Nash Equilibrium Problems

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Lecture Notes for
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Given a closed set $K \subseteq \mathbb{R}^n$ and a mapping $F : K \to \mathbb{R}^n$, the variational inequality, denoted $\text{VI}(K, F)$, is to find a vector $x \in K$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall y \in K.$$ 

The set of solutions to this problem is denoted $\text{SOL}(K, F)$. 
Given a mapping $F : \mathbb{R}_+^n \to \mathbb{R}^n$, the nonlinear complementarity problem, denoted NCP($F$), is to find a vector $x \in \mathbb{R}_+^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \text{and} \quad x^T F(x) = 0.$$  

The relationship of VI($\mathbb{R}_+^n, F$) and NCP($F$):

**Thm.** $x$ solves NCP($F$) $\iff$ $x \in \text{SOL}(\mathbb{R}_+^n, F)$.
Nash Equilibrium Problems:

Suppose that there are $N$ players in a noncooperative game. Each player has a cost function and strategy set that may depend on the other players’ actions.

For simplicity, we assume that player $i$’s strategy set is $K_i \subseteq \mathbb{R}^{n_i}$ which is independent of the other players’ actions. Player $i$’s cost function $\theta(x)$ depends on all players’ strategies, where

$$ x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{bmatrix}, \quad x^i \in \mathbb{R}^{n_i}. $$
Player $i$’s strategy is to minimize his cost function for any given tuple of other players’ strategies. That is,
\[ \min \, \theta_i(y^i, \tilde{x}^i) \]

s.t. \[ y^i \in K_i. \]
Denote the solution set of the above optimization problem by $S_i(\tilde{x}^i)$. This solution set depends on $\tilde{x}^i$, the other players’ strategies.

A **Nash equilibrium** is a tuple of strategies $x = (x^i : i = 1, 2, \ldots N)$ with the property that

$$\text{for each } i, \ x^i \in S_i(\tilde{x}^i).$$

In words, a Nash equilibrium is a tuple of strategies, one for each player, such that no player can lower the cost by unilaterally deviating his action from his designated strategy.
We have the following theorem.

**Theorem.** Let each $K_i \subseteq \mathbb{R}^{n_i}$ be a closed convex set. Suppose that for each fixed tuple $\tilde{x}^i$, the function $\theta_i(y^i, \tilde{x}^i)$ is convex and continuously differentiable in $y^i$. Then a tuple $x \equiv (x^i : i = 1, 2, \ldots N)$ is a Nash equilibrium if and only if $x \in \text{SOL}(K, F)$, where

$$K = \prod_{i=1}^{N} K_i \quad \text{and} \quad F(x) = (\nabla_{x^i} \theta_i(x))_{i=1}^{N}.$$
Proof: “$$\implies$$” By convexity, we know that $x$ is a Nash equilibrium if and only if for each $i = 1, 2, \ldots, N$, $x^i$ solves the following individual VI($K_i, \nabla_{x^i} \theta_i(x)$):

$$(y^i - x^i)^T \nabla_{x^i} \theta_i(x) \geq 0, \quad \forall y^i \in K_i.$$ 

\[ \downarrow \]

\[ x \in \text{SOL}(K, F). \]
Conversely, if \( x \equiv (x^i : i = 1, 2, \ldots N) \) solves the \( \text{VI}(K, F) \), then

\[
(y - x)^T F(x) \geq 0, \quad \forall y \in K.
\]

In particular, by taking
\[ y = \begin{bmatrix}
    x^1 \\
    x^2 \\
    \vdots \\
    x^{i-1} \\
    y^i \\
    x^{i+1} \\
    \vdots \\
    x^N
\end{bmatrix}, \text{ replacing } x^i \text{ by } y^i, \]
we know that the above VI becomes

\[(y^i - x^i)^T \nabla_{x^i} \theta_i(x) \geq 0, \quad \forall y^i \in K_i.\]

Hence, by convexity, \(x^i\) solves

\[
\min \theta(y^i, \tilde{x}^i) \\
\text{s.t.} \quad y^i \in K_i.
\]

Sometimes, the Nash equilibrium problem is called an “\(N\)-person nonzero-sum game”.
An Example in Nash Equilibrium
Define $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as follows

$$L(x, y) = p^T x + q^T y + \frac{1}{2} x^T P x + x^T R y - \frac{1}{2} y^T Q y, \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}^m,$$

where $p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$, $Q = Q^T > 0$, $Q \in \mathbb{R}^{m \times m}$, and $R \in \mathbb{R}^{n \times m}$. Let $X$ be a closed convex set in $\mathbb{R}^n$ and $Y$ be a closed convex set in $\mathbb{R}^m$. For each $x \in \mathbb{R}^n$, let

$$\varphi(x) = \max_{v \in \mathbb{R}^m} L(x, v)$$

and for each $y \in \mathbb{R}^m$, let

$$\phi(y) = \min_{u \in \mathbb{R}^n} L(u, y).$$
Suppose that there are two players in a noncooperative game. Player 1’s strategy, for each fixed but arbitrary player 2’s strategy $y \in Y$, is to

$$\min \quad \theta_1(x, y)$$

s.t. $x \in X$, 

where $\theta_1(x, y) = \varphi(x) + x^T S_1 y$ and $S_1 \in \mathbb{R}^{n \times m}$. Player 2’s strategy, for each fixed but arbitrary player 1’s strategy $x \in X$, is to

$$\min \quad \theta_2(x, y)$$

s.t. $y \in Y$, 

where $\theta_2(x, y) = -\phi(y) + x^T S_2 y$ and $S_2 \in \mathbb{R}^{n \times m}$. 
(i) Give explicit formulas of \( \varphi(x) \) and \( \phi(y) \) for each \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \).

(ii) Show that \( \theta_1(x, y) \) is a convex function in \( x \) and \( \theta_2(x, y) \) is a convex function in \( y \).

(iii) Model the problem of finding a Nash equilibrium of the above two players’ noncooperative game as a variational inequality problem.