1 + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{x_1, \ldots, x_m \in \mathbb{Z}} \tilde{C}^m (\ln t)^{m-1} \prod_{i=1}^{m} M(|I_i| + 2)
\]
\[
\leq 1 + \sum_{m=1}^{\infty} (\tilde{C} \ln t)^m \sum_{k=1}^{\infty} \lambda^k \frac{k!}{k!} \sum_{m=1}^{\infty} \sum_{x_1, \ldots, x_m \in \mathbb{Z}} \prod_{i=1}^{m} M(k_i + 2)
\]
\[
= 1 + \sum_{m=1}^{\infty} (\tilde{C} \ln t)^m \sum_{k=1}^{\infty} \lambda^k \frac{k!}{k!} \sum_{m=1}^{\infty} \sum_{x_1, \ldots, x_m \in \mathbb{Z}} \prod_{i=1}^{m} M(k_i + 2)
\]
\[
= 1 + \sum_{m=1}^{\infty} (\tilde{C} \ln t)^m \sum_{k=1}^{\infty} \lambda^k \frac{k!}{k!} M(k + 2) < C < \infty
\]
if \( c \) in \( \lambda_t = \frac{1}{\sqrt{\ln t}} \) is chosen small enough. Indeed, let \( V \) be a random variable with \( M(\alpha) = \mathbb{E}[|V|^\alpha] \). Since assumption (1.16) implies \( \mathbb{E}[e^{\lambda_t V}] < \infty \), we have
\[
\tilde{C} \ln t \sum_{k=1}^{\infty} \lambda^k \frac{k!}{k!} M(k + 2) = \mathbb{E} \left[ \sum_{k=1}^{\infty} \frac{e^{\lambda_t} \ln t}{(1 \vee \ln t)^{k!}} |V|^{k+2} \right]
\]
\[
\leq c \tilde{C} \mathbb{E} \left[ |V|^3 \sum_{k=1}^{\infty} \frac{e^{\lambda_t} \ln t}{(1 \vee \ln t)^{k!}} |V|^{k+2} \right] \leq c \tilde{C} \mathbb{E} \left[ |V|^6 \right] \mathbb{E} \left[ e^{\frac{2\ln t}{\sqrt{\ln t}}} |V| \right] ^{1/2} < 1
\]
if \( t \) is large enough and \( c \) is chosen small enough. This completes the proof. \( \square \)
4 Proof of Theorem 1.2

To prepare for the proof, we recall here the strategy for the lower bound on the annealed survival probability employed in [DGRS12], and we show how to rewrite the survival probability in terms of the range of random walks.

4.1 Strategy for lower bound

The lower bound on the annealed survival probabilities in Theorem 1.1 follows the same strategy as for the case of immobile traps in previous works. Denote \( B_t = \{ x \in \mathbb{Z}^d : \| x \| \leq r \} \). For a fixed time \( t \), we force the environment \( \xi \) to create a ball \( B_{Rt} \) of radius \( R_t \) around the origin, which is free of traps from time 0 up to time \( t \). We then force the random walk \( X \) to stay inside \( B_{Rt} \) up to time \( t \). This leads to a lower bound on the survival probability that is independent of \( \gamma \in (0, \infty] \).

To be more precise, we consider the following events:

- Let \( E_t \) denote the event that \( N_y = 0 \) for all \( y \in B_{Rt} \).
- Let \( F_t \) denote the event that \( Y_s^j \notin B_{Rt} \) for all \( y \notin B_{Rt} \), \( 1 \leq j \leq N_y \), and \( s \in [0, t] \).
- Let \( G_t \) denote the event that \( X \) with \( X(0) = 0 \) does not leave \( B_{Rt} \) before time \( t \).

Then, by the strategy outlined above, the annealed survival probability

\[
\mathbb{P}_0^X [Z_{t,X}] \geq \mathbb{P}(E_t \cap F_t \cap G_t) = \mathbb{P}(E_t) \mathbb{P}(F_t) \mathbb{P}(G_t),
\]

since \( E_t, F_t, \) and \( G_t \) are independent.

In order to lower bound (4.1), note that

\[
\mathbb{P}(E_t) = e^{-\nu(2R_t+1)^d}.
\]

To estimate \( \mathbb{P}(G_t) \), Donsker’s invariance principle implies that there exists \( \alpha > 0 \) such that for all \( t \) sufficiently large,

\[
\inf_{x \in B_{\sqrt{t}/2}} \mathbb{P}_0^X \left( X_s \in B_{\sqrt{t}} \forall s \in [0, t] \, , \, X_t \in B_{\sqrt{t}/2} \mid X_0 = x \right) \geq \alpha.
\]

Now if \( 1 \ll R_t \ll \sqrt{t} \) as \( t \to \infty \), then by partitioning the time interval \([0, t]\) into intervals of length \( R_t^2 \) and applying the Markov property at times \( tR_t^2 \), we obtain

\[
\mathbb{P}(G_t) \geq \mathbb{P}_0^X \left( X_s \in B_{R_t} \forall s \in [(i-1)R_t^2, iR_t^2] \, , \, X_i R_t^2 \in B_{R_t/2} \, , \, i = 1, 2, \ldots, \lfloor t/R_t^2 \rfloor \right) \geq \alpha^{\lfloor t/R_t^2 \rfloor} (1 + o(1)) e^{\nu \ln \alpha / R_t^2}.
\]

This actually gives the correct logarithmic order of decay for \( \mathbb{P}(G_t) \). Indeed, by Donsker’s invariance principle, uniformly in \( t \) large and \( X_0 = x \in B_{R_t} \),

\[
\mathbb{P}_0^X (X_s \notin B_{R_t} \text{ for some } s \in [0, R_t^2]) \geq \mathbb{P}_0^W (W_s \notin B_3 \text{ for some } s \in [0, 1]) =: \rho > 0,
\]

where \( W \) is a standard \( d \)-dimensional Brownian motion. Therefore by a similar application of the Markov inequality as in (4.3), we find that

\[
\mathbb{P}(G_t) \leq e^{\lfloor t/R_t^2 \rfloor \ln(1 - \rho)}.
\]

In dimension \( d = 1 \), which is our main focus, integrating out the Poisson initial distribution of the \( Y \)-particles gives

\[
\mathbb{P}(F_t) = \exp \left\{ -\nu \sum_{y \in \mathbb{Z} \setminus B_{R_t}} \mathbb{P}_0^Y (\tau^Y(B_{R_t}) \leq t) \right\} = \exp \left\{ -\nu \sum_{y \in \mathbb{Z} \setminus \{0\}} \mathbb{P}_0^Y (\tau^Y(\{0\}) \leq t) \right\} = \exp \left\{ -\nu \mathbb{P}_0^Y \left( |\text{Range}_{s \in [0,t]}(Y_s)| \right) - 1 \right\},
\]

(4.5)
where \( \tau^Y(B) \) denotes the first hitting time of a set \( B \subset \mathbb{Z}^d \) by \( Y \), and in the second equality, we used the assumption that \( Y \) makes nearest-neighbor jumps. Note that it was shown in [DGRS12] that \(- \ln \mathbb{P}(F_t) \sim \sqrt{\frac{8gd}{\pi t}}\).

Substituting the bounds (4.2)–(4.5) into (4.1), we find that in dimension \( d = 1 \), the optimal choice is \( R_t = t^\frac{1}{2} \), which is determined by the interplay between \( \mathbb{P}(E_t) \) and \( \mathbb{P}(G_t) \) as \( t \to \infty \). If this lower bound strategy is optimal, then under \( P_t^\gamma \), \( X \) will fluctuate on the scale of \( t^\frac{1}{2} \).

### 4.2 Rewriting in terms of the range

Averaging out the Poisson initial condition of \( \xi \), we can rewrite (1.3) as

\[
Z^\gamma_{t,X} = \exp \left\{ \nu \sum_{y \in \mathbb{Z}} (v_X(t, y) - 1) \right\},
\]

with

\[
v_X(t, y) = \mathbb{E}^Y_0 \left[ \exp \left\{ - \gamma L^Y_{t-X}(0) \right\} \right],
\]

where \( L^Y_{t-X}(0) = \int_0^t \delta_0(Y_s - X_s) \, ds \) is the local time of \( Y - X \) at 0, introduced in (1.10).

When \( \gamma = \infty \), we define \( v_X(t, Y) = \mathbb{P}^Y_0(L^Y_{t-X}(0) = 0) \), and it is easily seen that

\[
Z^\infty_{t,X} = \exp \left\{ - \nu \sum_{y \in \mathbb{Z}} \mathbb{P}^Y_0(L^Y_{t-X}(0) > 0) \right\} = \exp \left\{ - \nu \mathbb{E}^Y_0 \left[ \left| \text{Range}_{\gamma \in [0,t]}(Y_s - X_s) \right| \right] \right\}
\]

\[
= \exp \left\{ - \nu \mathbb{E}^Y_0 \left[ \left| \text{Range}_{\gamma \in [0,t]}(Y_s + X_s) \right| \right] \right\},
\]

where we used the assumption that \( Y \) is symmetric, and for any \( f \in \mathcal{D}([0, t], \mathbb{Z}) \),

\[
\text{Range}_{\gamma \in [0,t]}(f(s)) := \{ f(s) : s \in [0, t] \}
\]

denotes the range.

When \( \gamma < \infty \), \( Z^\gamma_{t,X} \) admits a similar representation in terms of the range of \( Y + X \). Indeed, let \( N_t := \{ J_1 < J_2 < \cdots \} \) be an independent Poisson point process on \([0, \infty)\) with rate \( \gamma \in (0, \infty) \), and define

\[
\text{SoftRange}_{\gamma \in [0,t]}(f(s)) := \{ f(J_k) : k \in \mathbb{N}, J_k \in [0, t] \}.
\]

Probability and expectation for \( N \) will be denoted by adding the superscript \( N \) to \( \mathbb{P} \) and \( \mathbb{E} \). We can then rewrite (4.6) as

\[
Z^\gamma_{t,X} = \exp \left\{ - \nu \sum_{y \in \mathbb{Z}} \mathbb{P}^Y_0(N(0 \in \text{SoftRange}_{\gamma \in [0,t]}(Y_s - X_s))) \right\}
\]

\[
= \exp \left\{ - \nu \mathbb{E}^Y_0 \left[ \left| \text{SoftRange}_{\gamma \in [0,t]}(Y_s + X_s) \right| \right] \right\}.
\]

### 4.3 Proof of Sub-diffusivity of \( X \)

To control the path measure \( P_t^\gamma \) (cf. (1.7)) we could try to proceed with the bounds outlined in (4.1)–(4.5). We cannot use (4.5) directly as we had assumed that \( Y \) was a simple random walk for that particular bound. To circumvent this we use Theorem 1.4. However note that, we are free to use the bounds (4.1)–(4.4) for \( Y \) as in (1.5). Using these with \( R_t = t^\frac{1}{2} \) and (4.10) we observe that

\[
P_t^\gamma(X \in \cdot) \leq \frac{\mathbb{E}^X_0 \left[ \exp \left\{ - \nu \mathbb{E}^Y_0 \left[ \left| \text{SoftRange}_{\gamma \in [0,t]}(Y_s + X_s) \right| \right] \right\} \right] \mathbb{I}_{X \in \cdot}}{e^{-ct^\frac{1}{2} \mathbb{P}(F_t)}}.
\]
When \( Y \) is as in (1.5) and satisfies (1.8) then using Theorem 1.4 we have
\[
\mathbb{P}(F_t) = \exp \left\{ -\nu \sum_{y \in \mathbb{Z} \setminus B_{R_t}} \mathbb{P}_y^Y(\tau^Y(B_{R_t}) \leq t) \right\} = \exp \left\{ -\nu \sum_{y \in \mathbb{Z} \setminus B_{R_t}} \mathbb{P}_y^Y(\tau^Y(y + B_{R_t}) \leq t) \right\}
\geq \exp \left\{ -\nu \left( \mathbb{E}_0^Y \left( \sup_{s \in [0,t]} Y_s - \inf_{s \in [0,t]} Y_s \right) \right) \right\} \geq \exp \left\{ -\nu \left( \mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]}(Y_s) \right| \right] + C \ln t \right) \right\},
\]
where we used translation invariance and (1.18). Since \( \ln t \ll t^{\frac{1}{2}} \), this and (4.11) implies that
\[
P_t^Y(X \in \cdot) \leq e^{c_1 t^{\frac{1}{2}}} \mathbb{E}_0^X \left\{ \exp \left\{ -\nu \left( \mathbb{E}_0^{Y,N} \left[ \left| \text{SoftRange}_{s \in [0,t]}(Y_s + X_s) \right| \right] - \mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]}(Y_s) \right| \right] \right) ) \right\} \right\} \mathbb{1}_{X \in \cdot}
\]
for \( t \) sufficiently large. This will be the starting point of our analysis of \( P_t^Y \).

**Proof of Theorem 1.2 for simple random walks.** We first bound the fluctuation of \( X \) under \( P_t^Y \) from below. Since \( Y \) is an irreducible symmetric random walk, Pascal’s principle (see [DGRS12, Prop. 2.1], in particular, [DGRS12, (38) & (49)] for \( \gamma < \infty \), implies that the expected (soft) range (cf. (4.8)-(4.9)) of a \( Y \) walk increases under perturbations, i.e., \( P_0^Y \)-a.s.,
\[
\mathbb{E}_0^{Y,N} \left[ \left| \text{SoftRange}_{s \in [0,t]}(Y_s + X_s) \right| \right] - \mathbb{E}_0^{Y} \left[ \left| \text{Range}_{s \in [0,t]}(Y_s) \right| \right] \geq 0.
\]
Also note that by the definition of \( F_t^Y (\cdot) \) in (1.12) and the definition of soft range in (4.9), we have
\[
F_t^Y(Y) = |\text{Range}_{s \in [0,t]}(Y_s)| - \mathbb{E}^N[|\text{SoftRange}_{s \in [0,t]}(Y_s)|].
\]
Furthermore, Proposition 2.1 applied to \( Y \), combined with the exponential Markov inequality, gives
\[
\mathbb{E}_0^Y \left[ F_t^Y(Y) \right] = \int_0^\infty \mathbb{P}_0^Y(F_t^Y(Y) \geq m) dm \leq \int_0^\infty C(\gamma) e^{-\frac{m(\gamma)}{C\ln m}} dm \leq C \ln t
\]
for all \( t \) large enough. Hence, in combination with (4.12), we obtain
\[
P_t^Y(\|X\|_t \leq \alpha t^{\frac{1}{2}})
\leq e^{c_1 t^{\frac{1}{2}}} \mathbb{E}_0^X \left\{ \exp \left\{ -\nu \left( \mathbb{E}_0^{Y,N} \left[ \left| \text{SoftRange}_{s \in [0,t]}(Y_s + X_s) \right| \right] - \mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]}(Y_s) \right| \right] \right) \right\} \right\}_1_{\|X\|_t \leq \alpha t^{\frac{1}{2}}}
\leq e^{c_1 t^{\frac{1}{2}}} \mathbb{E}_0^X \left\{ \exp \left\{ -\nu \left( \mathbb{E}_0^{Y,N} \left[ \left| \text{SoftRange}_{s \in [0,t]}(Y_s + X_s) \right| \right] - \mathbb{E}_0^{Y,N} \left[ \left| \text{SoftRange}_{s \in [0,t]}(Y_s) \right| \right] \right) \right\} \right\}_1_{\|X\|_t \leq t^{\frac{1}{2}}}
\leq e^{c_1 t^{\frac{1}{2}}} e^{\nu \mathbb{E}_0^Y |F_t^Y(Y)|} \mathbb{E}_0^X \left\{ \|X\|_t \leq \alpha t^{\frac{1}{2}} \right\}
\leq e^{c_1 t^{\frac{1}{2}}} + C \nu \ln t e^{\alpha^2 \ln(1-\rho) t^{\frac{1}{2}}} \rightarrow 0 \quad \text{as } t \rightarrow \infty,
\]
where we applied (4.4) in the last inequality, with \( \alpha > 0 \) chosen sufficiently small.

It remains to bound the fluctuation of \( X \) under \( P_t^Y \) from above. We divide into two cases: \( \gamma = \infty \) or \( \gamma \in (0,\infty) \).

**Case 1: \( \gamma = \infty \).** By (4.12), it suffices to show that
\[
\liminf_{t \rightarrow \infty} t^{-\frac{1}{4}+\epsilon} \inf_{X: \|X\|_t > 0} \left( \mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]}(Y_s + X_s) \right| \right] - \mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]}(Y_s) \right| \right] \right) > 0.
\]
Since $X$ and $Y$ are independent continuous time simple random walks, $Y + X$ is also a simple random walk, which allows us to write

$$
\mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]} (Y_s + X_s) \right| \right] = \mathbb{E}_0^Y \left[ \sup_{s \in [0,t]} (Y_s + X_s) - \inf_{s \in [0,t]} (Y_s + X_s) \right]
$$

$$
= \mathbb{E}_0^Y \left[ \sup_{s \in [0,t]} (Y_s + X_s) + \sup_{s \in [0,t]} (-Y_s - X_s) \right]
$$

$$
= \mathbb{E}_0^Y \left[ \sup_{s \in [0,t]} (Y_s + X_s) + \sup_{s \in [0,t]} (Y_s - X_s) \right],
$$

where the last equality follows since $Y$ is symmetric.

Now observe that on the event \( \{ \|X\|_t \geq t_{1/2}^{1/2 + \epsilon} \} \), for

$$
\sigma(X) := \arg\max_{s \in [0,t]} X_s \quad \text{and} \quad \tau(X) := \arg\min_{s \in [0,t]} X_s \in [0,t],
$$

where ties are broken by choosing the minimum value, one of the sets

$$
S := \{ s \in [0,t] : X_{\sigma(X)} - X_s \geq t_{1/2}^{1/2 + \epsilon}/2 \} \quad \text{and} \quad T := \{ r \in [0,t] : X_s - X_{\tau(X)} \geq t_{1/2}^{1/2 + \epsilon}/2 \}
$$

has Lebesgue measure $\lambda(S) \geq t/2$ or $\lambda(T) \geq t/2$. Without loss of generality, we can assume that $\lambda(S) \geq t/2$. We then have

$$
\mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]} (Y_s + X_s) \right| - \mathbb{E}_0^Y \left[ \left| \text{Range}_{s \in [0,t]} (Y_s) \right| \right] \right]
$$

$$
= \mathbb{E}_0^Y \left[ \sup_{s \in [0,t]} (Y_s + X_s) + \sup_{s \in [0,t]} (Y_s - X_s) - 2 \sup_{s \in [0,t]} (Y_s) \right]
$$

$$
\geq \mathbb{E}_0^Y \left[ \left( \sup_{s \in [0,t]} (Y_s + X_s) + \sup_{s \in [0,t]} (Y_s - X_s) - 2 \sup_{s \in [0,t]} (Y_s) \right) \mathbb{1}_{\sigma(Y) \in S} \mathbb{1}_{\tau_{\sigma(Y)} - \tau_{\sigma(X)} \leq t_{1/2}^{1/2}} \right]
$$

$$
\geq \mathbb{E}_0^Y \left[ (Y_{\sigma(X)} + X_{\sigma(X)} + Y_{\sigma(Y)} - X_{\sigma(Y)} - 2Y_{\sigma(Y)}) \mathbb{1}_{\sigma(Y) \in S} \mathbb{1}_{\tau_{\sigma(Y)} - \tau_{\sigma(X)} \leq t_{1/2}^{1/2}} \right]
$$

$$
\geq \left( t_{1/2}^{1/2 + \epsilon}/2 - t_{1/2}^{1/2} \right) \mathbb{P}_0^Y (Y_{\sigma(Y)} - Y_{\sigma(X)} \leq t_{1/2}^{1/2}, \sigma(Y) \in S),
$$

where the first inequality uses that the difference of the sup’s in the expectation is non-negative.

It remains to lower bound the probability

$$
\mathbb{P}_0^Y (Y_{\sigma(Y)} - Y_{\sigma(X)} \leq t_{1/2}^{1/2}, \sigma(Y) \in S) = \int_S \mathbb{P}_0^Y (Y_{\sigma(Y)} - Y_{\sigma(X)} \leq t_{1/2}^{1/2} | \sigma(Y) = r) \mathbb{P}_0^Y (\sigma(Y) \in dr). \quad (4.20)
$$

Note that $\mathbb{P}_0^Y (\sigma(Y) \in dr)$ is absolutely continuous with respect to the Lebesgue measure $\lambda(dr)$ with density

$$
\lim_{\delta \downarrow 0} \delta^{-1} \mathbb{P}_0^Y (\sigma(Y) \in [r, r + \delta]) = \rho \mathbb{P}_0^Y (Y_s \leq 0 \forall s \in [0,t - r]) \sum_{z < 0} p_Y(z) \mathbb{P}_0^Y (Y_s < 0 \forall s \in [0,r]), \quad (4.21)
$$

where $\rho$ is the jump rate of $Y$, $p_Y$ its jump kernel, $\sigma(Y)$ is the first time when $Y$ reaches its global maximum in the time interval $[0,t]$, and we used the observation that given $\sigma(Y) = r$ and the size of the jump at time $r$, $(Y_s)_{0 \leq s \leq r}$ and $(Y_s)_{r \leq s \leq t}$ are two independent random walks. By [LL10, Theorem 5.1.7], if $\tau_{[0,\infty)}$ denotes the first hitting time of $[0,\infty)$, then there exist $C_1, C_2 > 0$ such that for all $z < 0$ and $s > |z|^2$,

$$
C_1 \frac{|z|}{\sqrt{s}} \leq \mathbb{P}_z^Y (\tau_{[0,\infty)} \geq s) \leq C_2 \frac{|z|}{\sqrt{s}}. \quad (4.22)
$$

Substituting the lower bound into (4.21), we find that for any $\delta > 0$, there exists a constant $c(\delta) > 0$ such for all $t > 0$, we have

$$
\mathbb{P}_0^Y (\sigma(Y) \in dr) \geq \frac{c(\delta)\lambda(dr)}{t} \quad \text{on } [\delta t, (1 - \delta)t]. \quad (4.23)
$$
Since $\lambda(S) \geq t/2$ by assumption, to lower bound the probability in (4.20), it only remains to lower bound
\begin{equation}
\mathbb{P}^Y_0(Y_{\sigma(Y)} - Y_{\sigma(X)} \leq t^{1/4} | \sigma(Y) = r)
\end{equation}
uniformly in $r \in [\delta t, (1-\delta)t]$ with $|r - \sigma(X)| \geq \delta t$, and in $t > 0$, for any $\delta < 1/8$.

Note that conditioned on $\sigma(Y) = r$ and $Y_{\sigma(Y)} - Y_{\sigma(X)} = z < 0$, $(Y_{\sigma(Y)} - Y_{\sigma(X)})_{s \in [0, r]}$ and $(Y_{\sigma(Y) + s} - Y_{\sigma(Y)})_{s \in [0, t-r]}$ are two independent conditioned random walks, starting respectively at $z$ and $0$, and conditioned respectively to not visit $[0, \infty)$ and $[1, \infty)$. Note that such conditioned random walks are comparable to a Bessel-3 process, although we will only use random walk estimates. We will only consider the case $s := \sigma(X) - r > 0$, the case $s < 0$ is entirely analogous. We then get for (4.24) the lower bound
\begin{equation}
\frac{\mathbb{P}^Y_0(Y_s \geq -t^{1/4}, \tau_{1, \infty} \geq t - r)}{\mathbb{P}^Y_0(\tau_{1, \infty} \geq t - r)}.
\end{equation}

By the Markov property, the numerator in (4.25) equals
\begin{align*}
\sum_{x=-[t^{1/4}]}^0 \mathbb{P}^Y_0(Y_s = x, \tau_{1, \infty} \geq s) \mathbb{P}^Y_x(\tau_{1, \infty} \geq t - r - s) \\
\geq \sum_{x=-[t^{1/4}]}^0 \sum_{\sqrt{s} \leq y, z \leq 2\sqrt{s}} \mathbb{P}^Y_0(Y_{s/3} = y, Y_{2s/3} = z, Y_s = x, \tau_{1, \infty} \geq s) \mathbb{P}^Y_x(\tau_{1, \infty} \geq t - r - s) \\
\geq C \sum_{x=-[t^{1/4}]}^0 \sum_{\sqrt{s} \leq y, z \leq 2\sqrt{s}} \mathbb{P}^Y_0(Y_{s/3} = y, \tau_{1, \infty} > s/3) \mathbb{P}^Y_y(Y_{s/3} = z, \tau_{1, \infty} > s/3) \times \mathbb{P}^Y_x(\tau_{1, \infty} \geq t - r - s) \\
\geq C \sum_{x=-[t^{1/4}]}^0 \sum_{\sqrt{s} \leq y, z \leq 2\sqrt{s}} |x| \mathbb{P}^Y_0(Y_{s/3} \in [\sqrt{s}, 2\sqrt{s}], \tau_{1, \infty} > s/3) \mathbb{P}^Y_x(Y_{s/3} \in [\sqrt{s}, 2\sqrt{s}], \tau_{1, \infty} > s/3) \\
\geq C \sum_{x=-[t^{1/4}]}^0 |x| \mathbb{P}^Y_0(\tau_{1, \infty} > s/3) \mathbb{P}^Y_x(\tau_{1, \infty} > s/3) \\
\geq C \sum_{x=-[t^{1/4}]}^0 |x|^2 \geq Ct^{-2}t^{3/4} = Ct^{-5/4},
\end{align*}
where in the third inequality we applied the local limit theorem and (4.22), in the fourth inequality we used $s \geq \delta t$, and in the fifth inequality we used the fact that conditioned on $\{\tau_{1, \infty} > s/3\}$, $(Y_{us}/\sqrt{s})_{0 \leq u \leq 1/3}$ converges in distribution to a Brownian meander if $Y_0 \ll \sqrt{s}$ as $s \to \infty$ (cf. [B76]).

Since $t - r \geq \delta t$, again by (4.22), we find that
\begin{equation}
\frac{\mathbb{P}^Y_0(Y_s \geq -t^{1/4}, \tau_{1, \infty} \geq t)}{\mathbb{P}^Y_0(\tau_{1, \infty} \geq t - r)} \geq Ct^{-5/4}t^{1/4} = Ct^{-1/8}.
\end{equation}
Plugging this into (4.20) (recall (4.23)) and the resulting inequality into (4.19), we find that (4.17) holds, which concludes the proof for the case $\gamma = \infty$ and $X, Y$ are simple random walks.
Case 2: $\gamma \in (0, \infty)$. We first use (4.12) to upper bound

$$P_t^\gamma(\|X\| \geq t^{\frac{3}{2}+\epsilon})$$

$$\leq e^{c_1 t^{\frac{3}{2}} + C\nu c_1} \mathbb{E}_0^X \exp \left\{-\nu \left( \mathbb{E}_0^{Y,N} \left[ \left| \text{SoftRange}_{s \in [0,t]}(Y_s + X_s) \right| \right] - \mathbb{E}_0^{Y} \left[ \left| \text{Range}_{s \in [0,t]}(Y_s) \right| \right] \right) \right\} 1_{\|X\| \geq t^{\frac{3}{2}+\epsilon}}$$

$$= e^{c_1 t^{\frac{3}{2}} + C\nu c_1} \mathbb{E}_0^X \exp \left\{-\nu \left( \mathbb{E}_0^{Y} \left[ \left| \text{Range}_{s \in [0,t]}(Y_s + X_s) \right| \right] - \mathbb{E}_0^{Y} \left[ \left| \text{Range}_{s \in [0,t]}(Y_s) \right| \right] \right) \right\} \times \exp \left\{ \nu \mathbb{E}_0^{Y} [F_t^\gamma (X + Y)] \right\} 1_{\|X\| \geq t^{\frac{3}{2}+\epsilon}},$$

(4.27)

where we applied (4.14) to $Y + X$ in the last equality.

It is clear from the above bound that

$$P_t^\gamma(\|X\| \geq t^{\frac{3}{2}+\epsilon}, \mathbb{E}_0^Y [F_t^\gamma (X + Y)] \leq t^{\frac{3}{2}+\epsilon}) \rightarrow 0,$$

(4.28)

On the other hand, by the same calculations as in (4.16), we have

$$P_t^\gamma \left( \mathbb{E}_0^Y [F_t^\gamma (X + Y)] > t^{\frac{3}{2}+\epsilon} \right) \leq e^{c_1 t^{\frac{3}{2}} + C\nu c_1} \mathbb{E}_0^X \left( \mathbb{E}_0^Y [F_t^\gamma (X + Y)] > t^{\frac{3}{2}+\epsilon} \right)$$

$$\leq e^{c_1 t^{\frac{3}{2}} + C\nu c_1} \mathbb{E}_0^X \left( \mathbb{E}_0^Y [F_t^\gamma (X + Y)] \right)$$

$$\leq C(\gamma) e^{c_1 t^{\frac{3}{2}} + C\nu c_1} \mathbb{E}_0^X \left( \mathbb{E}_0^Y [F_t^\gamma (X + Y)] \right) \rightarrow 0,$$

(4.29)

where we have applied Jensen’s inequality and Proposition 2.1. Combined with (4.28), this concludes the proof for the case $\gamma \in (0, \infty)$. $\square$

Proof of Theorem 1.2 for general $X$ and $Y$. When $X$ and $Y$ are non-simple random walks, identities such as (4.18) fails because the range of the walk is no longer the interval bounded between the walk’s infimum and supremum. Theorem 1.4 allows us to salvage the argument.

The lower bound (4.16) on the fluctuation of $X$ under $P_t^\gamma$ remains valid, using (4.12) for $Y$ as in (1.5) and (1.8).

For the upper bound on the fluctuation of $X$ under $P_t^\gamma$, $\gamma \in (0, \infty)$, note that by (4.12),

$$P_t^\gamma(\|X\| \geq t^{\frac{3}{2}+\epsilon})$$

$$\leq e^{c_1 t^{\frac{3}{2}} + C\nu c_1} \mathbb{E}_0^X \left( \mathbb{E}_0^Y \left[ \sup_{s \in [0,t]} (Y_s + X_s) - \inf_{s \in [0,t]} (Y_s + X_s) \right] \right)$$

$$\leq e^{c_1 t^{\frac{3}{2}} + C\nu c_1} \mathbb{E}_0^X \left( \mathbb{E}_0^Y \left[ \sup_{s \in [0,t]} (Y_s + X_s) - 2 \sup_{s \in [0,t]} Y_s \right] \right) \times \exp \left\{ \nu \mathbb{E}_0^Y [G_t (Y + X)] + \nu \mathbb{E}_0^Y [F_t^\gamma (X + Y)] \right\} 1_{\|X\| \geq t^{\frac{3}{2}+\epsilon}}$$

(4.30)
where we recall from (1.15) that \( G_t(X) := \left( \sup_{0 \leq s \leq t} X_s - \inf_{0 \leq s \leq t} X_s \right) - \text{Range}_{s \in [0,t]}(X_s) \), and we used the symmetry of \( Y \) in the last equality. Note that when \( \gamma = \infty \), \( F_t^\gamma(Y + X) = 0 \).

The proof of (4.19) does not require \( X \) and \( Y \) to be simple random walks, and in particular, it implies that

\[
\liminf_{t \to \infty} t^{-\frac{1}{4} - \epsilon} \inf_{X : \|X\| \geq t^{\frac{11}{24} + \epsilon}} \mathbb{E}_0^Y \left[ \sup_{s \in [0,t]} (Y_s + X_s) + \sup_{s \in [0,t]} (Y_s - X_s) - 2 \sup_{s \in [0,t]} Y_s \right] > 0.
\]

Therefore it follows from (4.30) that

\[
P_t^\gamma(\|X\| \geq t^{\frac{11}{24} + \epsilon}, \mathbb{E}_0^Y [F_t^\gamma(X + Y)] \leq t^{\frac{1}{4} + \frac{\epsilon}{2}}, \mathbb{E}_0^Y [G_t(Y + X)] \leq t^{\frac{1}{4} + \frac{\epsilon}{2}}) \to 0 \quad \text{as} \quad t \to \infty.
\]

(4.31)

The argument for \( P_t^\gamma(\mathbb{E}_0^Y [F_t^\gamma(X + Y)] > t^{\frac{1}{4} + \frac{\epsilon}{2}}) \to 0 \in (4.29) \) is still valid, while the same argument as in (4.29) with \( F_t^\gamma(Y + X) \) replaced by \( G_t(Y + X) \), together with Theorem 1.4, shows that we also have \( P_t^\gamma(\mathbb{E}_0^Y [G_t(Y + X)] > t^{\frac{1}{4} + \frac{\epsilon}{2}}) \to 0 \) as \( t \to \infty \). This completes the proof. \( \square \)

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