Geometry of the random walk range conditioned on survival among Bernoulli obstacles

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June 22, 2018

Abstract

We consider a discrete time simple symmetric random walk among Bernoulli obstacles on $\mathbb{Z}^d$, $d \geq 2$, where the walk is killed when it hits an obstacle. It is known that conditioned on survival up to time $N$, the random walk range is asymptotically contained in a ball of radius $\varrho_N = CN^{1/(d+2)}$ for any $d \geq 2$. For $d = 2$, it is also known that the range asymptotically contains a ball of radius $(1 - \epsilon)\varrho_N$ for any $\epsilon > 0$, while the case $d \geq 3$ remains open. We complete the picture by showing that for any $d \geq 2$, the random walk range asymptotically contains a ball of radius $\varrho_N - \varrho'_N$ for some $\epsilon \in (0, 1)$. Furthermore, we show that its boundary is of size at most $\varrho_N^{-d-1}(\log \varrho_N)^a$ for some $a > 0$.

MSC 2000. Primary: 60K37; Secondary: 60K35.

Keywords. Bernoulli obstacles, random walk range, Faber–Krahn inequality, annealed law.

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1 Introduction

Let $S := (S_n)_{n \geq 0}$ be a discrete time simple symmetric random walk on $\mathbb{Z}^d$. We will use $P_x$ and $E_x$ to denote probability and expectation for $S$ with $S_0 = x \in \mathbb{Z}^d$, and we will omit the subscript $x$ when $x = 0$. Independently for each $x \in \mathbb{Z}^d$, an obstacle is placed at $x$ with probability $1 - p$ for some fixed $p \in (0, 1)$, which generates the so-called Bernoulli obstacle configuration and plays the role of a random environment. Probability and expectation for the obstacles will be denoted by $P$ and $E$, respectively. Let $O$ denote the set of sites occupied by obstacles. When there is no obstacle at the site $x \in \mathbb{Z}^d$, we will say $x$ is open. The random walk is killed at the moment it hits an obstacle (called hard obstacles), namely, at the stopping time

$$\tau_O := \min\{n \geq 0 : S_n \in O\}. \hspace{1cm} (1.1)$$

More generally, we will use $\tau_A$ to denote the first hitting time of a set $A \subset \mathbb{Z}^d$. We will write $E[f(S) : A] = E[f(S)1_A]$ and $E[g(O) : B] = E[g(O)1_B]$.

We are interested in $P \otimes P((S, O) \in \cdot \mid \tau_O > N)$, the so-called annealed law of $(S, O)$ conditioned on the random walk’s survival up to time $N$. For simplicity, we will denote

$$\mu_N((S, O) \in \cdot) := P \otimes P((S, O) \in \cdot \mid \tau_O > N). \hspace{1cm} (1.2)$$

In particular, we are interested in the law of the random walk range

$$S_{[0,N]} := \{S_i : 0 \leq i \leq N\} \hspace{1cm} (1.3)$$

under the conditioned measure $\mu_N$. It is worth noting that the marginal law of $\mu_N$ for the random walk admits a representation in terms of the range of the random walk:

$$\mu_N(S \in \cdot) = \frac{E \left[ p^{|S_{[0,N]}|} : S \in \cdot \right]}{E \left[ p^{|S_{[0,N]}|} \right]}, \hspace{1cm} (1.4)$$

where $|S_{[0,N]}|$ denotes the cardinality of the set $S_{[0,N]}$.

Let us review known results on this model. The first result dates back to Donsker–Varadhan’s work [DV79] which determined the leading order asymptotics of the denominator in (1.4), which can be regarded as the “partition function” of a self-attracting polymer model. The main result of [DV79] reads as

$$P \otimes P(\tau_O > N) = \exp \left\{ -c(d, p)N^{\frac{d}{d+2}}(1 + o(1)) \right\},$$

with $c(d, p) := \frac{d + 2}{2} \left( \log(1/p) \right)^{\frac{d}{d+2}} \left( \frac{2 \lambda_1}{d} \right)^{\frac{d}{d+2}}$, (1.5)
where $\lambda_1$ is the principal Dirichlet eigenvalue of $-\frac{1}{2d}\Delta$ in the ball of unit volume in $\mathbb{R}^d$ centered at the origin. In fact, Donsker–Varadhan studied this problem in the continuum setting first in [DV75] as the asymptotics of the moment generating function of the Wiener sausage. This corresponds to a space-time continuum analogue of a random walk among Bernoulli obstacles, known as a Brownian motion among Poissonian obstacles, where each obstacle takes a fixed shape, say a ball, and the centers of the obstacles follow a homogeneous Poisson point process on $\mathbb{R}^d$. This model has been studied extensively and most of the results can be found in the celebrated monograph by Sznitman [S98]. The core of the method of Sznitman, called the method of enlargement of obstacles, is translated to the discrete setting in [A95] and therefore most of the results in continuum setting can be converted to the discrete setting. For this reason, in this section we will not explicate in which setting a result has been proved.

The argument of Donsker–Varadhan indicates that the dominant contribution to the partition function comes from the strategy of finding a ball of optimal radius

$$\varrho_N := \left(\frac{2\lambda_1}{d \log(1/p)}\right)^{\frac{1}{d+2}} N^{\frac{1}{d+2}},$$

(1.6)

which is free of obstacles and the random walk is confined in that ball up to time $N$. It has been proved later that this is what happens under the annealed measure in [S91] and [B94] for $d = 2$ and [P99] for $d \geq 3$.

**Theorem A** (Confinement). For any $d \geq 2$, there exists $\epsilon_1 \in (0,1)$ and $x_N \in \mathbb{Z}^d$ depending only on the obstacle configuration $\mathcal{O}$, such that $x_N \in B(0; \varrho_N)$, the ball of radius $\varrho_N$ centered at $0$, and

$$\lim_{N \to \infty} \mu_N(S_{[0,N]} \subset B(x_N; \varrho_N + \varrho_1^\epsilon)) = 1.$$

(1.7)

The law of $\varrho_N^{-1} x_N$ converges to $\phi_{B(0;1)} dx$ as $N \to \infty$, where $\phi_{B(0;1)}$ is the $L^1$-normalized principal Dirichlet eigenfunction of $-\frac{1}{2d}\Delta$ in $B(0;1)$. Furthermore, for $d = 2$ and for any $\epsilon \in (0,1)$,

$$\lim_{N \to \infty} \mu_N(B(x_N; (1-\epsilon)\varrho_N) \subset S_{[0,N]}) = 1.$$

(1.8)

**Remark 1.1.** The above formulation of confinement is in fact far more precise than what Donsker–Varadhan’s argument suggests. Their argument is based on the large deviation principle for the empirical measure and thus it only indicates that the random walk spends most of the time in the ball. The interested reader can find a detailed explanation in [B02], Section 2.5.

It remains open to show that (1.8) also holds for dimensions $d \geq 3$, that the random walk range covers a full ball with radius almost $\varrho_N$ (see [B94], Conjecture 1.3)). Our first main result resolves this question.

**Theorem 1.2** (Ball covering). Let $d \geq 2$, and let $\varrho_N$ and $x_N$ be as in (1.6) and Theorem A, respectively. Then there exists $\epsilon_2 \in (0,1)$, such that

$$\lim_{N \to \infty} \mu_N(B(x_N; \varrho_N - \varrho_2^\epsilon) \subset S_{[0,N]}) = 1.$$

(1.9)

**Remark 1.3.** This theorem extends and refines (1.8) for general $d \geq 2$. In fact, we will first prove the extension of (1.8) to $d \geq 3$ as an intermediate step to the above refined result. The interested reader may jump to Section 4 after reading Subsections 2.1 and 2.2.
We proceed to the second main result of this paper, which is about the boundary of the range of the random walk under the annealed law. For any set $A \subseteq \mathbb{Z}^d$, we define its external boundary by
\[
\partial A := \{ y \in \mathbb{Z}^d \setminus A : \| y - x \| = 1 \text{ for some } x \in A \},
\]
where $\| \cdot \|$ denotes the Euclidean norm. Theorem A and Theorem B together imply that, conditioned on survival up to time $N$, the rescaled boundary of the random walk range, $\varrho_N^{-1} \partial S_{[0,N]}$, converges in probability to a unit sphere as $N$ tends to infinity, and $\partial S_{[0,N]}$ fluctuates on a scale of at most $\varrho_N^d$ with $\epsilon = \max\{\epsilon_1, \epsilon_2\} \in (0, 1)$. Identifying the precise scale of fluctuation is an extremely interesting, but also challenging question. The following theorem is a step in this direction, which bounds the size of $\partial S_{[0,N]}$.

**Theorem 1.4** (Boundary size). Let $d \geq 2$, and let $\varrho_N$ be defined as in (1.10). Then there exists $a > 0$, such that
\[
\lim_{N \to \infty} \mu_N(\|\partial S_{[0,N]}\| \leq \varrho_N^{d-1}(\log \varrho_N)^a) = 1.
\]

Let us mention a few more related works. There is a general framework containing our setting called the parabolic Anderson model, where the obstacles are replaced by independent and identically distributed random potential $\{\omega(x)\}_{x \in \mathbb{Z}^d}$. One is interested in what happens under the measures
\[
\mathbb{E} \otimes \mathbb{E} \left[ \exp \left\{ \sum_{k=1}^{N} \omega(S_k) \right\} : (S, \omega) \in \cdot \right] \quad \text{or} \quad \mathbb{E} \left[ \exp \left\{ \sum_{k=1}^{N} \omega(S_k) \right\} : S \in \cdot \right].
\]

Formally our model corresponds to the case where $\omega$ takes value 0 or $-\infty$. More generally if $\omega$ is non-positive, the above weighted measures can be interpreted as the law of random walk killed with probability $1 - e^{\omega(x)}$ when it visits $x$, conditioned to survive until $N$. Thus $\omega$ plays the role of repulsive impurities. On the other hand, positive $\omega$ corresponds to attractive impurities. There are various localization results depending on the distribution of $\omega$. See, for example, the recent monograph by König [K18] for an up-to-date review. The first measure is known as the annealed law, which is what we study in this paper, while the second measure in (1.12) conditions on the random potential and is called the quenched law.

In the case of Bernoulli obstacles, it has been proved recently in [DX17, DX18] that under the quenched law, with high probability, within $o(N)$ steps, the random walk reaches a ball of volume $\Theta(\log N)$ which is almost free of obstacles, and then stays close to that ball till time $N$. However, because the size of the ball is much smaller than in the annealed setting, it is unlikely that the random walk will be confined to the ball of localization once it is reached, and it will be an interesting problem to characterize the random walk range in the quenched setting. We will address this question elsewhere.

In what follows, we will use $c, c', C, C'$ to denote generic constants depending only on $d$ and $p$, whose values may change from line to line. For $G \subseteq \mathbb{R}^d$, we write $|G|$ for the number of points in $G \cap \mathbb{Z}^d$ and $\text{vol}(G)$ for the Euclidean volume.

## 2 Proof Outline

In this section, we list the main ingredients needed and outline the proof structure. An overview of how the rest of the paper is organized will be given at the end of the section. In what follows, many statements are supposed to hold with $\mu_N$-probability tending to one as $N$ tends to infinity, but we often make it implicit for brevity.
2.1 Path and Environment Switching

An argument that will be used repeatedly in our proof is path and environment switching. More precisely, if $A_1$, $A_2$ are two sets of random walk path configurations, and $E_1$, $E_2$ are two sets of obstacle configurations, then we can switch from $(A_1, E_1)$ to $(A_2, E_2)$ and bound

$$
\mu_N((S, \mathcal{O}) \in (A_1, E_1)) \leq \frac{\mathbb{P} \otimes \mathbb{P}(S \in A_1, \mathcal{O} \in E_1, \tau > N)}{\mathbb{P} \otimes \mathbb{P}(S \in A_2, \mathcal{O} \in E_2, \tau > N)}
$$

$$
= \frac{\mathbb{P}(\mathcal{O} \in E_1)}{\mathbb{P}(\mathcal{O} \in E_2)} \mathbb{E} \left[ \frac{\mathbb{P}(S \in A_1, \tau_{\mathcal{O}} > N) | \mathcal{O} \in E_1}{\mathbb{P}(S \in A_2, \tau_{\mathcal{O}} > N) | \mathcal{O} \in E_2} \right].
$$

The first factor determines the probability gain or cost in the environment when we switch from obstacle configurations in $E_1$ to $E_2$, while the second factor determines the gain or cost in the random walk when we switch from paths in $A_1$ to $A_2$. We will find suitable choices of $A_2$ and $E_2$ so that the gain in one factor will beat the cost in the other.

One way to bound the second factor in (2.1) is to find a coupling between two obstacle configurations $(\mathcal{O}_1, \mathcal{O}_2)$ with marginal distributions $\mathbb{P}(\cdot | \mathcal{O} \in E_1)$ and $\mathbb{P}(\cdot | \mathcal{O} \in E_2)$, and then bound $\mathbb{P}(S \in A_1, \tau_{\mathcal{O}_1} > N)/\mathbb{P}(S \in A_2, \tau_{\mathcal{O}_2} > N)$ uniformly with respect to $(\mathcal{O}_1, \mathcal{O}_2)$. This is possible because typically, $A_2$ and $E_2$ will be constructed by local modifications of paths in $A_1$ and obstacle configurations in $E_1$, respectively.

This type of comparison argument is much more useful in the study of the conditional measure $\mu_N$ than a direct analysis, since we only have the crude leading order asymptotics on the partition function in (2.1).

2.2 Proof Outline for the Weaker Version of Ball Covering Theorem

We first prove (2.1) for general $d \geq 2$, which will play an important role in the proof of Theorems 2.2 and 2.3. The key step is to show that if $x \in \mathcal{O}$, then there is a positive fraction of closed sites in its neighborhood.

**Lemma 2.1** (Density of obstacles). For each $x \in \mathbb{Z}^d$, $l > 0$, and $\delta > 0$, let

$$
E^\delta_l(x) := \left\{ x \in \mathcal{O} \text{ and } \frac{|\mathcal{O} \cap B(x; l)|}{|B(x; l)|} < \delta \right\}.
$$

Then there exists $\delta > 0$, such that

$$
\mu_N \left( \bigcup_{x \in B(0; 2g_N)} \bigcup_{(\log N)^2 \leq l \leq \delta N} E^\delta_l(x) \right) \to 0 \text{ as } N \to \infty
$$

faster than any negative power of $N$.

The proof of Lemma 2.1 will be based on path and environment switching arguments. Roughly speaking, if for some $x \in \mathcal{O}$, $B(x; l)$ contains few obstacles, then: either the walk visits $B(x; l)$ many times, in which case we remove all the obstacles in $B(x; l)$ and we will show that the gain in the random walk survival probability beats the loss from environment switching; or the walk visits $B(x; l)$ rarely, in which case we switch to typical obstacle configurations in $B(x; l)$ and force the walk to avoid $B(x; l)$, and we will show that the gain in environment switching beats the loss in path switching. A more precise outline and the proof will be given in Section 3.

Lemma 2.1 implies that if there is an obstacle inside the ball $B(x_N; (1 - \epsilon)g_N)$, then the confinement ball $B(x_N; g_N)$ contains order $g_N^d$ obstacles. This makes it too difficult for the random walk to survive and we can then deduce that the ball $B(x_N; (1 - \epsilon)g_N)$ is free of obstacles. More precisely, we have the following result.
Proposition 2.2. Let $d \geq 2$. Then for any $\epsilon > 0$, we have
\[
\lim_{N \to \infty} \mu_N \left( B(x_N; (1-\epsilon)g_N) \cap \mathcal{O} = \emptyset \right) = 1. \tag{2.4}
\]

Once we have this proposition, the covering property (1.8) readily follows. Indeed, if the random walk avoids a site $x \in B(x_N; (1-\epsilon)g_N)$ with positive probability uniformly in $N$, then we can close that site at little cost, which contradicts Proposition 2.2.

2.3 Reduction to the Cluster of “Truly”-Open Sites

The key idea in our proof of Theorems 1.2 and 1.4 is to approximate the range of the random walk, $S_{[0,N]}$, by a set of “truly”-open sites $\mathcal{T}$ that depends only on the obstacle configuration $\mathcal{O}$. Unlike sites in $S_{[0,N]}$, we can easily control the environment cost of creating a “truly”-open site, which facilitates the application of the switching argument in (2.1).

Definition 2.3 (“Truly”-open sites). Given an obstacle configuration $\mathcal{O}$ and $N \in \mathbb{N}$, a site $x \in \mathbb{Z}^d$ is called “truly”-open if
\[
P_x \left( \tau_{\mathcal{O}} > (\log N)^5 \right) \geq \exp \left\{ - (\log N)^2 \right\}. \tag{2.5}
\]
If the origin is “truly”-open, then we let $\mathcal{T}$ denote the largest connected component of “truly”-open sites inside $B(x_N; g_N + \epsilon_1 N)$ containing the origin, where $\epsilon_1$ is the constant appearing in Theorem A. Otherwise let $\mathcal{T} = \emptyset$.

Remark 2.4. A “truly”-open site is a site whose surrounding environment is atypically favorable for the random walk survival. If the environment is typical, then the probability in (2.5) would decay like $\exp\left\{ -c(\log N)^{5+o(1)} \right\}$ (cf. [SS8, Theorem 5.1 on p. 196]). Note that whether $x \in \mathbb{Z}^d$ is “truly”-open or not depends only on the obstacle configuration in the $l^1$-ball of radius $(\log N)^5$ centered at $x$.

The following two lemmas justifies the approximation of $\partial S_{[0,N]}$ by the boundary of “truly”-open sites $\partial \mathcal{T}$.

Lemma 2.5. Let $d \geq 2$. Then
\[
\lim_{N \to \infty} \mu_N \left( S_{[0,N]} \supset \left\{ x \in \mathcal{T} : \text{dist}(x, \partial \mathcal{T}) \geq (\log N)^5 \right\} \right) = 1. \tag{2.6}
\]
and
\[
\lim_{N \to \infty} \mu_N \left( \mathcal{T} \subset \left\{ x \in \mathbb{Z}^d : \text{dist}(x, S_{[0,N]}) \leq (\log N)^5 \right\} \right) = 1. \tag{2.7}
\]

Lemma 2.6. Let $d \geq 2$. Then
\[
\lim_{N \to \infty} \mu_N \left( S_{[0,N]} \subset \mathcal{T} \right) = 1. \tag{2.8}
\]
Indeed, (2.6) and (2.8) imply that
\[
\mu_N \left( \partial S_{[0,N]} \subset \bigcup_{x \in \partial \mathcal{T}} B(x; (\log N)^5) \right) \to 1 \tag{2.9}
\]
and therefore, Theorems 1.2 and 1.4 follow immediately from their analogues for $\mathcal{T}$.

Theorem 2.7. Let $d \geq 2$. Then there exists $\epsilon_2 \in (0, 1)$ such that
\[
\lim_{N \to \infty} \mu_N \left( B(x_N; g_N - \epsilon_2 N) \subset \mathcal{T} \right) = 1. \tag{2.10}
\]

Theorem 2.8. Let $d \geq 2$. Then there exists $a > 0$ such that
\[
\lim_{N \to \infty} \mu_N \left( |\partial \mathcal{T}| \leq \frac{d-1}{2} (\log g_N)^a \right) = 1. \tag{2.11}
\]
2.4 Proof Outline for the Cluster of “Truly”-Open Sites

In this subsection, we provide an outline for the proof of Theorems 2.4 and 2.5, assuming Lemmas 2.4 and 2.6.

Note that the random walk is confined in $T$ by Lemma 2.4. The geometry of $T$ under $\mu_N$ is then determined by an entropy-energy balance, namely, the number of possible configurations for $\partial T$, vs the probability that the random walk stays confined in $T$ up to time $N$ (equivalently, the principal Dirichlet eigenvalue for the discrete Laplacian on $T$). By definition, $T$ is contained in the confinement ball $B(x_N; \varrho_N + c_N^1)$ in Theorem 2.1. On the other hand, Proposition 2.5 implies that for any $\epsilon > 0$, $B(x_N; (1 - \epsilon)\varrho_N)$ is a ball of “truly”-open sites. Therefore, $\partial T$ is contained in the annulus
\begin{equation}
A(x_N; (1 - \epsilon)\varrho_N, \varrho_N + c_N^1) := B(x_N; \varrho_N + c_N^1) \setminus B(x_N; (1 - \epsilon)\varrho_N).
\end{equation}
We bound the entropy for $\partial T$ by proving the following weaker version of Theorem 2.5:

Proposition 2.9. Let $d \geq 2$. Then for any $b > 0$,
\begin{equation}
\lim_{N \to \infty} \mu_N(\partial T)^b \leq \varrho_N^{d-1+b} = 1.
\end{equation}
We prove Proposition 2.9 by considering the expected number of visits to $\bigcup_{x \in \partial T} B(x; (\log N)^6)$ by a random walk killed upon hitting $O$. It suffices to prove that
\begin{enumerate}
\item the expectation of the total number of visits to $\bigcup_{x \in \partial T} B(x; (\log N)^6)$ is bounded from above by $(\log N)^c$ for some $c > 0$;
\item uniformly in $x \in \partial T$, the expected number of visits to $B(x; (\log N)^6)$ is bounded from below by $N^{1-d+c}$ for any $\epsilon > 0$.
\end{enumerate}

Here we consider visits to $(\log N)^6$ neighborhood of $x \in \partial T$ because if the walk does not visit $B(x; (\log N)^6)$, then we can switch a “truly”-open site next to $x$ to be not “truly”-open by only modifying the obstacle configuration inside $B(x; (\log N)^6)$. The first item above follows from the fact that the random walk will typically be killed soon after visiting $x \in \partial T$. The second item is proved by the path and environment switching arguments. Roughly speaking, a site $x \in \partial T$ with atypically low expected number of visits is costly for the random walk to visit. Thanks to the confinement of $\partial T$ to the annulus in (2.12), $B(x; (\log N)^6)$ can be visited only by an excursion away from $B(x_N; (1 - \epsilon)\varrho_N)$ and we can gain a lot in the random walk probability by switching such excursions to those that stay inside $B(x_N; (1 - \epsilon)\varrho_N)$. This implies that the random walk does not visit the $(\log N)^6$ neighborhood of $x$. We can then gain further in the environment probability by switching $x$ to be not “truly”-open, which shows that such $x \in \partial T$ does not exist.

Proposition 2.9 provides a good enough bound on the entropy for $\partial T$ to allow us to strengthen the bound on the fluctuation of $\partial T$ in (2.6) to Theorem 2.1. More precisely, if $T^c$ contains a point in $B(x_N; \varrho_N + c_N^2)$, then Lemma 2.4 implies that $T$ differs significantly in volume from the confinement ball $B(x_N; \varrho_N + c_N^1)$. Recalling that $T \subset B(x_N; \varrho_N + c_N^1)$ by definition, we can then use the Faber–Krahn inequality to show that the principal Dirichlet eigenvalue on $T$ deviates so much from that of $B(x_N; \varrho_N + c_N^1)$ that the loss in survival probability dominates the entropy for $\partial T$.

Using Theorem 2.7 on the fluctuation of $\partial T$ as an input in place of the weaker Proposition 2.9, we can then repeat the proof of Proposition 2.9 and strengthen it to Theorem 2.1.
Organization of the paper. The rest of the paper is organized as follows. Section 3 is devoted to the proofs of Proposition 2.2 and (1.8) for general \( d \geq 2 \). In Section 4, we first prove Lemma 2.3 with an additional property for “truly”-open sites, and then prove Lemma 2.4 and derive Proposition 2.9 from Proposition 2.2. We will in fact formulate a lemma (Lemma 3.3) which unifies the derivation of Proposition 2.9 from Proposition 2.2 and the derivation of Theorem 2.8 from Theorem 2.7. Lastly, in Section 5, we conclude with the proof of Theorem 2.7. In the appendix, we prove some technical estimates on the Dirichlet eigenvalues and eigenfunctions for the generator of the random walk, as well as a lower bound on the survival probability slightly better than in (1.3).

3 Proof of the Weaker Version of Ball Covering Theorem

3.1 Proof of Proposition 2.2 and the Extension of (1.8)

In this subsection, we prove Proposition 2.2 and then (1.8) for general \( d \geq 2 \), assuming Lemma 2.1, which says that under the conditioned law \( \mathbb{N}(\cdot) \), obstacles cannot be too isolated.

We need another lemma which states that the size of the random walk range \( S_{[0,N]} \) satisfies a weak law of large numbers under \( \mathbb{N} \):

Lemma 3.1 (Size of random walk range). For all \( \epsilon > 0 \), we have

\[
\mu_N \left( \left| \frac{|S_{[0,N]}|}{|B(0, \rho_N)|} - 1 \right| > \epsilon \right) \to 0 \text{ as } N \to \infty
\]

faster than any negative power of \( N \).

Proof of Lemma 3.1. For a Brownian motion among Poissonian obstacles, the corresponding result is proved in [F08]. It is straightforward to adapt the argument there to the current discrete setting. Indeed, the key point of the argument therein was the following formula for the generating function

\[
\int \exp \{ \xi |S_{[0,N]}| \} \, d\mu_N = \frac{\mathbf{E} \left[ \exp \{ (\xi - \log(1/p)) |S_{[0,N]}| \} \right]}{\mathbf{E} \left[ \exp \{ -\log(1/p) |S_{[0,N]}| \} \right]},
\]

(3.2)

One can use (1.8) to derive the asymptotics of this for \( |\xi| < \log(1/p) \) and then (3.1) follows by standard exponential Chebyshev bounds.

Proof of Proposition 2.2. Thanks to Theorem A and Lemma 2.1, we may assume that \( S_{[0,N]} \subseteq B(x_N; \rho_N + \rho_N^\delta) \), and for any \( x \in O \cap B(0; 2\rho_N) \),

\[
\frac{|O \cap B(x; \epsilon \rho_N)|}{|B(x; \epsilon \rho_N)|} \geq \delta.
\]

(3.3)

Suppose that there is a point \( x \in B(x_N; (1-\epsilon)\rho_N) \cap O \). Then by (3.3), at least \( \delta \) fraction of sites in \( B(x; \epsilon \rho_N) \) are closed and hence are outside \( S_{[0,N]} \). Combined with \( S_{[0,N]} \subseteq B(x_N; \rho_N + \rho_N^\delta) \), this implies that the ratio \( |S_{[0,N]}|/|B(0; \rho_N)| \) stays strictly less than one, which has \( \mu_N \)-probability tending to zero as \( N \to \infty \) by Lemma 2.1.

Proof of (1.8) for general \( d \geq 2 \). We derive (1.8) as a consequence of the following lemma, which asserts that the random walk visits all \( x \) in the confinement ball such that \( B(x; (\log N)^3) \) is free of obstacles.
Lemma 3.2.

\[
\lim_{N \to \infty} \mu_N \left( \bigcup_{x \in B(0;2\varrho_N)} \{ \tau_x > N, \mathcal{O} \cap B(x; (\log N)^3) = \emptyset \} \right) = 0. \tag{3.4}
\]

Since we know from Proposition 2.2 that for any \(\epsilon > 0\), the ball \(B(x_N; (1 - \epsilon/2)\varrho_N)\) is free of obstacles with \(\mu_N\)-probability tending to one, Lemma 3.2 implies that

\[
\lim_{N \to \infty} \mu_N (B(x_N; (1 - \epsilon)\varrho_N) \subset S_{[0,N]}) = 1. \tag{3.5}
\]

Proof of Lemma 3.2. By the union bound,

\[
\mu_N \left( \bigcup_{x \in B(0;2\varrho_N)} \{ \tau_x > N, \mathcal{O} \cap B(x; (\log N)^3) = \emptyset \} \right) \\
\leq \sum_{x \in B(0;2\varrho_N)} \mu_N (\tau_x > N, \mathcal{O} \cap B(x; (\log N)^3) = \emptyset). \tag{3.6}
\]

Thus it suffices show that for any \(x \in B(0;2\varrho_N),\)

\[
\mu_N (\tau_x > N, \mathcal{O} \cap B(x; (\log N)^3) = \emptyset) \to 0 \text{ as } N \to \infty \tag{3.7}
\]

faster than any negative power of \(N\).

To this end, observe that the probability \(P(\tau_x \land \tau_{\mathcal{O}} > N)\) is independent of whether \(x \in \mathcal{O}\) or not. Therefore, we have

\[
\mu_N (\tau_x > N, \mathcal{O} \cap B(x; (\log N)^3) = \emptyset) = \frac{p}{1-p} \mu_N (\mathcal{O} \cap B(x; (\log N)^3) = \{x\}), \tag{3.8}
\]

where we have switched the environment at \(x\) conditionally on the random walk event \(\{\tau_x > N\}\). Lemma 2.1 then shows that the right-hand side decays faster than any negative power of \(N\).

The rest of this section is devoted to the proof of Lemma 2.1.

3.2 Proof Outline for Lemma 2.1

The proof of Lemma 2.1 is based on the environment and path switching argument in (2.1). The rough (and not fully accurate) heuristic is as follows. Suppose that the event \(E_\delta^\beta(x)\) occurs, i.e., \(x \in \mathcal{O}\) and \(|\mathcal{O} \cap B(x; l)|/|B(x; l)| < \delta\) for some \(\delta\) much smaller than the typical obstacle density \(1 - p\). Either the random walk spends a lot of time in \(B(x; l)\), in which case we remove all obstacles in \(B(x; l)\), and we expect the gain in the random walk survival probability to beat the environment cost of removing the obstacles; or the random walk spends little time in \(B(x; l)\), in which case we force the random walk not to visit \(B(x; l/4)\), change to typical obstacle configurations in \(B(x; l/4)\) and remove all obstacles in \(B(x; l) \setminus B(x; l/4)\), and we expect the probability gain in the environment to beat the cost in changing the random walk. To make this heuristic rigorous, we need the following ingredients.

Path decomposition. First, we decompose the random walk path \((S_n)_{0 \leq n \leq N}\) into successive crossings between the inner and outer shells of the annulus

\[
A(x; l/2, l) := B(x; l) \setminus B(x; l/2). \tag{3.9}
\]
More precisely, we define \( \bar{B}(x; l/2) := B(x; l/2) \cup \partial B(x; l/2) \) and stopping times
\[
\sigma_1 := \min\{n \geq 0: S_n \in \overline{B(x; l/2)}\} \wedge N, \tag{3.10}
\]
and for \( k \in \mathbb{N} \),
\[
\tau_k := \min\{n > \sigma_k: S_n \in B^c(x, l)\} \wedge N; \tag{3.11}
\]
\[
\sigma_{k+1} := \min\{n > \tau_k: S_n \in \overline{B(x; l/2)}\} \wedge N. \tag{3.12}
\]

We will perform path switching on the crossings from \( \overline{B(x; l/2)} \) to \( B(x; l)^c \), and perform environment switching in the ball \( B(x; l) \).

**Skeletal approximation of \( \mathcal{O} \cap B(x; l) \).** When the random walk spends a lot of time in \( B(x; l) \), we will remove all the obstacles in \( B(x; l) \). We need to estimate the gain in the random walk survival probability as a function of \( \mathcal{O} \cap B(x; l) \). This is not a very simple task and one of the difficulties is that the contribution from each obstacle is highly non-uniform, depending on others. Indeed, if an obstacle is well surrounded by others, then we gain little in the survival probability by removing it. In order to make the gain from an obstacle independent from others, we will only count the gain from a skeletal set \( \mathcal{X}_l(x) \subset \mathcal{O} \) with properties

\[
x \in \mathcal{X}_l(x) \text{ and all sites in } \mathcal{X}_l(x) \text{ are at least distance } l^{1/2d} \text{ away from each other; } \tag{3.13}
\]

\[
each y \in \mathcal{O} \cap B(x; l) \text{ is within distance } l^{1/2d} \text{ of some site in } \mathcal{X}_l(x). \tag{3.14}
\]

Such a set can be constructed iteratively. First include \( x \in \mathcal{X}_l(x) \) and remove all obstacles in \( B(x; l^{1/2d}) \). Next pick any one of the remaining obstacles at \( y \in B(x; l) \) that is closest to \( x \) and add it to \( \mathcal{X}_l(x) \), and remove all obstacles in \( B(y; l^{1/2d}) \). Repeat this procedure until no obstacles remain in \( B(x; l) \). Another difficulty is that we gain little in the random walk survival probability by removing obstacles in \( B(x; l) \) near \( \partial B(x; l) \) since we will only count the gain from the crossings \( \{S_{[\sigma_k, \tau_k]}\}_{k \in \mathbb{N}} \), that typically spend little time near \( \partial B(x; l) \). Therefore we focus on the obstacles deeply inside \( B(x; l) \) by setting
\[
\mathcal{X}_l^\circ := \mathcal{X}_l^\circ(x) := \mathcal{X}_l(x) \cap \overline{B(x; l/2)}. \tag{3.15}
\]

**Random walk estimates.** For \( D \subset \mathbb{Z}^d \), \( u \in D \) and \( v \in D \cup \partial D \), we denote
\[
p_n^D(u, v) := P_u(S_n = v, S_{[1,n-1]} \subset D). \tag{3.16}
\]
This is nothing but the discrete space-time Dirichlet heat kernel on \( D \) if \( v \in D \) and, while it is not for \( v \in \partial D \), it always satisfies the discrete heat equation in \( (n, u) \).

**Remark 3.3.** Since the symmetric simple random walk has period 2, we have \( p_n^D(u, v) = 0 \) when \( n + |u - v|_1 \) is an odd number. In what follows, we adopt a convention that \( p_n^D(u, v) \) is understood to be \( p_{n+1}^D(u, v) \) if \( n + |u - v|_1 \) is odd.

Now we are ready to state the gain in the random walk survival probability when we remove obstacles: for \( c_0, c_1 > 0 \) to be determined later in Lemma 3.4, uniformly in \( m \geq (c_0 l)^2 \), \( u \in B(x; l/2) \) and \( v \in \overline{B(x; l)} \),
\[
p_m^{B(x;l)\setminus \mathcal{O}}(u, v) \leq e^{-c_1([m/(c_0 l)^2] - 1)\Gamma(|\mathcal{X}_l^\circ|)} p_m^{B(x;l)}(u, v), \tag{RW1}
\]
where for \( k \in \mathbb{N} \),
\[
\Gamma(k) := \begin{cases} 
\left( \log \left( \frac{(c_0 l)^{3/2}}{2k} \right) \right)^{-1} & \forall 0, \ d = 2, \\
\frac{(d-2k)}{1 + (d-2k)^2/2d} & d \geq 3.
\end{cases} \tag{3.17}
\]
Roughly speaking, this estimate means that if the random walk stays in \( B(x; l) \), then in every \((c_0l)^2\) steps, it has more than \( c_1 \Gamma(|\mathcal{X}_l|) \) probability of hitting an obstacle (see Lemma 3.7). As mentioned at the beginning of this subsection, we force the crossing to avoid \( B(x; l/4) \) when the random walk spends little time in \( B(x; l) \). We need another random walk estimate to quantify the effect of this switching and also some others to deal with complementary cases. Those random walk estimates will be tagged as (RW1)–(RW5) and restated and proved in Lemma 3.4 in Subsection 3.4.

**Obstacles deep in the interior of** \( B(x; l) \). Note that in (RW1), the bound is in terms of \( |\mathcal{X}^o_l(x)| \), the number of skeletal points of \( \mathcal{O} \cap B(x; l) \) in \( B(x; l/2) \). Therefore we need to look for balls \( B(x; l) \) with sufficiently many obstacles deep in its interior, namely, \( |\mathcal{X}^o_l(x)| \geq \rho |\mathcal{X}_l(x)| \) for some \( \rho > 0 \). The next lemma guarantees the existence of such balls.

**Lemma 3.4.** Suppose that \( x \in \mathcal{O} \) and \( |\mathcal{O} \cap B(x; L)|/|B(x; L)| < \delta \) for some \( L \in \mathbb{N} \). Then we can find \( \rho > 0 \) independent of \( L \) and \( \delta \), such that there exists \( \sqrt{L} \leq l \leq L \) with \( |\mathcal{O} \cap B(x; l)|/|B(x; l)| < \delta \) and \( |\mathcal{X}^o_l(x)| \geq \rho \min\{|\mathcal{X}_l(x)|, \delta l^{d-1/2}\} \).

Therefore it suffices to prove Lemma 3.4 where we replace the event \( E^o_l(x) \) by the event
\[
E^{\delta, \rho}_l(x) := E^o_l(x) \cap \{|\mathcal{X}^o_l(x)| \geq \rho \min\{|\mathcal{X}_l(x)|, \delta l^{d-1/2}\} \},
\]
which will be carried out in the next subsection.

**Proof of Lemma 3.4.** Let \( j^* := \min \{ j \geq 0 : l = L/2^j \text{ satisfies } |\mathcal{X}^o_l(x)| \geq \rho \min\{|\mathcal{X}_l(x)|, \delta l^{d-1/2}\} \}. \tag{3.19} \)

We will show that \( l^* := L/2^{j^*} \) satisfies the desired properties if \( \rho > 0 \) is small enough.

If \( j^* = 0 \), then \( l^* = L \) works. Otherwise, for all \( l = L/2^j \) with \( 0 \leq j \leq j^* - 1 \), we have
\[
|\mathcal{X}^o_l(x)| < \rho \min\{|\mathcal{X}_l(x)|, \delta l^{d-1/2}\} \leq \rho |\mathcal{X}_l(x)|. \tag{3.20} \]

We claim that for all \( l \geq 1 \),
\[
|\mathcal{X}_l(x)| \leq C_d |\mathcal{X}^o_{2l}(x)| \text{ for some } C_d \text{ depending only on } d. \tag{3.21} \]
Together with (3.20), this implies that
\[
|\mathcal{X}^o_{2l}(x)| < \rho C_d |\mathcal{X}^o_l(x)| < \cdots < (\rho C_d)^{-1} |\mathcal{X}^o_{L}(x)| \leq (\rho C_d)^{-1} C_L^d. \tag{3.22} \]
Since \( |\mathcal{X}^o_l(x)|, \ldots, |\mathcal{X}^o_{2^j}(x)| \geq 1 \) because \( x \in \mathcal{O} \), we must have \( (\rho C_d)^{-1} \leq C_L^d \). We can then choose \( \rho > 0 \) small such that \( 2^{j^*} \leq \sqrt{L} \), and hence \( l^* = L/2^{j^*} \in [\sqrt{L}, L] \).

To bound the volume fraction of obstacles in \( B(x; l^*) \) when \( j^* \geq 1 \), we can apply (3.20) at \( l = L/2^{j^* - 1} = 2l^* \) to obtain
\[
|\mathcal{O} \cap B(x; l^*)| \leq |\mathcal{X}^o_{2l^*}(x)| \cdot |B(0; (2l^*)^{1/2d})| \leq \rho \delta (2l^*)^{d-1/2} |B(0; (2l^*)^{1/2d})| \leq \delta |B(x; l^*)|, \tag{3.23} \]
where the last inequality holds if \( \rho > 0 \) is chosen small.

Lastly, we prove the claim (3.21). Note that by our construction of \( \mathcal{X}_l(x) \) and \( \mathcal{X}^o_{2l}(x) \),
\[
\bigcup_{y \in \mathcal{X}_l(x)} B(y; l^{1/2d}) \subset \bigcup_{y \in \mathcal{O} \cap B(x; l)} B(y; l^{1/2d}),
\]
where \( \mathcal{O} \cap B(x; l) \subset \bigcup_{z \in \mathcal{X}^o_{2l}(x)} B(z; (2l)^{1/2d}). \tag{3.24} \]
Therefore by doubling the radii of the balls \( \{B(z;(2l)^{1/2d})\}_{z \in X_2^c(x)} \), we obtain

\[
\bigcup_{y \in X_l(x)} B(y;l^{1/2d}) \subset \bigcup_{z \in X_2^c(x)} B(z;2(2l)^{1/2d}).
\]

Since the balls \( \{B(y;l^{1/2d})\}_{y \in X_l(x)} \) are disjoint, a volume calculation then yields \((3.21)\).  

### 3.3 Proof of Lemma 3.4

As remarked after Lemma 3.4, if suffices to prove Lemma 3.4 with the event \( E_l^d(x) \) replaced by \( E_l^{d,p}(x) \). We will prove the following bound on \( \mu_N(E_l^{d,p}(x)) \), which immediately implies Lemma 3.4 by a union bound over all \( x \in B(0;2g_N) \) and \((\log N)^3 \leq l \leq g_N\).

**Lemma 3.5.** Let \( E_l^{d,p}(x) \) be defined as in \((3.18)\). There exist \( c_3 > 0 \) depending only on \( d \) and \( p \) such that for all \( l \in [\log N, g_N] \), we have

\[
\mu_N(E_l^{d,p}(x)) \leq \exp\{-c_3 l^{1/2}\}. \tag{3.25}
\]

**Proof of Lemma 3.5.** Recall the path decomposition introduced in Section 3.2, where we identified the successive crossings from \( B(x;l/2) = B(x;l/2) \cup \partial B(x;l/2) \) to \( B(x;l)^c \) during the time intervals \([\sigma_k, \tau_k], k \in \mathbb{N}\). Since these stopping times are truncated by \( N \), the duration \( \tau_k - \sigma_k \) can be zero. Henceforth, the word crossing refers to \( S_{[\sigma_k, \tau_k]} \) with \( \tau_k - \sigma_k > 0 \). In particular, the last crossing may be incomplete. To carry out the path and environment switching, we distinguish between three cases and in order to describe them, we need some more notation. We denote a sequence of numbers or vectors in bold face as \( a = (a_k)_{k \geq 1} \) and introduce the set of interlacing sequences

\[
I_N := \{(s,t): 0 \leq s_k \leq t_k \leq s_{k+1} \leq N \text{ for all } k \in \mathbb{N}\}. \tag{3.26}
\]

For \((s,t) \in \bigcup_{N \geq 1} I_N\), we write

\[
K(s,t) := \sup\{k \geq 1: t_k - s_k > 0\} \tag{3.27}
\]

which represents the number of crossings when \((s,t) = (\sigma, \tau)\). Now we are ready to describe the three cases. Recall that \( c_0 > 0 \) has already been chosen to satisfy Lemma 3.4. The constant \( \delta > 0 \) is to be determined later, depending only on the dimension \( d \) and the open probability \( p \).

1. There are many crossings and more than half of them are short \((\leq (c_0 l)^2)\), that is, \((\sigma, \tau)\) belongs to

\[
F_1 := \left\{(s,t) \in I_N: \left\{\frac{k \geq 1: 0 < t_k - s_k \leq (c_0 l)^2\} > \frac{1}{2} K(s,t) \vee \delta^{1/d} l^d\right\} \right\}. \tag{3.28}
\]

2. The total time duration of the long crossings \((> (c_0 l)^2)\) is long, that is, \((\sigma, \tau)\) belongs to

\[
F_2 := \left\{(s,t) \in I_N: \sum_{k \geq 1} (t_k - s_k) 1_{\{t_k - s_k > (c_0 l)^2\}} > \delta^{1/d} c_0^2 l^{d+2}\right\}. \tag{3.29}
\]
(3) The number of crossings as well as their total duration are small, that is, \((\sigma, \tau)\) belongs to
\[
F_3 := \left\{ (s, t) \in I_N : K(s, t) \leq 2\delta^{1/d} l^d \quad \text{and} \quad \sum_{k \geq 1} (t_k - s_k) \leq 2\delta^{1/d} c_0^2 l^{d+2} \right\}. \quad (3.30)
\]

These three cases exhaust all possibilities. Indeed, if \((s, t) \notin F_2\), then the number of long crossings is at most \(\delta^{1/d} l^d\), and their total duration is at most \(\delta^{1/d} c_0^2 l^{d+2}\). If in addition, \((s, t) \notin F_1\), then the number of short crossings is either less than the number of long crossings, or less than \(\delta^{1/d} l^d\); either way, it is bounded by \(\delta^{1/d} l^d\), and their total duration is at most \(\delta^{1/d} c_0^2 l^{d+2}\). Combining the short and long crossings, one finds that \((s, t) \in F_3\).

For each of the three cases above, by summing over all possible values of \((\sigma_k, \tau_k) \in F_i \quad (i \in \{1, 2, 3\})\) and the position of the walk at these times, we obtain
\[
\mathbb{P}(\tau_0 > N \quad \text{and} \quad (\sigma, \tau) \in F_1) = \sum_{(s, t) \in F_1} \sum_{u, v} \mathbb{P}_s^{B(x:l) \setminus O}(0, u_1) P_{s_1}^{B(x:l) \setminus O}(u_k, v_k) \prod_{k \geq 1} P_{s_k}^{B(x:l) \setminus O}(u_k, v_k) P_{s_k+1}^{B(x:l) \setminus O}(v_k, u_{k+1}), \quad (3.31)
\]

where \(u\) and \(v\) range over all the possible starting and ending points of crossings with \((\sigma, \tau) = (s, t)\). In particular, \(u_k \in \partial B(x;l/2)\) and \(v_k \in \partial B(x;l)\) as long as \(s_k < N\) and \(t_k < N\) respectively, except possibly \(u_1 = 0\) when \(0 \in B(x;l/2)\). For simplicity, we assume \(\delta^{1/d} l^d \in \mathbb{N}\) in this proof.

Case (1): In this case, we remove all the obstacles inside \(B(x;l)\) and lengthen all the short crossings by \(l^2\). We formalize this as the environment and path switching \((2, 1)\) by setting
\[
(A_1, A_2) := \left\{ (\sigma, \tau) \in F_1 \right\}; \quad (\tau_0 > N) \right\}; \quad (3.32)
\]
\[
(E_1, E_2) := (E_t^{\delta, \rho}(x), \{O \cap B(x;l) = \emptyset\}). \quad (3.33)
\]

Since \(|O \cap B(x;l)| \leq \delta |B(x;l)|\) on the event \(E_t^{\delta, \rho}(x)\), the cost of environment switching can be estimated as
\[
\frac{\mathbb{P}(E_t^{\delta, \rho})}{\mathbb{P}(O \cap B(x;l) = \emptyset)} \leq \frac{\mathbb{P}(|O \cap B(x;l)| \leq \delta |B(x;l)|)}{\mathbb{P}(O \cap B(x;l) = \emptyset)} \leq \frac{\delta |B(x;l)|}{i} \left( \frac{1 - p}{p} \right)^j \leq e^{c_3 (\log \frac{1}{p}) l^d} \quad (3.34)
\]

by using Stirling’s approximation. Alternatively, one can also interpret this as a consequence of Cramer’s large deviation principle.

On the other hand, since the short crossings are unlikely to happen, we gain in the random walk probability by lengthening them. More precisely, we will see in Lemma 4.3 that for \(t_k - s_k \leq (c_0 l)^2\), \(u_k \in B(x;l/2)\) and \(v_k \in \partial B(x;l)\),
\[
\frac{B(x;l) \setminus O}{P_{t_k-s_k}(u_k, v_k)} \leq \frac{B(x;l)}{P_{t_k-s_k}(u_k, v_k)} \leq \frac{1}{100} \quad (\text{RW2})
\]
where \( l^2 \) is to be understood as \( l^2 + 1 \) when \( l \) is odd as mentioned in Remark 3.3. It is easy to see that this change is harmless for the following argument and henceforth we will not mention this parity convention again. Setting

\[
I_k := 1_{\{0 < t_k - s_k \leq (\epsilon d)^2, u_k \in \partial B(x;l/2), v_k \in \partial B(x;l)\}},
\]

we can bound the product in the right-hand side of (3.31) by

\[
\prod_{k \geq 1} p_{l_k - s_k}^{B(x;l) \setminus \mathcal{O}}(u_k, v_k) p_{s_{k+1} - t_k}^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1})
\]

\[
\leq 100 \sum_{k \geq 1} I_k \prod_{k \geq 1} p_{l_k + \ell k - s_k}^{B(x;l)}(u_k, v_k) p_{s_{k+1} - t_k}^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1})
\]

\[
= 100 \sum_{k \geq 1} I_k \prod_{k \geq 1} p_{l_k - \tilde{s}_k}^{B(x;l)}(u_k, v_k) p_{s_{k+1} - \tilde{t}_k}^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1}),
\]

where \( \tilde{s}_k := s_k + l^2 \sum_{m < k} I_m \) and \( \tilde{t}_k := t_k + l^2 \sum_{m \leq k} I_m \). Let us consider the cases

\[
(\sigma, \tau) \in F_{1,j} := \left\{ (s, t) \in F_1 : \sum_{k \geq 1} I_k = j \right\}
\]

that is, exactly \( j \) crossings are lengthened, separately for \( j \in \{\delta^1/d^d, \delta^1/d^d + 1, \ldots, N\} \).

Summing (3.38) multiplied by \( p_{s_1}^{B(x;l)}(0, u_1) \) over \((s, t) \in F_{1,j} \) and \((u, v) \), we obtain

\[
\mathbf{P}(\tau_\mathcal{O} > N \text{ and } (\sigma, \tau) \in F_{1,j})
\]

\[
= \sum_{(s, t) \in F_{1,j}} \sum_{u, v} p_{s_1}^{B(x;l)}(0, u_1) \prod_{k \geq 1} p_{l_k - \tilde{s}_k}^{B(x;l)}(u_k, v_k) p_{s_{k+1} - \tilde{t}_k}^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1})
\]

\[
\leq 100^{-j} \sum_{(s, t) \in F_{1,j}} \sum_{u, v} p_{s_1}^{B(x;l)}(0, u_1) \prod_{k \geq 1} p_{l_k - \tilde{s}_k}^{B(x;l)}(u_k, v_k) p_{s_{k+1} - \tilde{t}_k}^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1}).
\]

In order to relate this last line to \( \mathbf{P}(\tau_\mathcal{O} > N) \), we rewrite (3.38) as a summation over

\[
(\tilde{s}, \tilde{t}) \in \tilde{F}_{1,j} := \{(\tilde{s}, \tilde{t}) : (s, t) \in F_{1,j}\}.
\]

Note that each \((\tilde{s}, \tilde{t})\) may come from different \((s, t)\)'s but with the same number of crossings \(K(s, t)\) and hence its pre-image has cardinality at most \(2^{K(s,t)} \leq 2^{2j+2} \) on \( F_{1,j} \). Recalling also that \( j \geq \delta^1/d^d \), it follows from (3.38) that

\[
\mathbf{P}(\tau_\mathcal{O} > N \text{ and } (\sigma, \tau) \in F_{1,j})
\]

\[
\leq 4 \cdot 25^{-j/d^d} \sum_{(\tilde{s}, \tilde{t}) \in \tilde{F}_{1,j}} \sum_{u, v} p_{s_1}^{B(x;l)}(0, u_1)
\]

\[
\times \prod_{k \geq 1} p_{l_k - \tilde{s}_k}^{B(x;l)}(u_k, v_k) p_{s_{k+1} - \tilde{t}_k}^{\mathbb{Z}^d \setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1}).
\]

The sum on the right-hand side is seen to be bounded by \( \mathbf{P}(\tau_{\mathcal{O} \setminus B(x;l)} > N + jl^2) \). Indeed, any \((\tilde{s}, \tilde{t}) \in \tilde{F}_{1,j}\) has terminal time \( \tilde{s}_{K(\tilde{s}, \tilde{t})+1} = N + jl^2 \) by construction and hence the above sum represents (a part of) the path decomposition before time \( N + jl^2 \).
Taking a sum of (3.40) over $j$, we obtain
\[
\mathbf{P}(\tau_0 > N \text{ and } (\sigma, \tau) \in F_1) = 4 \cdot 25^{-\delta^1/d} \sum_{j=\delta^1/d}^{N} \mathbf{P}(\tau_0 \setminus B(x;l) > N + jl^2) \tag{3.41}
\]
\[
\leq 4N \cdot 25^{-\delta^1/d} \mathbf{P}(\tau_0 \setminus B(x;l) > N).
\]
Since $\tau_0 \setminus B(x;l) = \tau_0$ on $\{O \cap B(x;l) = \emptyset\}$, recalling (3.33) and $l \geq \log N$ and choosing $\delta > 0$ small, we can use (3.34) to conclude
\[
\mu_N \left( (S, O) \in \left( \{(\sigma, \tau) \in F_1 \}, E^{1,\rho}_l(x) \right) \right) \leq e^{-c\delta^1/d}.
\tag{3.42}
\]

Case (2): In this case, we again remove all the obstacles in $B(x;l)$ and leave the crossings unchanged. We apply the same environment and path switching as in the previous case. Since the long crossings have higher probability of hitting the obstacles, removing the obstacles gives us a large gain in the random walk probability. In order to make it precise, note first that $(s, t) \in F_2$ implies
\[
\sum_{k \geq 1} \left| \frac{t_k - s_k}{(c_0 l)^2} \right| \geq \frac{1}{2} \sum_{k \geq 1} \frac{t_k - s_k}{(c_0 l)^2} 1\{t_k - s_k \geq (c_0 l)^2\} > \frac{\delta^1/d}{2} l^d.
\tag{3.43}
\]
Given this, we use the aforementioned (see also Lemma 3.4)
\[
P_{B(x;l)}^{B(x;l) \setminus O}(u_k, v_k) \leq e^{-c_1(|(t_k - s_k)/(c_0 l)^2| - 1)\Gamma(|X^\omega_t|)} P_{B(x;l)}^{B(x;l) \setminus O}(u_k, v_k)
\tag{RW1}
\]
repeatedly to obtain
\[
\mathbf{P}(\tau_0 > N \text{ and } (\sigma, \tau) \in F_2) = \sum_{(s, t) \in F_2} \sum_{u, v} P_{B(x;l)}^{B(x;l)}(0, u_1) \prod_{k \geq 1} P_{B(x;l)}^{B(x;l) \setminus O}(u_k, v_k) P_{s_k-1}^{B(x;l)}(u_k, v_k) P_{s_k+1-1}^{Z^d \setminus (O \cup B(x;l/2))}(v_k, u_k+1)
\leq e^{-c\delta^1/d \Gamma(|X^\omega_t|)} \sum_{(s, t) \in F_2} \sum_{u, v} P_{B(x;l)}^{B(x;l)}(0, u_1) \prod_{k \geq 1} P_{B(x;l)}^{B(x;l) \setminus O}(u_k, v_k) P_{s_k-1}^{B(x;l)}(u_k, v_k) P_{s_k+1-1}^{Z^d \setminus (O \cup B(x;l/2))}(v_k, u_k+1)
\leq e^{-c\delta^1/d \Gamma(|X^\omega_t|)} \mathbf{P}(\tau_0 \setminus B(x;l) > N)
\tag{3.44}
\]
uniformly in $O$, and we can replace $\tau_0 \setminus B(x;l)$ by $\tau_0$ on $\{O \cap B(x;l) = \emptyset\}$ as before.

We are going to show that the cost of removing the obstacles in $B(x;l)$ is much smaller than the above gain in the random walk probability. Recall that $|X^\omega_t| \geq \rho \min(\delta^{d-1/2}, |X|)$ on the event $E^{1,\rho}_l(x)$ and also note that (3.13) implies the bound $|X^\omega_t| \leq C\delta^{d-1/2}$ for some $C > 0$ depending only on the dimension. In the case $|X^\omega_t| \in [\rho \delta^{d-1/2}, C\delta^{d-1/2}]$, we have that $\Gamma(|X^\omega_t|)$ is bounded below by a positive constant, recalling the definition of $\Gamma$ in (3.17). Combining (3.44) with (3.43) and (3.41) and choosing $\delta$ small, we get
\[
\mu_N \left( (S, O) \in \left( \{(\sigma, \tau) \in F_2 \}, (E^{1,\rho}_l(x) \cap \{|X^\omega_t| \geq \rho \delta^{d-1/2}\}) \right) \right) \leq e^{-c\delta^1/d}.
\tag{3.45}
\]
In the other case $|X^\omega_t| \in [\rho |X|, \rho \delta^{d-1/2})$, instead, we have that for sufficiently small $\delta$,
\[
\Gamma(|X^\omega_t|) \geq c\delta^{-2/d} \mu^{1/2 - d/2} |X^\omega_t| / \log\delta
\tag{3.46}
\]
recalling (3.17) again. Indeed, for \( d \geq 3 \), using \(|X_i^{o}| \leq \rho \delta l^{d-1/2}\) in the denominator in (3.17) yields the above bound without \(|\log \delta|\); for \( d = 2 \), the argument of \( \log \) in (3.17) is large and the above bound follows from the fact that \( 1/ \log r = r^{1/2}(r/ \log r) \) and \( r/ \log r \) is increasing for \( r \) large. Given this lower bound on \( \Gamma(|X_i^{o}|) \), the gain from the random walk becomes

\[
\frac{\mathbb{P}(\tau_\mathcal{O} > N \text{ and } (\boldsymbol{\sigma}, \tau) \in \mathcal{F}_2)}{\mathbb{P}(\tau_\mathcal{O} \setminus B(x;l) > N)} \leq e^{-c_\delta - 1/d \delta l^{1/2}|X_i^{o}|/|\log \delta|}. \tag{3.47}
\]

Note that this gain is much smaller than the bound (3.34) on the cost of environment switching when \(|X_i^{o}| \) is small. Therefore we have to estimate the environment switching cost more carefully and this is done by considering separately the events \( \{|X_i^{o}| = k\} \) for \( k < \rho \delta l^{d-1/2} \).

In the case under consideration, \(|X_i^{o}| = k \) implies \(|X_i| \leq \rho^{-1}k \). Recall also that all the obstacles in \( B(x;l) \) are contained in \( \bigcup_{x \in X_i} B(x;l^{1/2d}) \) by (3.14). Therefore on each event \( \{|X_i^{o}| = k\} \), by counting the possible choices of \( X_i \) first and then the configurations inside \( \bigcup_{x \in X_i} B(x;l^{1/2d}) \), we can estimate the cost of environment switching as

\[
\frac{\mathbb{P}(E_i^{k,\rho} \cap \{X_i^{o} = k\})}{\mathbb{P}(\mathcal{O} \cap B(x;l) = \emptyset)} \leq \sum_{i=k}^{\rho^{-1}k} \left( \frac{|B(x;l)|}{i} \right)^j \sum_{j=0}^{i} \left( \frac{|B(x;l^{1/2d})|}{j} \right) \left( \frac{1-p}{p} \right)^j \tag{3.48}
\]

Combining this with (3.47) and choosing \( \delta \) small, we obtain

\[
\mu_N \left((S, \mathcal{O}) \in \left(\{\boldsymbol{\sigma}, \tau \in \mathcal{F}_2\}, (E_i^{k,\rho}(x) \cap \{|X_i^{o}| = k\})\right)\right) \leq e^{-c_\delta - 1/d \delta l^{1/2}|k|/|\log \delta|} \tag{3.49}
\]

for each \( k < \rho \delta l^{d-1/2} \). Finally we sum (3.48) and (3.49) for \( k \in \{1, 2, \ldots, \lfloor \rho \delta l^{d-1/2} \rfloor\} \) to obtain

\[
\mu_N \left( E_i^{k,\rho}(x) \times \{(\boldsymbol{\sigma}, \tau) \in \mathcal{F}_2\} \right) \leq e^{-c_\delta l^{1/2}}. \tag{3.50}
\]

Case (3): In this case, we remove all the obstacles in \( A(x;l/4, l) = B(x;l) \setminus B(x;l/4) \), change the obstacles configuration inside \( B(x;l/4) \) to typical configurations and force all the crossings to avoid \( B(x;l/4) \) after lengthening them by \( l^2 \). Complication arises when the origin is close to \( B(x;l/4) \) because then it costs a lot to force the first crossing to avoid \( B(x;l/4) \). We first deal with the simpler case \( 0 \not\in B(x;l/2) \) by applying the environment and path switching (2.4) with

\[
(A_1, A_2) := (\{(\boldsymbol{\sigma}, \tau) \in \mathcal{F}_3\}, \{\tau_\mathcal{O} \setminus \tau_{B(x;l/4)} > N\}); \tag{3.51}
\]

\[
(E_1, E_2) := (E_i^{k,\rho}(x), \{\mathcal{O} \cap A(x;l/4, l) = \emptyset\}). \tag{3.52}
\]

The gain from the environment switching can be estimated as

\[
\frac{\mathbb{P}(E_i^{k,\rho})}{\mathbb{P}(\mathcal{O} \cap A(x;l/4, l) = \emptyset)} \leq \frac{\mathbb{P}(|\mathcal{O} \cap B(x;l)| \leq k|B(x;l)|)}{\mathbb{P}(\mathcal{O} \cap A(x;l/4, l) = \emptyset)} \leq p^{k|B(x;l/4)|} \sum_{i=k}^{\rho^{-1}k} \left( \frac{|B(x;l)|}{i} \right) \left( \frac{1-p}{p} \right)^j \tag{3.53}
\]

by using Stirling’s approximation (or Cramer’s large deviation principle as before).
On the other hand, if we force the random walk to stay in $A(x; l/4, l)$ instead of $B(x; l)$, the extra cost per step should be measured by the difference of the principal Dirichlet eigenvalues of the discrete Laplacian in $A(x; l/4, l)$ and $B(x; l)$, which is of order $l^{-2}$. In fact, we will see in Lemma 4.4 that uniformly in $u_k \in \partial B(x; l/2)$ and $v_k \in \partial B(x; l)$,

$$p^{B(x;l)\setminus \mathcal{O}}_{tk-s_k}(u_k, v_k) \leq e^{c_2((tk-s_k)(l^{-2}+1))}p^{A(x;l/4)\setminus \mathcal{O}}_{tk-s_k+2l}(u_k, v_k). \tag{RW3}$$

If $s_k < N$ and $t_k = N$ for some $k \in \mathbb{N}$, then this (last) crossing may be incomplete and its endpoint $v_k$ may be in $B(x; l/4)$. In that case, the path switching should be done differently and we change the endpoint of the last crossing to $\bar{v}_k := v_k + (5l/8)e_1$. The cost is bounded similarly as

$$p^{B(x;l)\setminus \mathcal{O}}_{tk-s_k}(u_k, v_k) \leq e^{c_2((tk-s_k)(l^{-2}+1))}p^{A(x;l/4)\setminus \mathcal{O}}_{tk-s_k+2l}(u_k, \bar{v}_k). \tag{RW4}$$

We define $(\bar{s}_k, \bar{t}_k)_{k \geq 1}$ as the starting and ending times of switched crossings, similarly to Case (1), and also

$$(\bar{v}_k, \bar{u}_{k+1}) := \begin{cases} (v_k + (5l/8)e_1, v_k + (5l/8)e_1), & \text{if } s_k < N, t_k = N \text{ and } v_k \in B(x; l/4), \\ (v_k, u_{k+1}), & \text{otherwise}. \end{cases} \tag{3.54}$$

Then using the above estimates and recalling the definition of $F_3$, we can bound the product in the right-hand side of (3.31) by

$$\prod_{k \geq 1} p^{B(x;l)\setminus \mathcal{O}}_{tk-s_k}(u_k, v_k)p_{sk+1-l_k}^{2d\setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1}) \leq \exp \left\{ c_2 \sum_{k \geq 1} \frac{t_k - s_k}{l^2} + K(s, t) \right\} \prod_{k \geq 1} p^{A(x;l/4)\setminus \mathcal{O}}_{tk-s_k}(\bar{u}_k, \bar{v}_k)p_{sk+1-l_k}^{2d\setminus (\mathcal{O} \cup B(x;l/2))}(\bar{v}_k, \bar{u}_{k+1}) \tag{3.55}$$

$$= e^{c_3l^2/4d} \prod_{k \geq 1} p^{A(x;l/4)\setminus \mathcal{O}}_{tk-s_k}(\bar{u}_k, \bar{v}_k)p_{sk+1-l_k}^{2d\setminus (\mathcal{O} \cup B(x;l/2))}(\bar{v}_k, \bar{u}_{k+1}).$$

Note that each $(\bar{s}_k, \bar{t}_k)$ has pre-image of cardinality at most $2^{2l/4d}$. Therefore summing (3.55) over $(s, t, u, v)$ separately according to the number of crossings as in Case (1), we can obtain

$$P(\tau_\mathcal{O} > N \text{ and } (\sigma, \tau) \in F_3)$$

$$= \sum_{(s, t) \in F_3} \sum_{u, v} p^{B(x;l)\setminus \mathcal{O}}_{s_1}(0, u_1) \prod_{k \geq 1} p^{B(x;l)\setminus \mathcal{O}}_{tk-s_k}(u_k, v_k)p_{sk+1-l_k}^{2d\setminus (\mathcal{O} \cup B(x;l/2))}(v_k, u_{k+1}) \leq e^{c_3l^2/4d} \sum_{(\bar{s}, \bar{t}, \bar{u}, \bar{v})} \sum_{u, v} p^{B(x;l)\setminus \mathcal{O}}_{\bar{s}_1}(0, u_1) \prod_{k \geq 1} p^{B(x;l)\setminus \mathcal{O}}_{tk-s_k}(\bar{u}_k, \bar{v}_k)p_{sk+1-l_k}^{2d\setminus (\mathcal{O} \cup B(x;l/2))}(\bar{v}_k, \bar{u}_{k+1}) \leq CNe^{c_3l^2/4d}P(\tau_{\mathcal{O}\cup B(x;l/4)} > N) \tag{3.56}$$

uniformly in $\mathcal{O}$ in the case $0 \notin B(x; l/2)$. Recalling (3.33) and $l \geq \log N$ and using (2.1), we conclude that in this case

$$\mu_N \left( (S, \mathcal{O}) \in \left\{ (\sigma, \tau) \in F_3 \right\}, E^S_1(\rho(x)) \right) \leq e^{-cd}. \tag{3.57}$$

Finally, we deal with the case $0 \in B(x; l/2)$. In this case, the starting point of first crossing may be close to (or even inside) $B(x; l/4)$ and we want to ensure that the random walk gets
away from that ball quickly. To this end, we fix a path \(\pi(x; l) \subset B(x; l/2)\) of length \(l\) from 0 to \(n e_1 \in \partial B(x; l/2)\) \((n \in \mathbb{N})\) and modify the environment and path switching as follows (see Figure 3):

\[
(A_1, A_2) := \left\{ \{\sigma, \tau\} \in F_3 \left| S_{[0,l]} = \pi(x; l), \tau_{\mathcal{O}} \wedge (l + \tau_{B(x;l/4)} \circ \theta_1) > N \right. \right\}; \\
(E_1, E_2) := \left\{ e^{k,\rho}(x), \{ \mathcal{O} \cap \left\{ A(x;l/4, l) \cup \pi(x; l) \right\} = \emptyset \} \right\},
\]

(3.58)

(3.59)

where \(l + \tau_{B(x;l/4)} \circ \theta_1\) is the first hitting time to \(B(x; l/4)\) after time \(l\). Let us explain the difference from the previous case \(0 \not\in B(x; l/2)\). For the environment, we need to keep \(\pi(x; l)\) empty, which has a cost of \(e^{-cl}\), but is negligible compared with the original gain \(e^{cld}\) in (3.55).

For the random walk, only the first crossing, that is \(S_{[0,\tau_1]}\) in this case, is switched differently. In the present case, note that \(v_1 \in \partial B(x; l)\) since by the total duration constraint on \(F_3\), we have \(t_1 \leq 2^{1/d} \delta^{d+2} < N\) for small \(\delta\). We switch the paths with \(\tau_1 = t_1, S_{\tau_1} = v_1\) to those go from 0 to \(n e_1\) following \(\pi(x; l)\) in \(l\) steps and then to go from \(n e_1\) to \(v_1\) inside \(A(x;l/4, l)\) in \(t_1 + 2l^2\) steps afterward. The probability to follow \(\pi(x; l)\) in the first \(l\) steps is \((2d)^{-l}\) and combining this with the estimate (see Lemma 3.6)

\[
p_{t_1}^{B(x;l)}(0, v_1) \leq e^{c_2(\lfloor l + \lfloor l/2 \rfloor^2 + 1 \rfloor)} p_{l+2l^2}^{A(x;l/4, l)}(n e_1, v_1),
\]

(RW5)

we obtain the following bound on the switching cost of the first crossing:

\[
p_{t_1}^{B(x;l)}(0, v_1) \leq (2d)^l p_l^{\pi(x;l)}(0, n e_1) e^{c_2(\lfloor l + \lfloor l/2 \rfloor^2 + 1 \rfloor)} p_{l+2l^2}^{A(x;l/4, l)}(n e_1, v_1) \\
\leq (2d)^l e^{c_2(\lfloor l + \lfloor l/2 \rfloor^2 + 1 \rfloor)} p_{l+2l^2+1}^{A(x;l/4, l), \pi(x;l)}(0, v_1).
\]

(3.60)

Note that the term \(c_2(\lfloor l + \lfloor l/2 \rfloor^2 \rfloor\) already appeared in (3.55). The extra cost of \((2d)^l\) is again negligible compared with the \(e^{cld}\) gain from the environment.

Therefore simply by setting \(\tilde{t}_1 := t_1 + 2l^2 + l\) and changing \((\tilde{s}_k, \tilde{t}_k)_{k \geq 2}\) accordingly, we can use (3.60) to follow the same argument as before to extend (3.56) to the case \(0 \not\in B(x; l/2)\).

### 3.4 Random walk estimates

In this subsection, we restate and prove the random walk estimates used in the proof of Lemma 3.4. Recall the notation \(B(x;l) = B(x;l) \cup \partial B(x;l)\) and that \(p_n(u, v)\) is understood to be \(p_{n+1}(u, v)\) if \(n + |u + v|\) is odd by the convention introduced in Remark 3.3.

**Lemma 3.6.** Let \(\lambda_l^\alpha(x)\) and \(\Gamma\) be defined as in (5.13) and (5.14), respectively. There exist \(c_0, c_1, c_2 > 0\) such that the following hold for all sufficiently large \(l\):

1. Uniformly in \(m \geq (c_0 l)^2, \quad u \in B(x;l/2), \quad v \in \overline{B(x;l)}, \quad\)

\[
p_m^{B(x;l)} \mathcal{O}(u, v) \leq e^{-c_1(\lfloor m/(c_0 l)^2 \rfloor - 1) \Gamma(\lfloor m/4 \rfloor)} p_m^{B(x;l)}(u, v).
\]

(RW1)

2. Uniformly in \(m \leq (c_0 l)^2\) and \(u \in \overline{B(x;l/2)}, \quad v \in \partial B(x;l), \quad\)

\[
p_m^{B(x;l)}(u, v) \leq \frac{1}{100} p_m^{B(x;l)}(u, v).
\]

(RW2)

3. Uniformly in \(m > 0\) and \(u \in \partial B(x;l/2), \quad v \in \partial B(x;l), \quad\)

\[
p_m^{B(x;l)}(u, v) \leq e^{c_2(m^{-2} + 1)} p_m^{A(x;l/4, l)}(u, v).
\]

(RW3)
Figure 1: A schematic figure of the switching configuration from (a) to (b) in Case (3). The balls are \( B(x; l/4), B(x; l/2) \) and \( B(x; l) \) from inside. There are 4 crossings from \( B(x; l/2) \) to \( B(x; l)^c \), including the last incomplete crossing. Both \( S_0 = 0 \) and \( S_N \) are in \( B(x; l/4) \) in (a). The paths from \( \bullet \) to \( \circ \) are crossings and are lengthened. Observe that we cannot make the second crossing avoid \( B(x; l/4) \) without lengthening it as illustrated in (b). The paths from \( \circ \) to \( \bullet \) are unchanged. The first polygonal segment of the path in (b) represents \( \pi(x; l) \).

4. Uniformly in \( m > 0 \) and \( u \in \partial B(x; l/2), v \in B(x; l/4) \),

\[
p_m B(x;l)(u, v) \leq e^{c_2(ml^{-2} + 1)} p_m A(x;l/4)(u, v + (5l/8)e_1). \tag{RW4}
\]

5. Suppose \( 0 \in B(x; l/2) \) and let \( n \in \mathbb{N} \) be such that \( n e_1 \in \partial B(x; l/2) \). Then uniformly in \( m > 0 \), and \( v \in \partial B(x; l) \),

\[
p_m B(x;l)(0, v) \leq e^{c_2(ml^{-2} + 1)} p_m A(x;l/4)(n e_1, v). \tag{RW5}
\]

In the proof of this lemma, we will use the following estimate on the Dirichlet heat kernel: for any \( c \in (0, 1) \), there exists \( C > 0 \) such that uniformly in \( r \in \mathbb{N}, \ k \in [cr^2, r^2/c] \) and \( u, w \in B(x; r) \),

\[
C \leq \frac{p_k B(x;r)(u, w)}{r^{-d-2} \text{dist}(u, \partial B(x; r)) \text{dist}(w, \partial B(x; r))} \leq \frac{1}{C}. \tag{3.61}
\]

This can be found for example in [11, Proposition 6.9.4], where it is stated only for the case \( k = r^2 \) but the argument therein can easily be adapted to the above uniform estimate.

The first assertion \((\text{RW1})\) will be a direct consequence of the following lemma.

**Lemma 3.7.** For any \( c_0 \in (0, 1) \), there exists \( c_1 > 0 \) independent of \( \mathcal{O} \) such that for any \( l \in \mathbb{N} \) and \( u, w \in B(x; l) \),

\[
p_{(cd)^2} B(x;l)\mathcal{O} u, w \leq e^{-c_1 \Gamma(|X_l^c|))} p_{(cd)^2} B(x;l)(u, w). \tag{3.62}
\]

**Remark 3.8.** This lemma holds for arbitrary \( c_0 \in (0, 1) \). In Lemma \ref{lem:compareheatkernel}, it is only \((\text{RW2})\) which imposes a restriction on \( c_0 \).

**Proof of Lemma 3.7.** We write \( l_0 \) for \( c_0 l \) in this proof to ease the notation. It suffices to prove

\[
\min_{u, w \in B(x; l)} \mathbf{P}_u \{\tau_{\mathcal{O}} \leq l_0^2 \mid S_{l_0} = w, \tau_{B(x; l)^c} > l_0^2\} \geq c_1 \Gamma(|X_l^c|). \tag{3.63}
\]
for some $c_1 > 0$, since $1 - \lambda \leq e^{-\lambda}$. The proof of this relies on the so-called second moment method. Let us introduce

$$T := \sum_{m \in [l_0^2/4, 3l_0^2/4]} 1_{\{S_m \in \mathcal{X}_l^o\}}.$$  \hspace{1cm} (3.64)

We will show the following:

$$\mathbb{E}_u \left[ T \left| S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right. \right] \geq c|\mathcal{X}_l^o|l_0^{2-d},$$  \hspace{1cm} (3.65)

$$\mathbb{E}_u \left[ T^2 \left| S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right. \right] \leq \frac{C(|\mathcal{X}_l^o|l_0^{2-d})^2}{\Gamma(|\mathcal{X}_l^o|)}.$$  \hspace{1cm} (3.66)

Given these bounds, the desired bound follows via the Paley–Zygmund inequality as

$$\mathbb{P}_u \left( \tau_B \leq l_0^2 \right| S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right) \geq \mathbb{P}_u \left( T > 0 \left| S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right. \right) \geq \frac{\mathbb{E}_u \left[ T \right| S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right]}{\mathbb{E}_u \left[ T^2 \left| S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right. \right]} . \hspace{1cm} (3.67)$$

First moment (3.65): Note first that uniformly in $m \in [l_0^2/4, 3l_0^2/4]$, $z \in B(x;l/2)$ and $u, w$ as in the statement,

$$\mathbb{P}_u \left( S_m = z, S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right) = p_m^{B(x;l)}(u,z)p_m^{B(x;l)}(z,w) \geq \frac{C}{l_0^{d+1}} p_m^{B(x;l)}(u,w).$$  \hspace{1cm} (3.68)

Indeed, by (3.61), there exists $C > 0$ such that uniformly in $m, u, w$ and $z$ as above,

$$p_m^{B(x;l)}(u,z) \geq \frac{C}{l_0^{d+1}} \text{dist}(u, \partial B(x;l)),$$  \hspace{1cm} (3.69)

$$p_m^{B(x;l)}(z,w) \geq \frac{C}{l_0^{d+1}} \text{dist}(w, \partial B(x;l)),$$  \hspace{1cm} (3.70)

and

$$p_m^{B(x;l)}(u,w) \leq \frac{1}{C l_0^{d+2}} \text{dist}(u, \partial B(x;l)) \text{dist}(w, \partial B(x;l)).$$  \hspace{1cm} (3.71)

The bound (3.68) follows from these three bounds. Summing (3.68) over $m \in [l_0^2/4, 3l_0^2/4]$ and $z \in \mathcal{X}_l^o$ yields the following equivalent of (3.65):

$$\mathbb{E}_u \left[ T : S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right] \geq c|\mathcal{X}_l^o|l_0^{2-d} \mathbb{P}_u \left( S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right) .$$  \hspace{1cm} (3.72)

Second moment (3.66): We begin with

$$\mathbb{E}_u \left[ T^2 : S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right]$$

$$= 2 \sum_{l_0^2/4 \leq i \leq j \leq 3l_0^2/4} \sum_{z_1, z_2 \in \mathcal{X}_l^o} \mathbb{P}_u \left( S_i = z_1, S_j = z_2, S_{l_0}^2 = w, \tau_{B(x;l)^c} > l_0^2 \right)$$

$$= 2 \sum_{k, m \geq l_0^2/4} \sum_{z_1, z_2 \in \mathcal{X}_l^o} p_k^{B(x;l)}(u,z_1)p_m^{B(x;l)}(z_1, z_2)p_{l_0^2-k-m}^{B(x;l)}(z_2, w).$$ \hspace{1cm} (3.73)
It follows from (3.61) as before that for the parameters appearing above,
\[ p_k^{B(x;l)}(u, z_1)p_{l_0^2-k-m}^{B(x;l)}(z_2, v) \leq \frac{c}{l_0^d} p_{l_0^2}^{B(x;l)}(u, v). \] (3.74)
Substituting this into (3.73) and performing the summation over \( z_2 \) and \( k \), we find that
\[ E_u \left[ T^2 : S_{l_0^2} = v, \tau_{B(x;l)} > l_0^2 \right] \leq \frac{c}{l_0^d} \sum_{m \leq l_0^2} \sum_{z \in X_i^o} P_z(S_m = X_i^o). \] (3.75)
For the probability appearing in the summation, we claim
\[ \sup_{z \in X_i^o} P_z(S_m = X_i^o) \leq c \begin{cases} m^{-d/2}, & m < l_0^{1/d}, \\ l_0^{-1/2}, & m \in \left[l_0^{1/d}, \frac{|X_i^o|^2}{l_0^{1/d}}\right], \\ |X_i^o|^{-d/2}, & m \in \left(\frac{|X_i^o|^2}{l_0^{1/d}}, \frac{l_0^2}{2}\right]. \end{cases} \] (3.76)
Substituting this bound into (3.75), we obtain
\[ E_u \left[ T^2 : S_{l_0^2} = v, \tau_{B(x;l)} > l_0^2 \right] \leq \frac{C(|X_i^o|^{2-d})}{\Gamma(|X_i^o|)} \frac{p_{l_0^2}^{B(x;l)}(u, v)}{l_0^d} \] (3.77)
which is equivalent to (3.69).

It remains to show (3.69). First, the on-diagonal term \( P_z(S_m = z) \leq c m^{-d/2} \) is always smaller than the right-hand side of (3.76). Henceforth, we shall focus on the points \( w \in X_i^o \setminus \{z\} \) which are at least \( l_0^{-1/2d} \) away from \( z \). By a standard Gaussian estimate on the transition probability of the symmetric simple random walk [111, Theorem 6.28],
\[ \sum_{w \in X_i^o \setminus \{z\}} P_z(S_m = w) \leq \sum_{n=1}^{\infty} \sum_{w \in X_i^o \cap A(z; n^{-1/2d}, (n+1)^{-1/2d})} \frac{C}{m^{d/2}} \exp\left\{-\frac{|w - z|^2}{Cm}\right\}. \] (3.78)
This right-hand side is maximized when the annuli are filled from inside out but since the points in \( X_i^o \) are at least \( l_0^{-1/2d} \) away from each other, the \( n \)-th annulus contains at most \( Cn^{d-1} \) points. This leads us to the bound
\[ \sum_{w \in X_i^o \setminus \{z\}} P_z(S_m = w) \leq \frac{C|X_i^o|^{1/d}}{m^{d/2}} \sum_{n=1}^{\infty} n^{d-1} \exp\left\{-\frac{n^2 l_0^{1/d}}{Cm}\right\}. \] (3.79)
The desired bound (3.76) follows by a simple computation considering the cases \( m < l_0^{1/d}, m \in \left[l_0^{1/d}, \frac{|X_i^o|^2}{l_0^{1/d}}\right] \) and \( m > \frac{|X_i^o|^2}{l_0^{1/d}}\) separately.

**Proof of Lemma 3.6.** We only consider the case when \( l, 5l/8 \) and \( n \) are all even.

The first assertion (3.62) follows immediately from Lemma 5.7. Indeed, using (3.62) in the Chapman–Kolmogorov identity, we have
\[ p_{m}^{B(x;l)}(u, v) = \sum_{w \in B(x;l)} p_{(m-l_0)}^{B(x;l)}(u, w)p_{m-l_0}^{B(x;l)}(w, v) \leq e^{-d\Gamma(|X_i^o|)} \sum_{w \in B(x;l)} p_{m-l_0}^{B(x;l)}(u, w)p_{m-l_0}^{B(x;l)}(w, v) \] (3.80)
and (3.63) follows by iteration.
Let us proceed to prove the second assertion \((\text{RW2})\). Note first that by a standard Gaussian heat kernel bound [BT, Theorem 6.28], for any \(u, w \in B(x; l)\) with \(|u - w| \geq cl\),

\[
p_k^{B(x;l)}(u, w) \leq p_k^{cl}(u, w)
\]

\[
\leq Ck^{-d/2} \exp \left\{ -\frac{|u - w|^2}{Ck} \right\}
\]

\[
\leq Cl^{-d} \left( \frac{l^2}{k} \right)^{d/2} \exp \left\{ -c \frac{l^2}{k} \right\}.
\] (3.81)

On the other hand, for \(m \in [l^2, 2l^2]\), \(u \in \overline{B(x; l/2)}\) and \(w \in \partial B(x; 3l/4)\), we have

\[
p_m^{B(x;l)}(u, w) \geq Cl^{-d}
\] (3.82)

by (3.81). Combining this with (3.81), we obtain the comparison

\[
\max_{k \leq (cl)^2} \max_{u \in \overline{B(x; l/2)}, w \in \partial B(x; 3l/4)} \frac{p_k^{B(x;l)}(u, w)}{p_k^{B(x;l)}(u, w)} \leq \frac{1}{100}
\] (3.83)

for sufficiently small \(c_0\). Suppose \(m \leq (c_0l)^2\) for \(c_0\) satisfying the above. By decomposing the random walk path upon the last visit to \(\partial B(x; 3l/4)\), we get

\[
p_m^{B(x;l)}(u, v) = \sum_{k=1}^{m} \sum_{w \in \partial B(x; 3l/4)} p_k^{B(x;l)}(u, w)p_{m-k}^{A(x;3l/4,l)}(w, v)
\]

\[
\leq \frac{1}{100} \sum_{k=1}^{m} \sum_{w \in \partial B(x; 3l/4)} p_k^{B(x;l)}(u, w)p_{m-k}^{A(x;3l/4,l)}(w, v)
\] (3.84)

\[
= \frac{1}{100} p_m^{B(x;l)}(u, v).
\]

This concludes the proof of (RW2).

The proofs of (RW3)–(RW5) rely on the fact that there exists \(c_2 > 0\) such that for any \(m \in \mathbb{N}, u \in \partial B(x; l/2)\) and \(w \in A(x; 3l/8, 7l/8)\),

\[
p_{m+l^2}^{A(x; 3l/4, l)}(u, w) \geq \exp\{ -c_2(|m/2^2| + 1) \} p_m^{B(x;l)}(u, w).
\] (3.85)

In order to prove this, we bound the left-hand side from below by the probability that \(S_{[0,l^2]} \subset A(x; l/4, l), S_{[2, m+l^2]} \subset B(w; l/8)\) and \(S_{k^2} \in B(w; l/16)\) for each \(k \in \{1, 2, \ldots, \lfloor m/2^2 \rfloor \}\), which can be written as

\[
\sum_{z_1, \ldots, z_{\lfloor m/2^2 \rfloor} \in \overline{B(w; l/16)}} p_{l^2}^{A(x; 3l/4,l)}(u, z_1) \left( \prod_{j=1}^{\lfloor m/2^2 \rfloor - 1} p_{l^2}^{B(w;l/8)}(z_j, z_{j+1}) \right)
\]

\[
\times p_{m-\lfloor m/2^2 \rfloor l^2+16}^{B(w;l/8)}(z_{\lfloor m/2^2 \rfloor}, w).
\] (3.86)

Since \(m - \lfloor m/2^2 \rfloor l^2 + l^2 \in [l^2, 2l^2]\) and all the points \(u, z_j\)’s and \(w\) are at least \(l/16\) away from the corresponding boundaries, by (3.81), all the heat kernels appearing in this expression is bounded from below by \(Cl^{-d}\) regardless where \(z_j\)’s are in \(B(w; l/16)\). Therefore we find the bound

\[
p_{m+l^2}^{A(x; 3l/4,l)}(u, w) \geq \exp\{ -c_2(|m/2^2| + 1) \} l^{-d}
\] (3.87)
for some $c_2 > 0$. Recalling (4.31), we have $p^{B(x,l)}_m(u,w) \leq c l^{-d}$ and we conclude the proof of (4.33).

Given (4.33), the third bound (RW4) can be proved in the same way as in the proof of (RW2) via the last visit decomposition. In order to prove the bound (RW4), we first replace $m$ by $m + l^2$ and choose $u \in \partial B(x; l/2)$ and $w = v + (5l/8)e_1$ for $v \in B(x; l/4)$ in (4.33) to obtain

$$
\exp\{c_2\left(\lfloor ml^{-2} \rfloor + 2\right)\} p^{A(l/x; l)}_{m+l^2}(u,v + (5l/8)e_1) \geq p^{B(x,l)}_{m+l^2}(u,v + (5l/8)e_1). 
$$

(3.88)

We can further bound the right-hand side from below by $c_p p^{B(x,l)}_m(u,v)$ using either (4.33) ($m \leq (c_0l)^2$) or the parabolic Harnack inequality from [D94] ($m > (c_0l)^2$). This yields (RW4) by making $c_2$ larger. The proof of (RW3) is almost the same and left to the reader. 

\[ \square \]

\section{Random Walk Range and “Truly”-Open Sites}

In this section, we prove various properties of $\mathcal{T}$ and its relation with the random walk range $S_{[0,N]}$, which will pave the way for the proof of Theorems 1.2 and 2.5. First, we prove Lemma 4.1 in Subsection 4.1, which shows that the random walk must visit the interior of $\mathcal{T}$, and sites in $\mathcal{T}$ are well-approximated by sites in $S_{[0,N]}$. We then explain in Subsection 4.2 how Lemma 4.2, Proposition 4.3, and the deduction of Theorem 4.4 from Theorem 4.5 all follow from the same key Lemma 4.6 on the probability of visiting certain sites that are costly for survival. The proof of Lemma 4.6 is then given in Subsection 4.3 using path decomposition and switching, with the basic setup presented earlier in Section 4.1.

\subsection{Proof of Lemma 4.1}

In this subsection, we give the proof of Lemma 4.1. First we show that “truly”-open sites are rare in the following sense.

\begin{lemma} \text{ (“truly”-open sites are rare).} For any $v \in \mathbb{Z}^d$ and all sufficiently large $N$, \end{lemma}

$$
\mathbb{P}(v \text{ is “truly”-open}) \leq \exp\{-(\log N)^2\}. 
$$

(4.1)

\begin{proof} \text{ of Lemma 4.1.} \end{proof}

Recall that $x \in \mathbb{Z}^d$ is “truly”-open if

$$
\mathbb{P}_x (\tau_\partial > (\log N)^5) \geq \exp\{-(\log N)^2\}. 
$$

(4.2)

By using Donsker–Varadhan’s asymptotics (1.5), we obtain

$$
\mathbb{P} (\mathbb{P}_x (\tau_\partial > (\log N)^5) \geq \exp\{-(\log N)^2\}) 
$$

$$
\leq \exp\{-(\log N)^2\} \mathbb{E}\left[ \mathbb{P}_x (\tau_\partial > (\log N)^5) \right] 
$$

$$
\leq \exp\{-(\log N)^2\} \exp\left\{-c(\log N)^{\frac{5d}{d+2}}\right\}. 
$$

(4.3)

Since the power of $5d/(d+2) > 2$ for $d \geq 2$, we are done. \[ \square \]

\begin{proof} \text{ of Lemma 4.6.} \end{proof}

The proof of the first assertion (2.6) is simple. Indeed, since a “truly”-open site is open by definition, for any site in

$$
\{x \in \mathcal{T} : \text{dist}(x, \partial \mathcal{T}) \geq (\log N)^5\}, 
$$

(4.4)

its $(\log N)^5$ neighborhood is free of obstacles. Therefore (2.6) follows from Lemma 6.2.
The second assertion (2.7) can be restated as
\[
\lim_{N \to \infty} \mu_N \left( \bigcap_{w \in T} \{ \tau_{B(w; (\log N)^5)} \leq N \} \right) = 1. \tag{4.5}
\]

We are going to show that for any \( w \in B(0; 3\varrho_N) \),
\[
\mu_N \left( w \text{ is “truly”-open, } \tau_{B(w; (\log N)^5)} > N \right) \leq \exp \left\{ - (\log N)^2 \right\}, \tag{4.6}
\]
from which (4.5) follows by the union bound. But since whether \( w \) is “truly”-open or not depends only on the configuration of obstacles inside \( B(w; (\log N)^5) \) and hence independent of \( \mathbb{P}(\tau_{B(w; (\log N)^5)} \land \tau_O > N) \), we have
\[
\mathbb{P} \otimes \mathbb{P} \left( w \text{ is “truly”-open, } \tau_{B(w; (\log N)^5)} \land \tau_O > N \right) \leq \exp \left\{ - (\log N)^2 \right\} \mathbb{P} \otimes \mathbb{P}(\tau_O > N) \tag{4.7}
\]
by using Lemma 4.1.

4.2 Proof of Lemma 2.6, Proposition 2.9 and Theorem 2.8

The proof of Lemma 2.6 and Proposition 2.9 turn out to be quite similar, both involving random walk path switching to avoid sites that are costly for survival. As explained in Subsection 2.4, to bound \( |\partial T| \) and prove Proposition 2.8, it suffices to give an upper bound on the expected total number of visits to \( \bigcup_{x \in \partial T} B(x; (\log N)^6) \), as well as a uniform lower bound on the expected number of visits to \( B(x; (\log N)^6) \) over all \( x \in \partial T \). More precisely, define
\[
G_T(u, x) := \mathbb{E}_u \left[ \sum_{n=0}^{\tau_T} 1_{S_n \in B(x; (\log N)^6)} \right], \quad u \in T, x \in \partial T, \tag{4.8}
\]
which is the expected number of visits to \( B(x; (\log N)^6) \) before the walk is killed. We also introduce the set where the above expected number of visits is too small:
\[
\mathcal{L}(l) := \bigcup_{x \in \partial T, G_T(x, x) \leq \varphi(N, l)} B(x; (\log N)^6), \tag{4.9}
\]
where
\[
\varphi(N, l) := \begin{cases} \varrho_N^{-\epsilon}, & \text{if } l = \epsilon \varrho_N, \\ (\log N)^{-c_5}, & \text{if } l = \varrho_N / \log N \end{cases} \tag{4.10}
\]
with \( c_4 \in (0, 1) \) and \( c_5 > 0 \) to be chosen later.

We first claim that the expected number of visits to the neighborhood of \( \partial T \) is not too large.

Lemma 4.2. There exists \( c_6 > 0 \) such that
\[
\sum_{x \in \partial T} G_T(x_N, x) \leq (\log N)^{c_6}. \tag{4.11}
\]

We will then show that on the event of confinement in \( B(x_N; \varrho_N + l) \) and \( B(x_N; \varrho_N - l/4) \) being free of obstacles, the probability for the random walk to visit \( T^c \) or \( \mathcal{L}(l) \) is asymptotically negligible.
Lemma 4.3. There exists $\epsilon_0 > 0$ such that the following holds: let $l := \epsilon \eta_N$ with $\epsilon \leq \epsilon_0$ or $l := \eta_N / \log N$, and assume that

$$
\lim_{N \to \infty} \mu_N \left( \tau_{B(x_N; \eta_N + l)} > N, \mathcal{O} \cap B(x_N; \eta_N - l/4) = \emptyset \right) = 1.
$$

Then

$$
\lim_{N \to \infty} \mu_N \left( \tau_{T \cup \mathcal{L}(l)} \leq N \right) = 0.
$$

Let us present three consequences of these two lemmas before giving proofs.

Proof of Lemma 2.6. Since Theorem A and Proposition 2.2 imply

$$
\lim_{N \to \infty} \mu_N \left( \tau_{B(x_N; (1+\epsilon) \eta_N)} > N, \mathcal{O} \cap B(x_N; (1 - \epsilon/4) \eta_N) = \emptyset \right) = 1
$$

for any $\epsilon > 0$, Lemma 2.6 immediately follows from Lemma 4.3 with $l = \epsilon \eta_N$.

Proof of Proposition 2.9. By Lemma 2.5 (cf. (4.5)), we know that the random walk does visit $B(v; (\log N)^6)$ for each $v \in \partial T$, and together with Lemma 4.7, this implies that we must have $\mathcal{L}(\epsilon \eta_N) = \emptyset$. This means that we have a uniform lower bound $\min_{x \in \partial T} G_T(x_N, x) \geq \eta_N^{1-d-c_6 \epsilon}$ and hence

$$
\sum_{x \in \partial T} G_T(x_N, x) \geq |\partial T| \eta_N^{1-d-c_6 \epsilon}.
$$

Combining with Lemma 4.2, we conclude that $|\partial T| \leq \eta_N^{d-1+c_6 (\log N)^c_6}$, and since $\epsilon > 0$ can be taken arbitrarily small, Proposition 2.9 follows.

Proof of Theorem 2.8 assuming Theorem 2.7. Observe that once we have proved Theorem 2.7, we may take $l = \eta_N / \log N$ in Lemma 4.3. Then the same argument as above yields Theorem 2.8.

We close this subsection with the proof of Lemma 4.2 which is fairly simple. The proof of Lemma 4.3 is much more involved and will take up the next two subsections.

Proof of Lemma 4.2. Let us define the stopping times

$$
\xi_1 := \inf \left\{ n \geq 0 : \text{dist}(S_n, \partial T) < (\log N)^6 \right\}
$$

and for $k \geq 1$,

$$
\xi_{k+1} := \inf \left\{ n \geq \xi_k + 2(\log N)^{10} : \text{dist}(S_n, \partial T) < (\log N)^6 \right\}.
$$

We can then bound the left-hand side of (4.14) by

$$
\sum_{x \in \partial T} G_T(x_N, x) \leq (\log N)^C \mathbb{E} \left[ \max \left\{ k : \xi_k < \tau_{\mathcal{O}} \right\} \right]
$$

for some $C > 0$. Observe that whenever the random walk visits $(\log N)^6$ neighborhood of $\partial T$, there is more than $c(\log N)^{-6d}$ probability of exiting $T$ within the next $(\log N)^{12}$ steps by the local central limit theorem. And once the random walk exits $T$, it will hit $\mathcal{O}$ in the next $(\log N)^{12}$ steps with high probability by the definition of $T$. Therefore $\max \{ k : \xi_k < \tau_{\mathcal{O}} \}$ is stochastically dominated by a geometric random variables with parameter $c(\log N)^{-6d}$ and the desired bound follows.
4.3 Path decomposition

In order to prove Lemma 4.3, what will be relevant is the behavior of the random walk near \( \partial \mathcal{T} \). Since \( v \in \mathbb{Z}^d \) is "truly"-open if its \((\log N)^5\) neighborhood is open and we assume \( \mathcal{O} \cap B(x_N; g_N - 1/4) = \emptyset \), we know that \( \partial \mathcal{T} \) lies near \( \partial B(x_N; g_N) \). This motivates us to decompose the random walk paths according to the crossings of a thin annulus near the boundary of the confinement ball \( B(x_N; g_N) \).

Similarly to (4.9)–(4.12), we decompose a random walk \( (S_n)_{0 \leq n \leq N} \) by using successive crossings between the inner and outer shells of the annulus

\[
A(x_N; g_N - 2l, g_N - l) = B(x_N; g_N - l) \setminus \overline{B(x_N; g_N - 2l)},
\]

where we will choose \( l > 0 \) to be either \( \epsilon g_N \) or \( g_N / \log N \). To this end, we introduce the stopping times

\[
\sigma_1 := \min \{ n \geq 0 : S_n \in \overline{B(x_N; g_N - 2l)} \} \wedge N,
\]

and for \( k \in \mathbb{N} \),

\[
\begin{align*}
\tau_k &:= \min \{ n > \sigma_k : S_n \in B(x_N; g_N - l) \} \wedge N, \\
\sigma_{k+1} &:= \min \{ n > \tau_k : S_n \in \overline{B(x_N; g_N - 2l)} \} \wedge N.
\end{align*}
\]

In what follows, we will decompose the random walk paths into the pieces \( (S_{[\tau_k, \tau_{k+1}])_{k \geq 1}} \) and the role of \( (\sigma_k)_{k \geq 1} \) is auxiliary. More precisely, the paths that visit a costly site \( v \in \mathcal{T} \cup \mathcal{L}(l) \) (cf. (4.4)) during \( [\tau_k, \tau_{k+1}] \) are going to be switched to the paths that stay inside \( B(x_N; g_N - l/2) \) during \( [\tau_k, \tau_{k+1}] \).

We use bold face letters to denote sequences of numbers as in Subsection 3.3. For a non-decreasing sequence \( t = (t_k)_{k \geq 1} \) of integers, by slightly abusing our notation in Subsection 3.3, we write

\[
K(t) := \sup \{ k \geq 1 : t_{k+1} - t_k > 0 \}
\]

which represents the number of crossings from \( \partial B(x_N; g_N - l) \) to \( \partial B(x_N; g_N - 2l) \) and back to \( \partial B(x_N; g_N - l) \) when \( t = \tau \). We have the following control on \( K(\tau) \).

**Lemma 4.4.** There exists \( c_3 > 0 \) depending only on the dimension \( d \) such that if

\[
\lim_{N \to \infty} \mu_N \left( \tau_{B(x_N; g_N + l)}^c > N, \mathcal{O} \cap B(x_N; g_N - l/4) = \emptyset \right) = 1
\]

for some \( l \in [g_N / \log N, c_3 g_N] \), then

\[
\lim_{N \to \infty} \mu_N \left( K(\tau) \leq N l^{-2} \right) = 1.
\]

**Proof of Lemma 4.4.** Let us fix \( l \in [g_N / \log N, c_3 g_N] \) and suppose that \( K(\tau) > N l^{-2} \), or equivalently, \( \sigma_{[N l^{-2}]} < N \). Then we find that

\[
\begin{align*}
\mathbb{P} \left( \tau_{B(x_N; g_N + l)}^c > N, K(\tau) > N l^{-2} \right) \\
\leq \sup_{u \in \partial B(x_N; g_N - l)} \mathbb{P}_u \left( \tau_{B(x_N; g_N - 2l)} < \tau_{B(x_N; g_N + l)}^c \right)^{N l^{-2} - 1}
\end{align*}
\]

by using the strong Markov property at each \( \tau_k \) with \( k \leq N l^{-2} - 1 \). Since

\[
\sup_{u \in \partial B(x_N; g_N - l)} \mathbb{P}_u \left( \tau_{B(x_N; g_N - 2l)} < \tau_{B(x_N; g_N + l)}^c \right)
\]
is bounded away from one for all large $N$, by choosing $c_3$ sufficiently small and recalling (4.3), we find that the right-hand side of (4.30) decays faster than $\mathbb{P} \otimes \mathbb{P}(\tau_{\Omega} > N)$.  

It is also useful to know that the random walk does not end up near the boundary of the confinement ball at time $N$.

**Lemma 4.5.**

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} \mu_N (S_N \in B(x_N; (1 - \epsilon) \varrho_N)) = 1.$$  \hspace{1cm} (4.28)

**Proof of Lemma 4.5.** By Theorem 4.3 and Proposition 2.2, it suffices to show that

$$\mathbb{P} \otimes \mathbb{P} \left( S_N \notin B(x_N; (1 - \epsilon) \varrho_N) \mid \tau_{\Omega \cup B(x_N; \varrho_N + \varrho_N^3)^c} > N, \Omega \cap B(x_N; (1 - \epsilon) \varrho_N) = \emptyset \right)$$  \hspace{1cm} (4.29)

tends to zero as $N \to \infty$ and $\epsilon \to 0$. Let us write $B = B(x_N; \varrho_N + \varrho_N^3) \setminus \Omega$ in this proof to ease the notation. We use the eigenfunction expansion to get

$$\mathbb{E} [f(S_N) : \tau_B^c > N] = \sum_{k \geq 1} \left( 1 - \lambda_B^{RW,(k)} \right)^N \left\langle \phi_B^{RW,(k)}, f \right\rangle \phi_B^{RW,(k)} (0)$$  \hspace{1cm} (4.30)

for any bounded function $f$, where $\lambda_B^{RW,(k)}$ and $\phi_B^{RW,(k)}$ are the $k$-th smallest eigenvalue and corresponding eigenfunction with $\|\phi_B^{RW,(k)}\|_2 = 1$ for the generator of the random walk killed upon exiting $B$. On the event $\Omega \cap B(x_N; (1 - \epsilon) \varrho_N) = \emptyset$, each of the eigenvalues $\lambda_B^{RW,(k)}$ and eigenfunctions $\phi_B^{RW,(k)} (k \in \mathbb{N})$, after proper rescaling, should be close to the eigenvalues of the Dirichlet Laplacian on the unit ball. Based on this observation, one can in fact prove (see Lemma A.2 in the appendix) that

$$\lambda_B^{RW,(2)} - \lambda_B^{RW,(1)} \geq c_T \varrho_N^{-2},$$  \hspace{1cm} (EV)

$$\left\| \phi_B^{RW,(1)} \right\|_\infty \leq c_S \varrho_N^{-d/2},$$  \hspace{1cm} (EF)

which are well-known for the eigenvalues and eigenfunctions for the continuum Laplacian. It follows from (4.31) that the terms with $k \geq 2$ in (4.30) are negligible for bounded $f$ by a standard argument. See the proof of Lemma 4.3 in the appendix. By setting $f = 1_{B(x_N; (1 - \epsilon) \varrho_N)\setminus}^c$ and $f = 1$ in (4.30), we find that

$$\mathbb{P} (S_N \notin B(x_N; (1 - \epsilon) \varrho_N) \mid \tau_B^c > N) = \frac{\left\langle \phi_B^{RW,(1)}, 1_{B(x_N; (1 - \epsilon) \varrho_N)\setminus} \right\rangle}{\left\langle \phi_B^{RW,(1)}, 1 \right\rangle} + o(1)$$  \hspace{1cm} (4.31)

as $N \to \infty$. Since (4.3) and the fact $\phi_B^{RW,(1)} \geq 0$ imply that

$$\left\langle \phi_B^{RW,(1)}, 1_{B(x_N; (1 - \epsilon) \varrho_N)\setminus} \right\rangle \leq c_\epsilon \varrho_N^{d/2},$$  \hspace{1cm} (4.32)

$$\left\langle \phi_B^{RW,(1)}, 1 \right\rangle \geq c_S^{-1} \varrho_N^{-d/2} \left\| \phi_B^{RW,(1)} \right\|_2 = c_S^{-1} \varrho_N^{-d/2},$$  \hspace{1cm} (4.33)

the first term on the right-hand side of (4.31) vanishes as $\epsilon \to 0$.  

**Remark 4.6.** With a little more effort, it is possible to show that the eigenfunction $\phi_B^{RW,(1)}$ converges, after proper rescaling, to the eigenfunction of the Dirichlet Laplacian on the unit ball in $L^2$. See, for example, [12] for a further discussion on related problems.
4.4 Proof of Lemma 4.3

Proof of Lemma 4.3. Referring to (4.12) and Lemmas 4.1 and 4.2, let us fix \( \epsilon \in (0, c_3) \), \( l \in \{\varrho_N / \log N, \epsilon \varrho_N\} \) and introduce good events

\[
E_{10} := \{ \mathcal{O} \cap B(x_N; \varrho_N - l/4) = \emptyset \}, \\
E_{11} := \{ K(\sigma, \tau) \leq Nl^{-2} \}, \\
E_{12} := \{ x_N \in B(0; (1 - 2\epsilon)\varrho_N), S_N \in B(x_N; (1 - 2\epsilon)\varrho_N) \}
\]

and define \( E_{\text{good}} := E_{10} \cap E_{11} \cap E_{12} \), for which we have \( \lim_{N \to \infty} \mu_N(E_{\text{good}}) = 1 \). We are going to prove

\[
\mathbb{P} \otimes \mathbb{P} \left( \tau \leq N < \tau_\mathcal{O}, E_{\text{good}} \right) \leq c(\log N)^{-1} \mathbb{P} \otimes \mathbb{P} \left( \tau_\mathcal{O} > N \right) 
\]

and for any \( v \in B(0; 3\varrho_N) \),

\[
\mathbb{P} \otimes \mathbb{P} \left( v \in \partial T, G_T(x_N, v) \leq \varrho_N^{1-d} \varphi(N, l), \tau_{\mathcal{O} \cup B(v; (\log N)^6)} \leq N < \tau_\mathcal{O}, E_{\text{good}} \right) \\
\leq c(\log N)^{-1} \mathbb{P} \otimes \mathbb{P} \left( v \in \partial T, \tau_\mathcal{O} \wedge \tau_{B(v; (\log N)^6)} > N \right). 
\]

First, the bound (4.37) implies that \( \lim_{N \to \infty} \mu_N(\tau \leq N) = 0 \). Second, since \( \mathbb{P}(\tau_\mathcal{O} \wedge \tau_{B(v; (\log N)^6)} > N) \) is independent of the obstacle configuration in \( B(v; (\log N)^6) \), in particular whether each site \( w \in B(v; 1) \) is “truly”-open or not, Lemma 4.3 implies

\[
\mathbb{P} \otimes \mathbb{P} \left( v \in \partial T, \tau_\mathcal{O} \wedge \tau_{B(v; (\log N)^6)} > N \right) \\
\leq c(\log N)^{-1} \sum_{w \in B(v; 1)} \mathbb{P} \otimes \mathbb{P} \left( w \text{ is “truly”-open, } \tau_\mathcal{O} \wedge \tau_{B(v; (\log N)^6)} > N \right) \\
\leq \exp \left( - (\log N)^2 \right) \mathbb{P} \otimes \mathbb{P} \left( \tau_\mathcal{O} > N \right). 
\]

Therefore, by substituting this bound into (4.37) and summing over \( v \in B(0; 3\varrho_N) \), it follows that \( \lim_{N \to \infty} \mu_N(\tau_L(l) \leq N) = 0 \).

The strategy of the proofs of (4.34) and (4.35) is to show by a path switching argument that it is better for the random walk not to visit \( T \), or \( B(v; (\log N)^6) \) with \( v \in \partial T \) and \( G_T(x_N, v) \leq \varrho_N^{1-d} \varphi(N, l) \). Note that on the event \( E_{10} \), we have

\[
T \cup B(v; (\log N)^6) \subset B(x_N; \varrho_N - l/2) 
\]

and hence it is natural to use the path decomposition in terms of the crossings from \( \partial B(x_N; \varrho_N - l) \) to \( B(x_N; \varrho_N - 2l) \) introduced in Subsection 4.3. The random walk can visit \( T \cup B(v; (\log N)^6) \) only on a crossing \([\tau_k, \sigma_{k+1}]\) \((k \in \mathbb{N})\) and if it happens, we want to switch such a crossing to the one that avoids \( T \cup B(v; (\log N)^6) \). However, it turns out to be easier to switch the path on the entire interval \([\tau_k, \sigma_{k+1}]\). More precisely, for \( u \in \partial B(x_N; \varrho_N - l) \) and \( u' \in \partial B(x_N; \varrho_N - l) \cup B(x_N; (1 - 2\epsilon)\varrho_N) \), we are going to compare

\[
p_T^{\text{visit}}(u, u') := \begin{cases} 
\mathbb{P}_u \left( \tau_{T \cup B(v; (\log N)^6)} < \tau_1 = t < \tau_\mathcal{O}, S_t = u' \right), & \text{if } u' \in \partial B(x_N; \varrho_N - l), \\
\mathbb{P}_u \left( \tau_{T \cup B(v; (\log N)^6)} < \tau_1 = \tau_\mathcal{O} \wedge \tau_1, S_t = u' \right), & \text{if } u' \in B(x_N; (1 - 2\epsilon)\varrho_N) 
\end{cases} 
\]

with

\[
p_T^{\text{avoid}}(u, u') := \begin{cases} 
\mathbb{P}_u \left( \tau_{T \cup B(v; (\log N)^6)} > \tau_1 = t, S_t = u' \right), & \text{if } u' \in \partial B(x_N; \varrho_N - l), \\
\mathbb{P}_u \left( \tau_{T \cup B(v; (\log N)^6)} \wedge \tau_1 > t, S_t = u' \right), & \text{if } u' \in B(x_N; (1 - 2\epsilon)\varrho_N). 
\end{cases} 
\]
These are the probabilities that conditionally on $S_{\tau_k} = u$, the random walk path during $[\tau_k, \tau_{k+1}]$ either visits or avoids $\mathcal{T}^c$ and $B(v; (\log N)^6)$ and ends at $u'$ at time $\tau_{k+1} = \tau_k + t$. The two cases $u' \in \partial B(x_N; \varrho_N - l)$ and $u' \in B(x_N; (1 - 2\epsilon)\varrho_N)$ correspond to $k < K(\tau)$ and $k = K(\tau)$ respectively, where for the latter case recall that we are working on the event $E_{\text{vis}}$. The key comparison estimate we will prove is the following: if $v \in \partial \mathcal{T}$, $G_\mathcal{T}(x_N, v) \leq \varrho_N^{1-d} \varphi(N, l)$ and $E_{\text{vis}}$ holds, then

$$p_t^{\text{visit}}(u, u') \leq \varrho_N^{-d}(\log N)^{-3} p_t^{\text{avoid}}(u, u'). \quad (4.43)$$

Let us first see how we can deduce the desired bounds (4.37) and (4.38) from (4.43). We assume (4.42) and $G_\mathcal{T}(x_N, v) \leq \varrho_N^{1-d} \varphi(N, l)$. For $j \geq 1$ and $k = (k_i)_{i=1}^j \in \mathbb{N}^j$, consider the event that the crossings with indices $k \in \mathcal{K}$ visit $\mathcal{T}^c \cup B(v; (\log N)^6)$ and others do not. Its probability can be bounded as

$$\sum_{K} \sum_{t_1 < t_2 < \ldots < t_{K+1} = N} p_t^{B(x_N; \varrho_N - l)}(0, u_1) \prod_{k=1}^{K} \left( p_t^{\text{avoid}}(u_k, u_{k+1})_{1 \neq k} + p_t^{\text{visit}}(u_k, u_{k+1})_{1 \in k} \right) \leq \varrho_N^{-d}(\log N)^{-3} \mathbb{P} \left( \tau_{\mathcal{T}^c \cup B(v; (\log N)^6)} > N + 2j \varrho_N^2 \right), \quad (4.44)$$

where in the first line, $u_1, \ldots, u_K \in \partial B(x_N; \varrho_N - l)$, $u_{K+1} \in B(x_N; (1 - 2\epsilon)\varrho_N)$ and for each $k \in \mathcal{K}$, we have used (4.42) to get the extra multiplicative factor $\varrho_N^{-d}(\log N)^{-3}$ by lengthening the crossing duration by $2\varrho_N^2$. Recalling that we are assuming $E_{\text{vis}}$, we may restrict our consideration to $K \leq Nl^{-2} \leq \varrho_N^{d}(\log N)^2$. Therefore there are at most $K^j \leq \varrho_N^{d}(\log N)^{2j}$ choices of $k = (k_i)_{i=1}^j$ and thus it follows that

$$\mathbb{P} \left( \text{exactly } j \text{ crossings visit } \mathcal{T}^c \cup B(v; (\log N)^6), E_{\text{vis}}, \tau_{\mathcal{O}} > N \right) \leq (\log N)^{-j} \mathbb{P} \left( \tau_{\mathcal{T}^c \cup B(v; (\log N)^6)} > N \right). \quad (4.45)$$

Since $\mathcal{O} \subset \mathcal{T}^c$, summing over $j \geq 1$, we obtain (4.47) and (4.38).

It remains to prove (4.47). Recall first that (4.43) holds on the event $E_{\text{vis}}$. In particular, during $[\sigma_1, \tau_1]$, the random walk stays inside $B(x_N; \varrho_N - l)$ and can visit neither $\mathcal{O}$ nor $\mathcal{T}^c \cup B(v; (\log N)^6)$. Based on this observation, both cases in (4.43) can be described as follows: the random walk starting from $u \in \partial B(x_N; \varrho_N - l)$ visits $\mathcal{T}^c \cup B(v; (\log N)^6)$ and $\mathcal{B}(x_N; \varrho_N - 2l)$ in this order without hitting $\mathcal{O}$, and then stays inside $B(x_N; \varrho_N - l)$ before it ends at $u'$ at time $t$. Therefore, using the strong Markov property at $\sigma_1$, the first hitting time of $B(x_N; \varrho_N - 2l)$, we obtain

$$p_t^{\text{visit}}(u, u') = E_u \left[ p_t^{B(x_N; \varrho_N - l)}(S_{\sigma_1}, u'): \tau_{\mathcal{T}^c \cup B(v; (\log N)^6)} < \tau_1 < t \land \tau_{\mathcal{O}} \right]. \quad (4.46)$$

Similarly, by (4.44), for the random walk to avoid $\mathcal{T}^c \cup B(v; (\log N)^6)$, it suffices to stay inside $B(x_N; \varrho_N - l/2)$ and hence

$$p_t^{\text{avoid}}(u, u') \geq E_u \left[ p_t^{B(x_N; \varrho_N - l)}(S_{\sigma_2}, u'): \tau_{B(x_N; \varrho_N - 2l)} \land \tau_1 > \varrho_N^2, S_{\varrho_N^2} \in B(x_N; \varrho_N/2) \right]. \quad (4.47)$$

When we replace $p^{\text{visit}}$ by $p^{\text{avoid}}$, we basically switch the path $S_{[0, \sigma_1]}$ to paths of length $\varrho_N^2$ that stays inside $B(x_N; \varrho_N - l/2)$, does not exit $B(x_N; \varrho_N - l)$ after hitting $\mathcal{B}(x_N; \varrho_N - 2l)$ and ends in $B(x_N; \varrho_N/2)$ at time $\varrho_N^2$. After time $\varrho_N^2$, we let the random walk continue to evolve and first exit (after $\varrho_N^2$) from $B(x_N; \varrho_N - l)$ at time $t + 2\varrho_N^2$.  

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We will prove in Lemma 4.7 below the following four estimates (RW6)–(RW9) in order to control the gain from this switching. The first three estimates show that we gain a lot by switching the first piece \( S_{[0,\sigma_1]} \): On the event \( \{x_N \in B(0; (1-2\epsilon)\bar{c}_N) \cap \{O \cap B(x_N; \bar{c}_N - l/4) = \emptyset \} \) for any \( v \in \partial T \), we have

\[
\sup_{u \in \partial B(x_N; \bar{c}_N - l)} P_u (\tau_T < \sigma_1 < \tau_\Omega) \leq \exp \{- (\log N)^2 \}, \tag{RW6}
\]

\[
\sup_{u \in \partial B(x_N; \bar{c}_N - l)} P_u (\tau_B(v; (\log N)^6) < \sigma_1 < \tau_\Omega) \leq G_T(0, v)^2 \frac{\bar{c}_N^{d-1}}{l} (\log N)^{8d}, \tag{RW7}
\]

and there exists \( c_9 > 0 \) such that

\[
\inf_{u \in \partial B(x_N; \bar{c}_N - l)} P_u \left( \tau_B(x_N; \bar{c}_N - 1/2)^c \cap \tau_1 \geq \bar{c}_N^2, S_{\bar{c}_N^2} \in B(x_N; \bar{c}_N^2 / 2) \right) \geq c_9 \frac{l}{\bar{c}_N^2}. \tag{RW8}
\]

Substituting the last estimate (RW8) into (4.11), we find that uniformly in \( u \in \partial B(x_N; \bar{c}_N - l) \),

\[
p_{t+2\bar{c}_N}^{\text{avoid}} (u, u') \geq c_9 \frac{l}{\bar{c}_N^2} \inf_{y \in \partial B(x_N; \bar{c}_N^2 / 2)} p_{t+\bar{c}_N^2}^{B(x_N; \bar{c}_N - l)}(y, u'). \tag{4.48}
\]

The fourth estimate controls the possible cost caused by switching the second piece \( S_{[\sigma_1, \tau_1]} \): There exists \( c_{10} > 0 \) such that for all \( t > \sigma_1, u' \in \partial B(x_N; \bar{c}_N - l) \) and \( w \in \partial B(x_N; \bar{c}_N - 2l) \),

\[
p_{t-\sigma_1}^{B(x_N; \bar{c}_N - l)}(w, u') \leq c_{10} \left( \frac{\bar{c}_N^2}{l} \right)^{c_{10}} \exp \left\{ c_{10} \frac{\sigma_1}{\bar{c}_N^2} \right\} \inf_{y \in \partial B(x_N; \bar{c}_N^2 / 2)} p_{t+\bar{c}_N^2}^{B(x_N; \bar{c}_N - l)}(y, u'). \tag{RW9}
\]

We defer the proofs of (RW6)–(RW9) to the next subsection and now complete the proof of (4.11). Note that the cost in (RW9) becomes large if \( \sigma_1 \) is large. We first exclude the case where \( \sigma_1 \) is atypically large by using a tail bound for \( \sigma_1 \). For typical values of \( \sigma_1 \), the cost in (RW9) is not too large and we can use (RW6)–(RW8) to prove (4.11). Let us first consider the case \( \sigma_1 \geq c_{11} l^2 \log N \) in (4.10), where \( c_{11} > 0 \) is to be determined later. By using (RW9), we find that

\[
E_u \left[ p_{t-\sigma_1}^{B(x_N; \bar{c}_N - l)}(S_{\sigma_1}, u') : c_{11} l^2 \log N \leq \sigma_1 < t \wedge \tau_\Omega \right] \leq c_{10} \left( \frac{\bar{c}_N}{l} \right)^{c_{10}} \inf_{y \in \partial B(x_N; \bar{c}_N^2 / 2)} p_{t+\bar{c}_N^2}^{B(x_N; \bar{c}_N - l)}(y, u') \times E_u \left[ \exp \left\{ c_{10} \frac{\sigma_1}{\bar{c}_N^2} \right\} : c_{11} l^2 \log N \leq \sigma_1 < t \wedge \tau_\Omega \right].
\tag{4.49}
\]

Observe that up to time \( \sigma_1 \), the walk is confined in an annulus of width \( 3l \) and hence \( P_u(\sigma_1 \geq n) \leq \exp\{-cn l^{-2}\} \). This bound yields that if \( l \leq c \bar{c}_N \) for sufficiently small \( \epsilon > 0 \), then

\[
E_u \left[ \exp \left\{ c_{10} \frac{\sigma_1}{\bar{c}_N^2} \right\} : c_{11} l^2 \log N \leq \sigma_1 < t < \tau_T \right] \leq \sum_{n \geq c_{11} l^2 \log N} \exp \left\{ c_{10} \frac{n}{\bar{c}_N^2} - \frac{n}{l^2} \right\} \leq \sum_{n \geq c_{11} l^2 \log N} \exp \left\{ -(c - c_{10}) \frac{n}{l^2} \right\} \leq N^{-cc_{11}/2}. \tag{4.50}
\]
We choose \(c_1\) so large that the above right-hand side is less than \(\varrho_N^{-3d}\). Then by substituting the above into (1.43) and comparing with (1.45), we obtain
\[
\mathbb{E}_u \left[ p_{l-\sigma_1}^{B(x_N; \varrho N^{-l})}(S_{\sigma_1}, u') : c_{11} l^2 \log N \leq \sigma_1 < t < \tau_0 \right]
\leq \varrho_N^{-2d} \inf_{y \in B(x_N; \varrho N/2)} p_{t+\varrho N^2}^{B(x_N; \varrho N^{-l})}(y, u')
\leq \frac{1}{2} \varrho_N^{-d} (\log N)^{-3p_{\text{avoid}}}(u, u').
\]

Next, we consider the case \(\sigma_1 < c_{11} l^2 \log N\) in (1.40). In this case, we have a deterministic upper bound on the exponential factor in (RW9) and hence
\[
\mathbb{E}_u \left[ p_{l-\sigma_1}^{B(x_N; \varrho N^{-l})}(S_{\sigma_1}, u') : \tau_{T_{c\cup B(v; (\log N)^6)}} < \sigma_1 < c_{11} l^2 \log N \land t \land \tau_0 \right]
\leq c_{10} \left( \frac{\varrho N}{l} \right)^{c_{10}} \exp \left\{ c_{10} c_{11} \left( \frac{1}{\varrho N} \right) ^2 \log N \right\} \inf_{y \in B(x_N; \varrho N/2)} p_{l+\varrho N}^{B(x_N; \varrho N^{-l})}(y, u')
\times \mathbb{P}_u \left( \tau_{T_{c\cup B(v; (\log N)^6)}} < \sigma_1 < c_{11} l^2 \log N \land t \land \tau_0 \right).
\]

Now we use (RW9) and (RW7) to see that on the event \(\{G_T(x_N, v) < \varrho_N^{-d} \varphi(N, l)\},\)
\[
\mathbb{P}_u \left( \tau_{T_{c\cup B(v; (\log N)^6)}} < \sigma_1 < c_{11} l^2 \log N \land t \land \tau_0 \right)
\leq \mathbb{P}_u \left( \tau_{T_{c\cup B(v; (\log N)^6)}} < \sigma_1 < t \land \tau_0 \right) + \mathbb{P}_u \left( \tau_{T_{B(v; (\log N)^6)}} < \sigma_1 < t \land \tau_0 \right)
\leq 2 \varrho_N^{-2d} \varphi(N, l)^2 \mathbb{P}_u \left( \tau_{T_{c\cup B(v; (\log N)^6)}} < \sigma_1 < t \land \tau_0 \right)
\leq 2 \varrho_N^{-2d} \varphi(N, l)^2 \log N)^{8d}.
\]

Substituting this into (1.54) and comparing with (1.45) as in the previous case, we arrive at
\[
\mathbb{E}_u \left[ p_{l-\sigma_1}^{B(x_N; \varrho N^{-l})}(S_{\sigma_1}, u') : \tau_{T_{c\cup B(v; (\log N)^6)}} < \sigma_1 < c_{11} l^2 \log N \land t \land \tau_0 \right]
\leq c_{12} \left( \log N \right)^{c_{12} \varphi(N, l)^2} \exp \left\{ c_{12} \left( \frac{l}{\varrho N} \right)^2 \log N \right\} p_{l+2\varrho N}^{\text{avoid}}(u, u')
\leq \frac{1}{2} \varrho_N^{-d} (\log N)^{-3p_{\text{avoid}}}(u, u')
\]
by setting
\[
\varphi(N, l) = \begin{cases} N^{-c_{12} \epsilon}, & \text{if } l = \epsilon \varrho N, \\ (\log N)^{-c_{12} - 4}, & \text{if } l = \varrho N / \log N. \end{cases}
\]

Gathering (1.54) and (1.51), we get (1.53) and we are done.

\section{4.5 Random walk estimates II}

In this subsection, we prove the random walk estimates (RW6)–(RW7) used in Subsection 4.4. Recall the definition of the stopping times \(\sigma_k\) and \(\tau_k\) (\(k \in \mathbb{N}\)) in (2.20)–(2.22).

**Lemma 4.7.** Suppose that \(l \in [\varrho N / \log N, c_3 \varrho N]\), \(x_N \in B(0; (1-2\epsilon)\varrho N)\) and \(\mathcal{O} \cap B(x_N; \varrho N - l/4) = \emptyset\). Then the following hold:

1. For \(u \in \partial B(x_N; \varrho N - l)\) and \(v \in \partial T\),
\[
\mathbb{P}_u \left( \tau_{T_{c\cup B(v; (\log N)^6)}} < \sigma_1 < \tau_0 \right) \leq \exp \left\{ - (\log N)^2 \right\} \tag{RW6}
\]
\[
\mathbb{P}_u \left( \tau_{T_{B(v; (\log N)^6)}} < \sigma_1 < \tau_0 \right) \leq G_T(0, v) \frac{\varrho N}{l} (\log N)^{8d}. \tag{RW7}
\]
2. There exists \( c_9 > 0 \) such that for \( u \in \partial B(x_N; \varrho_N - l) \),
\[
    P_u \left( \tau_{B(x_N; \varrho_N - l/2)}^c \wedge \tau_1 \geq \varrho_N^2 N, S_{\varrho_N^2} \in B(x_N; \varrho_N/2) \right) \geq c_9 \frac{l}{\varrho_N^2}. \tag{RW8}
\]

3. There exists \( c_{10} > 0 \) such that uniformly in \( 0 \leq m < n \), \( w \in \partial B(x_N; \varrho_N - 2l) \) and \( u' \in \partial B(x_N; \varrho_N - l) \),
\[
    P_{n-m}^{B(x_N; \varrho_N-l)}(w, u') \leq c_{10} \left( \frac{\varrho_N}{l} \right)^c \exp \left\{ c_{10} m \varrho_N^{-2} \right\} \inf_{y \in B(x_N; \varrho_N/2)} P_{n-m}^{B(x_N; \varrho_N-l)}(y, u'). \tag{RW9}
\]

Let us explain the intuitions behind these bounds before delving into the proof. The first assertion (RW6) follows readily from the definition of the “truly”-open set. The second assertion (RW7) is based on the following observation. The probability for the random walk to visit \( B(v; (\log N)^\delta) \) without hitting \( \mathcal{O} \) is bounded by \( G_T(u, v) \). Then it has to come back to \( w \) but by the time reversal, the probability is again bounded by \( G_T(u, v) \). Finally, the factor \( l^{-1} \) comes from the fact that the random walk hits \( B(x; \varrho_N - 2l) \) for the first time at \( w \). We need the extra poly-logarithmic factor to change the starting points in \( G_T(u, v) \) and \( G_T(w, v) \) to \( x_N \) by using the elliptic Harnack inequality. The third assertion (RW8) is a slight modification of (RW7). The fourth assertion (RW9) basically says that it is easier for the random walk to go from \( y \in B(x_N; \varrho_N/2) \) to \( u' \in \partial B(x_N; \varrho_N - l) \) than from \( w \in \partial B(x_N; \varrho_N - 2l) \), without exiting \( B(x_N; \varrho_N - l) \). There are two reasons why we have a large factor on the right-hand side. First, if \( w \) and \( u' \) are close to each other and \( n - m \) is of order \( l^2 \), then it is in fact better to start from \( w \); second, if both \( m \) and \( n \) are large, then we have to include the cost for the random walk to stay in \( B(x_N; \varrho_N - l) \) for extra time \( m + \varrho_N^2 \).

**Proof of Lemma 4.7.** The left-hand side of (RW6) is bounded by
\[
    P_u(\tau_{T^c} < \sigma_1 < \tau_{\mathcal{O}}) \leq \sup_{x \in T^c} P_x(\sigma_1 < \tau_{\mathcal{O}}). \tag{4.56}
\]
by the strong Markov property applied at \( \tau_{T^c} \). Since we assume \( \mathcal{O} \cap B(x_N; \varrho_N - l/4) = \emptyset \), we have \( T^c \subset B(x_N; \varrho_N - l/2) \) and hence \( \sigma_1 > (\log N)^\delta \) whenever the random walk starts from \( T^c \). The bound (RW6) follows from the definition of \( T^c \).

Next, the left-hand side of (RW7) is bounded by
\[
    \sum_{w \in \partial B(x_N; \varrho_N - 2l)} \sum_{y \in B(v; (\log N)^\delta)} P_u(\tau_y < \sigma_1 < \tau_{\mathcal{O}}, S_{\sigma_1} = w). \tag{4.57}
\]
By reversing the time on \([\tau_y, \sigma_1] \), we have that for each \( y \in B(v; (\log N)^\delta) \),
\[
    P_u(\tau_y < \sigma_1 < \tau_{\mathcal{O}}, S_{\sigma_1} = w) \leq P_u(\tau_{B(v; (\log N)^\delta)} < \sigma_1 \wedge \tau_{\mathcal{O}}) P_w(\tau_{B(v; (\log N)^\delta)} < \sigma_1 \wedge \tau_{\mathcal{O}}). \tag{4.58}
\]
We further bound the second factor on the right-hand side by using the strong Markov property at \( \tau_1 \) as
\[
    \sum_{z \in \partial B(x_N; \varrho_N - l)} P_w(S_{\tau_1} = z, \tau_1 < \sigma_1) P_z(\tau_{B(v; (\log N)^\delta)} < \sigma_1 \wedge \tau_{\mathcal{O}}) \leq P_w(\tau_1 < \sigma_1) \max_{z \in \partial B(x_N; \varrho_N - l)} P_z(\tau_{B(v; (\log N)^\delta)} < \sigma_1 \wedge \tau_{\mathcal{O}}) \tag{4.59}
\]
\[
    \leq \frac{c}{l} \max_{z \in \partial B(x_N; \varrho_N - l)} P_z(\tau_{B(v; (\log N)^\delta)} < \sigma_1 \wedge \tau_{\mathcal{O}}),
\]
where in the last inequality we have used a gambler’s ruin type estimate (see [LL10, (6.14)] for a similar estimate). Substituting this into (4.58) and summing over $y \in B(v; (\log N)^6)$, we find that

\[
P_u \big( \tau_{B(v; (\log N)^6)} < \sigma_1 < \tau_\partial, S_{\sigma_1} = w \big) \leq \frac{c}{l} (\log N)^{5d} \max_{z \in \partial B(x_N; \rho_N - l)} P_z \big( \tau_{B(v; (\log N)^6)} < \sigma_1 \wedge \tau_\partial \big)^2 \leq \frac{c}{l} (\log N)^{5d} \max_{z \in \partial B(x_N; \rho_N - l)} G_T(z, v)^2. \tag{4.60}
\]

We are going to shift the variable $z \in \partial B(x_N; \rho_N - l)$ to $x_N$ by applying the following elliptic Harnack inequality to the function $G_T(\cdot, v)$, which is harmonic in $B(x_N; \rho_N - l/2)$: There exists $c_{13} > 0$ such that for any $x \in \mathbb{Z}^d$, $r \in \mathbb{N}$ sufficiently large and any non-negative harmonic function $f$ on $B(x; r)$,

\[
\sup_{B(x; 0.9r)} f(y) \leq c_{13} \inf_{B(x; 0.9r)} f(y). \tag{4.61}
\]

See [LL10, Theorem 6.3.9], for example.

To compare $G_T(z, v)$ with $G_T(x_N, v)$, we will apply (4.61) iteratively as follows. First, let $l_1 = l$ and $z_1$ be the point on $\partial B(z; l_j)$ closest to $x_N$. Applying (4.61) to $G_T(\cdot, v)$ on the ball $B(z_1; l_1)$ gives $G_T(z, v) \leq c G_T(z_1, v)$. We can now iterate this procedure. For $j \geq 2$, let $l_j := 2^{j-1} l$ and $z_{j+1}$ be the point on $\partial B(z_j; l_j)$ closest to $x_N$, and apply (4.61) to $G_T(\cdot, v)$ on the ball $B(z_j; l_j)$. The iteration is stopped at the first $J$ such that $z_J \in B(x_N; 2\rho_N/3)$. See Figure 2. Noting that $J \leq c \log(\rho_N/l)$, we have

\[
G_T(z, v) \leq c_{13} G_T(z_1, v) \leq \ldots \leq c_{13}^J G_T(z_J, v) \leq \left(\frac{\rho_N}{l}\right)^c G_T(x_N, v), \tag{4.62}
\]

\[
\begin{array}{c}
\partial B(x_N; \rho_N) \\
\partial B(x_N; \rho_N - l) \\
\partial B(x_N; 2\rho_N/3)
\end{array}
\]

Figure 2: The sequence $(z_j)_{j=1}^J$ constructed in the proof of (RWW7). The balls around $z_j$ have geometrically growing radii and the construction terminates at $J = 4$ in this picture.
where in the last inequality, we have applied (4.61) in $B(x_N; \varrho_N - l/2)$ to bound $G_T(z_J, v)$ by $c_{13}G_T(x_N, v)$. Since $l \geq \varrho_N / \log N$, by substituting (4.62) into (4.61) and summing over $w \in \partial B(x_N; \varrho_N - 2l)$, we get the desired bound (RWS).

In order to prove the lower bound (RWS), we let the random walk obey the following strategy: pick $u' \in \partial B(x_N; \varrho_N - 3l/2) \cap B(u; l)$ and

1. $S_{l_2} \in B(u'; l/4)$ without exiting $A(x_N; \varrho_N - 2l, \varrho_N - l/2)$;
2. $S_{\varrho_N} \in B(x_N; \varrho_N/2)$ without exiting $B(x_N; \varrho_N - l)$.

In this way, the condition $\tau_B(x_N; \varrho_N - l/2) \wedge \tau_1 \geq \varrho_N^2$ holds and hence the left-hand side of (RWS) is bounded from below by the probability of the above strategy. With the help of (4.61), one can find $c > 0$ such that

$$P_u \left( S_{l_2} \in B(u'; l/4), S_{[0,l_2]} \subset A(x_N; \varrho_N - 2l, \varrho_N - l/2) \right) \geq c, \quad (4.63)$$

$$\inf_{y \in B(u'; l/4)} P_y \left( S_{\varrho_N} \not\in B(x_N; \varrho_N/2), S_{[0,\varrho_N]} \subset B(x_N; \varrho_N - l) \right) \geq c \frac{l}{\varrho_N}. \quad (4.64)$$

Collecting these bounds, we get (RWS).

Finally we prove (RWS). In the case $n + \varrho_N^2 \leq 2\varrho_N^2$, we have

$$p^B_{n+m}(y, u') \geq \mathbb{E}_u \left[ p^B_{n+\varrho_N - \tau_B(x_N; \varrho_N - 3l/2), y} \left( S_{\tau_B(x_N; \varrho_N - 3l/2)}, y \right) \right], \quad (4.65)$$

$$\geq cl^{-d} \left( \tau_B(x_N; \varrho_N - 3l/2) < (n - m) \wedge \tau_B(x_N; \varrho_N - l)^c \right),$$

where we have used the strong Markov property at $\tau_B(x_N; \varrho_N - 3l/2)$ and applied (4.61) to the transition probability appearing inside the expectation, noting that $n + \varrho_N^2 - \tau_B(x_N; \varrho_N - 3l/2) \in [\varrho_N^2, 2\varrho_N^2]$ under the conditions $n + \varrho_N^2 \leq 2\varrho_N^2$ and $\tau_B(x_N; \varrho_N - 3l/2) < (n - m) \wedge \tau_B(x_N; \varrho_N - l)^c$.

Similarly, we have

$$p^B_{n-m}(w, u') = \mathbb{E}_u \left[ p^B_{n-m - \tau_B(x_N; \varrho_N - 3l/2), y} \left( S_{\tau_B(x_N; \varrho_N - 3l/2), y} \right) \right], \quad (4.66)$$

$$\leq cl^{-d} \left( \tau_B(x_N; \varrho_N - 3l/2) < (n - m) \wedge \tau_B(x_N; \varrho_N - l)^c \right),$$

where we have used the strong Markov property at $\tau_B(x_N; \varrho_N - 3l/2)$ and the estimate

$$p^B_{n-m}(w, u') = \sup_{k \in \mathbb{N}, |x - y| \geq l/2} \left( S_{\tau_B(x_N; \varrho_N - 3l/2), y} \right) \leq c l^{-d}, \quad (4.67)$$

which follows in the same way as in (4.81). Combining the above two bounds, we get (RWS) in this case.
In the other case \( n + \varrho_N > 2\varrho_N^2 \), we use the following parabolic Harnack inequality from [DGR, H(C_H) in Theorem 1.7]: For all \( x_0 \in \mathbb{Z}^d \), \( s \in \mathbb{R} \), \( r > 200 \) and every non-negative \( u(t, x) \) that satisfies the discrete heat equation on \( \mathbb{Z} \cap [s, s + 100r^2] \times B(x_0; r) \),

\[
\sup_{(t_1, x_1) \in \mathbb{Z} \cap [s+0.01r^2, s+0.1r^2] \times B(x_0; 0.99r)} u(t_1, x_1) \leq 100 \inf_{(t_2, x_2) \in \mathbb{Z} \cap [s, s+100r^2] \times B(x_0; 0.99r)} u(t_2, x_2).
\]  

(4.68)

We use this to first shift the spatial variable \( w \) to \( y \in B(x_N; \varrho_N/2) \) and then the time variable in the transition probability kernel in (4.72) in the same way as in the proof of (4.71) (see Figure 3): let \( w_0 := w \), and for \( j \geq 0 \) let \( l_j := 2^{j-1}l \) and \( w_{j+1} \) be the point on \( \partial B(w_j; l_j) \) closest to \( x_N \) and

\[
J := \min \{ j \geq 0 : w_j \in B(x_N; 2\varrho_N/3) \}. \tag{4.69}
\]

Note that \( \varrho_N/3 \leq l_J \leq 2\varrho_N/3 \) and therefore \( J \leq c \log(\rho_N/l) \). As a first step, we switch from \( w = w_0 \) to \( w_1 \) and use the bound

\[
\rho_{n-m}^B(x_N; \varrho_N - l) B(w_n; u') \leq c_{14} \frac{\varrho_N}{l} \rho_{n-m+l_j^2}^B(w_1, u')
\]  

(4.70)

with some \( c_{14} > 0 \), which can be verified by applying either (4.67) to the left-hand side and (4.64) to the right-hand side when \( n - m < \varrho_N^2 \), or (4.68) with \( x_0 = w_1 \), \( s = n - m - 0.05l_j^2 \) and \( r = l_j \) to the function \( u(t, x) := p_{t_j}^B(x_N; \varrho_N - l) (x, u') \) restricted to \( B(w_1; l_1) \), when \( n - m > \varrho_N^2 \). Next, noting that \( w_j \) keeps distance at least \( l_{j+1}/2 \) from \( \partial B(w_{j+1}; l_{j+1}) \) for any \( j < J \), we can apply (4.68) with \( x_0 = w_j \), \( s = n - m + 0.95l_j^2 \) and \( r = l_j \) to the function \( u(t, x) := p_t^B(x_N; \varrho_N - l) (x, u') \) restricted to \( B(w_j; l_j) \) to obtain

\[
p_{n-m+l_j^2}^B(w_j, u') \leq 100p_{n-m+l_{j+1}^2}^B(w_{j+1}, u') \quad \text{for } j \geq 1,
\]  

(4.71)

\[
p_{n-m+l_j^2}^B(w_j, u') \leq 100p_{n-m+\varrho_N^2}^B(y, u'),
\]  

(4.72)

Figure 3: The sequence \( (w_j)_{j=1}^J \) constructed in the proof of (4.67). The balls around \( w_j \) have geometrically growing radii and the construction terminates at \( J = 4 \) in this picture.
where in the second bound, we have used \( p_j^2 \leq 4\varrho_N^2/9 \). By using (4.71)–(4.72) and recalling the upper bound on \( J \), we get

\[
P^{(x_N;\varrho_N-l)}_{n-m}(w, u') \leq cJ\varrho_N \frac{p_{n-m}^{(x_N;\varrho_N-l)}}{p_{n-m}^{(x_N;\varrho_N-l)}}(w, u') \leq \left( \frac{\varrho_N}{l} \right)^c p^{(x_N;\varrho_N-l)}_{n-m+2\varrho_N^2}(y, u'). \tag{4.73}
\]

Now another iteration of (4.68) leads us to

\[
P^{(x_N;\varrho_N-l)}_{n-m+2\varrho_N^2}(y, u') \leq 100p^{(x_N;\varrho_N-l)}_{n-m+2\varrho_N^2}(y, u') \leq \cdots \leq 100|m\varrho_N^2|^{-1}p^{(x_N;\varrho_N-l)}_{n-m+|m\varrho_N^2|\varrho_N^2}(y, u') \leq 100|m\varrho_N^2|p^{(x_N;\varrho_N-l)}_{n+\varrho_N^2}(y, u'), \tag{4.74}
\]

where in the \( k \)-th inequality, we choose \( x_0 = x_N \), \( s = n - m + (k - 0.05)\varrho_N^2 \), \( r = \varrho_N - l \) and \( u(t, x) := p^{(x_N;\varrho_N-l)}_{t}(x, u') \) in (4.68). The desired estimate follows from (4.73) and (4.74). \( \blacksquare \)

## 5 Proof of the Refined Version of Ball Covering Theorem

In this section, we prove Theorems 2.7. As we discussed in Subsection 4.3, below Lemma 2.6, Theorem 2.8 follows from Theorem 2.7. To strengthen Proposition 2.4 to Theorem 2.7, we first show that the volume of \( T \) is very close to \( |B(0; \varrho_N)| \) with the help of the Faber–Krahn inequality and a control on the number of the possible shapes of \( T \) provided by Proposition 2.4. Now if there is a non-“true”-open site \( x \in B(x_N; \varrho_N - \varrho_N^2) \), then there exists a closed site near \( x \). Lemma 2.4 then implies that there are in fact many closed sites around \( x \), which leads us to a contradiction.

In order to carry out the proof, we will compare \( \lambda^{\text{RW},(k)}_T \), the \( k \)-th smallest eigenvalue for the generator of the random walk killed upon exiting \( T \), with its counterpart for the continuum Laplacian, and then apply the Faber–Krahn inequality. We define the continuous hull of \( T \subset \mathbb{Z}^d \) by

\[
\tilde{T} := \{ x \in \mathbb{R}^d : \text{dist}_\infty(x, T) < 2 \} \tag{5.1}
\]

and denote the \( k \)-th smallest Dirichlet eigenvalue of \(-\frac{1}{2\varrho_N^2} \Delta\) in \( \tilde{T} \) and the corresponding eigenfunction by \( \lambda^{(k)}_T \) and \( \varphi^{(k)}_T \), respectively. We will prove the following comparison in the appendix.

**Lemma 5.1.** There exists \( c > 0 \) such that for any \( x \in \mathbb{Z}^d \) and \( T \supset B(x; \varrho_N/2) \),

\[
\lambda^{\text{RW},(k)}_T \geq \lambda^{(k)}_T - c\varrho_N^{-4}. \tag{5.2}
\]

Moreover, for each \( k \in \mathbb{N} \), there exists \( \gamma_k > 0 \) such that

\[
\left| \lambda^{\text{RW},(k)}_{B(0;\varrho_N)} - \lambda^{(k)}_{B(0;\varrho_N)} \right| \leq \gamma_k \varrho_N^{-3}. \tag{5.3}
\]

Let us recall a lower bound on the survival probability which is well-known in the continuum setting \([299, (33)]\) and for the two dimensional continuous time random walk \([394, Proposition 2.1] \). Lemma 5.2 below is the analogue in our discrete time setting, which we prove in the appendix for completeness. Recall that we denote the Euclidean volume of \( G \subset \mathbb{R}^d \) by \( \text{vol}(G) \).
Lemma 5.2. There exists $c > 0$ such that for all sufficiently large $N > 0$,

$$
\mathbb{P} \otimes \mathbb{P} \left( \tau_{0} > N \right) \geq \exp \left\{ \log \left( \frac{1}{B(0; \varrho_{N})} \right) \log p - N \lambda_{B(0; \varrho_{N})}^{(1)} - cg_{N}^{-4} \right\}.
$$

(5.4)

Using this lemma, we can show that the volume of $\mathcal{T}$ is very close to $\text{vol}(B(0; \varrho_{N}))$.

Lemma 5.3. For any $b \in (0, 1)$,

$$
\lim_{N \to \infty} \mu_{N} \left( \left| \text{vol}(\mathcal{T}) - \text{vol}(B(0; \varrho_{N})) \right| < \varrho_{N}^{(d-1)/2 + 2b} \right) = 1.
$$

(5.5)

Proof of Lemma 5.3. Referring to Theorem K and Propositions 2.2 and 2.9, we introduce the set of possible shapes of $\mathcal{T}$ for $b \in (0, 1)$:

$$
\mathcal{T}_b := \{ T \subset \mathbb{Z}^d : B(x; 0.9 \varrho_{N}) \subset T \subset B(x; \varrho_{N} + \varrho_{N}^{b}) \text{ for some } x \in B(0; \varrho_{N}), \text{ and } |\partial T| \leq g_{N}^{d-1+b} \}
$$

(5.6)

so that $\lim_{N \to \infty} \mu_{N}(\mathcal{T} \in \mathcal{T}_b) = 1$. The cardinality of this set is bounded by

$$
|\mathcal{T}_b| \leq \exp\{cg_{N}^{d-1+2b}\}
$$

(5.7)

simply by considering the choice of $g_{N}^{d-1+b}$ points from $B(0; 3 \varrho_{N})$. Having controlled the entropy, we estimate next the probability $\mu_{N}(T = T)$ for each $T \in \mathcal{T}_b$. With the help of the eigenfunction expansion, one finds the upper bound

$$
\mathbb{P} \otimes \mathbb{P} \left( \tau_{T^c} > N, \mathcal{T} = T \right) \leq \mathbb{P}(\mathcal{O} \cap T = \emptyset) \mathbb{P}(\tau_{T^c} > N)
$$

$$
\leq |T|^{1/2} \exp \left\{ |\text{vol}(\mathcal{T})| \log p - N \lambda_{\text{RW},(1)} \right\}
$$

(5.8)

for general $T \subset \mathbb{Z}^d$. See, for example, [K16, (2.21)]. Observe that since $\mathcal{T} \subset \bigcup_{x \in \mathbb{Z}^d} \{ x + [0, 2]^d \}$, we have $|T| \geq \text{vol}(\mathcal{T}) - c|\partial T|$ and hence for $T \in \mathcal{T}_b$,

$$
|T| \geq \text{vol}(\mathcal{T}) - cg_{N}^{d-1+b}.
$$

(5.9)

On the other hand, by (5.2) and the Faber–Krahn inequality (see, for example, [CS1, pp.87–92]), for any $T \in \mathcal{T}_b$, we have

$$
\lambda_{\text{RW},(1)} \geq \lambda_{\mathcal{T}}^{(1)} - cg_{N}^{-4}
$$

$$
\geq \lambda_{|\mathcal{T}|} - cg_{N}^{-4},
$$

(5.10)

where for $r > 0$, we denote by $\lambda_{r}$ the principal Dirichlet eigenvalue of $-\frac{1}{2\varrho} \Delta$ in a ball with volume $r$. Substituting (5.11) and (5.11) into (5.8), we obtain

$$
\mathbb{P} \otimes \mathbb{P} \left( \tau_{T^c} > N, \mathcal{T} = T \right)
$$

$$
\leq |T|^{1/2} \exp \left\{ \text{vol}(\mathcal{T}) \log p - N \lambda_{|\mathcal{T}|} + cg_{N}^{d-1+b} \right\}.
$$

(5.11)

Suppose that $T \in \mathcal{T}_b$ satisfies $|\text{vol}(\mathcal{T}) - \text{vol}(B(0; \varrho_{N}))| \geq g_{N}^{(d-1)/2 + 2b}$. Then, since the function

$$
r \mapsto r \log p - N \lambda_{r} = r \log p - \frac{N \lambda_{1}}{r^{2/d}}
$$

(5.12)
is twice-differentiable and maximized at \( \text{vol}(B(0; \varrho_N)) \) (cf. (1.6)), one finds by the Taylor expansion that
\[
\text{vol}(\bar{T}) \log p - N \lambda_{\bar{T}} \\
\leq \text{vol}(B(0; \varrho_N)) \log p - N \lambda_{B(0; \varrho_N)}^{(1)} - c \| \text{vol}(\bar{T}) \| - \text{vol}(B(0; \varrho_N)) \|^2 \\
\leq \text{vol}(B(0; \varrho_N)) \log p - N \lambda_{B(0; \varrho_N)}^{(1)} - c \varrho_N^{d-1+4b}.
\]
Substituting this into (5.11) and comparing with Lemma 5.2, we obtain \( \mu_N(T = T) \leq \exp\{-c \varrho_N^{d-1+4b}\} \). Thanks to (5.10), we can use the union bound to conclude the proof of Lemma 5.3. \( \blacksquare \)

**Proof of Theorems 2.7** Thanks to Lemma 5.4, we can restrict our consideration to the event
\[
\bigcap_{x \in B(0;2\varrho_N)} \bigcap_{l \leq \varrho_N} \left\{ x \in \mathcal{O} \text{ and } \frac{|O \cap B(x; l)|}{|B(x; l)|} < \delta \right\},
\]
that is, any \( x \in B(0;2\varrho_N) \) is either open or has \( \delta \)-fraction of closed sites in its \( l \)-neighborhood for all \( l \leq (\log N)^3, \varrho_N \). In addition, by Lemma 5.4 with \( b = \epsilon_1/2 \) where \( \epsilon_1 \) is as in Theorem 5, we can further assume that
\[
\big| \text{vol}(\bar{T}) - \text{vol}(B(0; \varrho_N)) \big| < \varrho_N^{(d-1)/2+\epsilon_1}.
\]
Let \( \epsilon_2 > (d - 1 + \epsilon_1)/d \) and suppose that there exists \( x \in B(x_N; \varrho_N - \varrho_N^2) \setminus T \). Then there exists at least one closed site \( y \in B(x; (\log N)^5) \), and by (5.12), more than \( \delta \) fraction of the points in the ball \( B(y; \varrho_N^2/2) \subset B(x_N; \varrho_N) \) must be closed. Recalling (5.3) and that \( \bar{T} \subset B(x_N; \varrho_N + 2\varrho_N^2) \) by the definition of \( T \), we find that
\[
\text{vol}(\bar{T}) \leq |T| + \varrho_N^{d-1+b} \\
\leq |B(x_N; \varrho_N + 2\varrho_N^2)| - \delta|B(y; \varrho_N^2/2)| + \varrho_N^{d-1+\epsilon_1} \\
\leq \text{vol}(B(0; \varrho_N)) + c \varrho_N^{d-1+\epsilon_1} - c' \varrho_N^{d+2} \\
\leq \text{vol}(B(0; \varrho_N)) - c \varrho_N^{d-1+\epsilon_1},
\]
which contradicts (5.11). \( \blacksquare \)

**Remark 5.4** (Finer asymptotics of survival probability). There is a conjecture on the precise second order asymptotics of the survival probability in the literature: there exists \( a_1 > 0 \) such that
\[
\mathbb{P} \otimes \mathbb{P}(\tau_\mathcal{O} > N) = \exp \left\{ -c(d, p) N^{\frac{d}{d+4}} - a_1 N^{\frac{d}{d+4}} + o(N^{\frac{d}{d+4}}) \right\}.
\]
See [LS8] and the bottom of p. 76 in [HOP2] for more information. Lemma 5.4 gives a lower bound of this form while the currently best known upper bound is
\[
\mathbb{P} \otimes \mathbb{P}(\tau_\mathcal{O} > N) \leq \exp \left\{ -c(d, p) N^{\frac{d}{d+4}} + N^{\frac{d+4}{d+4}} \right\}
\]
for some \( \epsilon \in (0, 1) \). See [HOP2, (2.40)] or [SOS, Theorem 5.6 on page 208].

Based on what we have proved, we can get a refined upper bound on the survival probability. Theorem 2.8 implies that \( \lim_{N \to \infty} \mu_N(T \in T_{0+}) = 1 \) with
\[
T_{0+} := \left\{ T \subset \mathbb{Z}^d : B(x; 0.9\varrho_N) \subset T \subset B(x; \varrho_N + \varrho_N^{\epsilon_1}) \text{ for some } x \in B(0; \varrho_N) \text{ and } |\partial T| \leq \varrho_N^{d-1}(\log N)^a \right\},
\]
and...
Just as in (5.7), we have

\[ |T_{0+}| \leq \exp\{c\varrho_N^{d-1}(\log N)^{a+1}\} \tag{5.20} \]

and then by Lemma 2.4 and a variant of (5.11), we obtain

\[
\mathbb{P} \otimes \mathbb{P}(\tau_0 > N) \\
\sim \mathbb{P} \otimes \mathbb{P}(\tau_{\tau^c} > N, \tau \in T_{0+}) \\
\leq |T_{0+}| \sup_{T \in \tau_{0+}} |T|^{1/2} \exp\left\{ \text{vol}(\overline{T}) \log p - N\lambda_{B_{\overline{T}}} + c\varrho_N^{d-1}(\log N)^{a+1} \right\} \\
\leq \exp\left\{ \text{vol}(B(0; \varrho_N)) \log p - N\lambda_{B(0; \varrho_N)} + c\varrho_N^{d-1}(\log N)^{a+1} \right\} \\
= \exp\left\{ -c(d, \log(1/p))N \frac{d}{\varphi} + cN \frac{d-1}{\varphi} (\log N)^{a+1} \right\} .
\]

### A Estimates for eigenvalues and eigenfunctions

In this appendix, we collect some estimates on eigenvalues and eigenfunctions, including Lemma A.2. Recall that \(\lambda_T^{(k)}\) and \(\phi_T^{(k)}\) are the \(k\)-th smallest Dirichlet eigenvalue and corresponding eigenfunction with \(\|\phi_T^{(k)}\|_2 = 1\) for \(-\frac{1}{2d}\Delta\) in \(T \subset \mathbb{R}^d\) and \(\lambda_T^{RW,(k)}\) and \(\phi_T^{RW,(k)}\) are their discrete space counterparts.

We begin with the following comparison lemma which includes Lemma 5.1.

**Lemma A.1.** For any \(d \geq 1\) and \(k \geq 1\), there exists \(\gamma_k > 0\) such that for all sufficiently large \(R > 0\), the following bounds hold:

\[
\left| \lambda_T^{RW,(k)} - \lambda_B^{(k)(0;R)} \right| \leq \gamma_k R^{-3}, \tag{A.1}
\]

\[
\phi_T^{RW,(1)}(0) \geq \gamma_1^{-1} R^{-d/2}, \tag{A.2}
\]

and for any \(x \in \mathbb{Z}^d\) and \(T \supset B(x; \varrho_N/2)\),

\[
\lambda_T^{RW,(1)} \geq \lambda_T^{(1)} - \gamma_1 \varrho_N^{-4}. \tag{A.3}
\]

**Proof of Lemma A.1.** The first assertion can be found in [W55], (3.27) and (6.11)]. The second assertion follows from [B93, Lemma 2.1(b)] which states that

\[
\left| \phi_B^{(1)(0;R)}(0) - \phi_T^{RW,(1)}(0) \right| \leq cR^{-d/2-1}, \tag{A.4}
\]

and the fact that \(\phi_B^{(1)(0;R)}(0) \geq c^{-1} R^{-d/2}\). The third assertion follows from [W55, (6.9)], which states that

\[
\lambda_T^{(1)} \leq \lambda_T^{RW,(1)}(1 + c\varrho_N^{-2}), \tag{A.5}
\]

and the bound \(\lambda_T^{RW,(1)} \leq c\varrho_N^2\) that follows from the assumption \(T \supset B(x; \varrho_N/2)\).

Next we restate and prove (EV) and (EF).

**Lemma A.2.** There exist \(c_7, c_8 > 0\) such that if \(B(x; (1 - \epsilon)\varrho_N) \subset B \subset B(x; \varrho_N + \varrho_N^1)\) for some \(x \in \mathbb{Z}^d\) and \(\epsilon > 0\) sufficiently small depending only on the dimension \(d\), then the following bounds hold:

\[
\lambda_B^{RW,(2)} - \lambda_B^{RW,(1)} \geq c_7 \varrho_N^{-2}, \tag{EV}
\]

\[
\left\| \phi_B^{RW,(1)} \right\|_\infty \leq c_8 \varrho_N^{-d/2}. \tag{EF}
\]
Proof of Lemma A.2. In order to show the first assertion (A.4), recall first that the continuum eigenvalue satisfies the scaling relation $\lambda_{B(0; R)}^{(k)} = R^{-2} \lambda_{B(0; 1)}^{(k)}$. Combining this with (A.1), we get the following bounds:

$$\begin{align*}
\lambda_{B(0; 1)}^{\text{RW}, (1)} &\leq \lambda_{B(0; \epsilon N + \epsilon^3 N)}^{\text{RW}, (1)} \\
&\leq \lambda_{B(0; \epsilon N + \epsilon^3 N)}^{(1)} + \gamma \epsilon^3 N \\
&\leq \epsilon^{-2} \lambda_{B(0; 1)}^{(1)} + c \epsilon^3 N, \quad (A.6)
\end{align*}$$

and

$$\begin{align*}
\lambda_{B(0; 1)}^{\text{RW}, (2)} &\geq \lambda_{B(0; (1-\epsilon) \epsilon N)}^{\text{RW}, (2)} \\
&\geq \lambda_{B(0; (1-\epsilon) \epsilon N)}^{(2)} - \gamma \epsilon^3 N \\
&\geq \epsilon^{-2} \lambda_{B(0; 1)}^{(2)} - c \epsilon^3 N. \quad (A.7)
\end{align*}$$

Since $\lambda_{B(0; 1)}^{(1)} < \lambda_{B(0; 1)}^{(2)}$, the desired bound (A.4) follows for sufficiently small $\epsilon > 0$.

Next, we show the second assertion (A.5). By the eigenvalue equation for the semigroup, it follows for any $x \in B$ that

$$\begin{align*}
\phi_{B}^{\text{RW}, (1)}(x) &= \left(1 - \lambda_{B}^{\text{RW}, (1)}\right)^{-[1/\lambda_{B}^{\text{RW}, (1)}]} \sum_{y \in B} P_{B, 1/\lambda_{B}^{\text{RW}, (1)}}^{B}(x, y) \phi_{B}^{\text{RW}, (1)}(y) \\
&\leq \left(1 - \lambda_{B}^{\text{RW}, (1)}\right)^{-[1/\lambda_{B}^{\text{RW}, (1)}]} \left\|P_{B, 1/\lambda_{B}^{\text{RW}, (1)}}^{B}(x, \cdot)\right\|_{2} \left\|\phi_{B}^{\text{RW}, (1)}\right\|_{2}, \quad (A.8)
\end{align*}$$

where we have used the Schwarz inequality in the second line. Then by the symmetry of the transition kernel $P$, the Chapman–Kolmogorov identity and the normalization $\left\|\phi_{B}^{(1)}\right\|_{2} = 1$, we can further rewrite (A.8) as

$$\begin{align*}
\left|\phi_{B}^{\text{RW}, (1)}(x)\right| &\leq \left(1 - \lambda_{B}^{\text{RW}, (1)}\right)^{-[1/\lambda_{B}^{\text{RW}, (1)}]} \left\|P_{B, 1/\lambda_{B}^{\text{RW}, (1)}}^{B}(x, x)\right\|_{2}^{1/2} \\
&\leq c \left(\lambda_{B}^{\text{RW}, (1)}\right)^{d/4}. \quad (A.9)
\end{align*}$$

Since we have $\lambda_{B}^{\text{RW}, (1)} \geq c \epsilon_{N}^{-2}$ similarly to (A.10), the last line is bounded by $c \epsilon_{N}^{-d/2}$. ■

Proof of Lemma 5.2. Note first that by the eigenfunction expansion, we have

$$\begin{align*}
P \otimes P (\tau_{\mathcal{O}} > N) \\
&\geq P(\mathcal{O} \cap B(0; \epsilon N) = \emptyset) P(\tau_{B(0; \epsilon N)} > N) \\
&= P(B(0; \epsilon N)) \sum_{1 \leq k \leq |B(0; \epsilon N)|} \left(1 - \lambda_{B(0; \epsilon N)}^{\text{RW}, (k)}\right)^{N} \left\langle \phi_{B(0; \epsilon N)}^{\text{RW}, (k)}, 1 \right\rangle \phi_{B(0; \epsilon N)}^{\text{RW}, (k)}(0). \quad (A.10)
\end{align*}$$

Using (A.4), $1 - \lambda \geq \exp\{-\lambda - \lambda^2\}$ for small $\lambda > 0$, (A.2) and $\phi_{B(0; \epsilon N)}^{\text{RW}, (1)} \geq 0$, we can bound the term with $k = 1$ in the above sum from below by

$$\left(1 - \lambda_{B(0; \epsilon N)}^{\text{RW}, (1)}\right)^{N} \phi_{B(0; \epsilon N)}^{\text{RW}, (1)}(0)^{2} \geq \exp\{-N \lambda_{B(0; \epsilon N)}^{\text{RW}, (1)} - cN \epsilon_{N}^{-3}\}. \quad (A.11)$$
On the other hand, we can use (E.1), (A.6) and $1 - \lambda \leq \exp\{-\lambda\}$ to bound the sum over $k \geq 2$ by

$$
\left| \sum_{2 \leq k \leq |B(0; \varrho N)|} \left( 1 - \lambda^{\text{RW},(k)}_{B(0; \varrho N)} \right)^N \left\langle \phi^{\text{RW},(k)}_{B(0; \varrho N)}, \phi^{\text{RW},(k)}_{B(0; \varrho N)}(0) \right\rangle \right|
\leq \exp \left\{ -N \lambda^{\text{RW},(2)}_{B(0; \varrho N)} \left( \sum_{k \geq 2} \left\langle \phi^{\text{RW},(k)}_{B(0; \varrho N)}, \phi^{\text{RW},(k)}_{B(0; \varrho N)}(0) \right\rangle \right)^2 \right\}^{1/2} \left( \sum_{k \geq 2} \phi^{\text{RW},(k)}_{B(0; \varrho N)}(0) \right)^{1/2} \tag{A.12}
$$

$$
= \exp \left\{ -N \lambda^{\text{RW},(1)}_{B(0; \varrho N)} - cN\varrho^2 \right\} \|1_{B(0; \varrho N)}\|_2 \|1(0)\|_2.
$$

This is negligible compared with (A.11) and hence we obtain

$$
\mathbb{P} \otimes \mathbb{P} (\tau_0 > N) \geq \exp \left\{ |B(0; \varrho N)| \log(1/p) - N \lambda^{\text{RW},(1)}_{B(0; \varrho N)} - cN\varrho^{-3} \right\}. \tag{A.13}
$$

Finally, substituting the bound $||B(0; \varrho N)| - \text{vol}(B(0; \varrho N))|| \leq c\varrho^{-d-1}$ and (A.11) into the above, we arrive at the desired bound.

Acknowledgement J. Ding is supported by NSF grant DMS-1757479 and an Alfred Sloan fellowship. R. Fukushima is supported by JSPS KAKENHI Grant Number 16K05200. R. Sun is supported by NUS Tier 1 grant R-146-000-253-114.

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