Integer Programming Models for MA4260 Model Building in Operations Research

Defeng Sun
Department of Mathematics
National University of Singapore
Republic of Singapore
Office: S14-04-25; Phone: 6516-3343
Email: matsundf@nus.edu.sg

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Refer to Chapter 9, H. P. Williams’ Book:

I. The Use of Discrete Variables

Integer variables may serve a number of purposes

- Indivisible (Discrete) Variables: number of aeroplanes, cars, houses or people.

- Decision variables: $\delta = 0, 1$;

$$\gamma = \begin{cases} 
0 & \text{not built} \\
1 & \text{depot of type A built} \\
2 & \text{depot of type B built}
\end{cases}$$
Indicator variables
\( \delta \) — \( \{0, 1\} \) variable, linked to continuous variables:

\( x \) — the quantity of an ingredient to be included in a blend.

\( \delta \) — indicator variable to distinguish the state where \( x = 0 \) and the state where \( x > 0 \).

By introducing the following constraint we can force \( \delta \) to take the value 1 when \( x > 0 \):

\[
x - M\delta \leq 0,
\]

where \( M \) is a constant coefficient representing a known upper bound for \( x \).
Logically, we have achieved the condition

\[ x > 0 \rightarrow \delta = 1. \]  \hspace{1cm} (2)

In some cases, we may also wish to impose the condition

\[ x = 0 \rightarrow \delta = 0, \]  \hspace{1cm} (3)

or equivalently

\[ (3) \iff \delta = 1 \rightarrow x > 0. \]  \hspace{1cm} (4)
(2) + (3) [or (4)] impose the condition

\[ \delta = 1 \leftrightarrow x > 0. \]  \hspace{1cm} (5)

Realistically, a threshold imposed:

\[ (4) \Rightarrow \delta = 1 \rightarrow x \geq m > 0. \]  \hspace{1cm} (6)

The above condition can be imposed by the constraint

\[ x - m\delta \geq 0. \]  \hspace{1cm} (7)
Example 1. The Fixed Charge Problem

$x$ — quantity of a product at a marginal cost per unit of $c_1$.

$c_2$ — set-up cost if the product is manufactured at all.

$x = 0$, the total cost $= 0$.

$x > 0$, the total cost $= c_1x + c_2$. [Not continuous at $x = 0$.]
Introduce an indicator variable $\delta$ so that if any of the product is manufactured $\delta = 1$:

$$\text{total cost} = c_1 x + c_2 \delta$$

with constraints

$$x - M\delta \leq 0, \quad \delta \in \{0, 1\}.$$  

[By using (1)]
Example 2. Blending

$x_A$ — the proportion of ingredient $A$ to be included in a blend.

$x_B$ — the proportion of ingredient $B$ to be included in a blend.

Extra condition

A is included $\implies$ B must be included.
Introduce a 0 − 1 indicator variable: \( \delta = 1 \) if \( x_A > 0 \):

\[
x_A - \delta \leq 0. \quad (8)
\]

[A special case of (8).]

Use \( \delta \) to impose the condition

\[
\delta = 1 \rightarrow x_B > 0. \quad (9)
\]
Choose some proportionate level $m$ (say 1/100) below which we will regard $B$ as out of the blend. This gives

$$x_B - m\delta \geq 0.$$  \hspace{1cm} (10)

Hence, two extra conditions (8) and (10) are imposed by introducing the 0 − 1 variable $\delta$. 
An extension of the extra condition:

A is included $\iff$ B is also included.

In addition to (8) and (10), we need to impose two more constraints:

$$x_B - \delta' \leq 0$$

and

$$x_A - m\delta' \geq 0,$$

where $\delta'$ is another $0 - 1$ indicator variable: $\delta' = 1$ if $x_B > 0$: 
• Use indicator variables to show whether an inequality holds or does not hold.

Indicate

\[ \sum_j a_j x_j \leq b \]

holds or does not hold.

First, model

\[ \delta = 1 \rightarrow \sum_j a_j x_j \leq b. \] (11)
\( (1 - \delta) = 0 \rightarrow \sum_j a_j x_j - b \leq 0. \)

Thus, it can be represented by the constraint

\[
\sum_j a_j x_j - b \leq M(1 - \delta)
\]

i.e.,

\[
\sum_j a_j x_j + M\delta \leq M + b, \tag{12}
\]

where \( M \) is an upper bound for the expression \( \sum_j a_j x_j - b \). [When \( \delta = 0 \), no constraint imposed because \( M \geq \sum_j a_j x_j - b \).]
We will now consider how to model the reverse of the constraint (11), i.e.,

\[ \sum_{j} a_j x_j - b \leq 0 \rightarrow \delta = 1. \]  

(13)

This is conveniently expressed as

\[ \delta = 0 \rightarrow \sum_{j} a_j x_j - b \not\leq 0, \]  

(14)

i.e.,

\[ \delta = 0 \rightarrow \sum_{j} a_j x_j - b > 0. \]  

(15)
We must rewrite

$$\sum_j a_jx_j - b > 0 \text{ as } \sum_j a_jx_j - b \geq \varepsilon,$$

where $\varepsilon$ is some small tolerance value beyond which we will regard the constraint as having been broken.

[Should be coefficients $a_j$ be integers as well as the variables $x_j$, as often happens in the type of situation, there is no difficulty as $\varepsilon$ can be taken as 1.]
(15) may now be written as

\[ \delta = 0 \rightarrow - \sum_{j} a_j x_j + b + \varepsilon \leq 0, \tag{16} \]

which can be modelled as

\[ - \sum_{j} a_j x_j + b + \varepsilon \leq (-m + \varepsilon)\delta, \]
i.e.,

$$\sum_j a_jx_j - (m - \varepsilon)\delta \geq b + \varepsilon,$$

where $m$ is a lower bound for expression

$$\sum_j a_jx_j - b.$$
Should we wish to indicate whether a “≥” inequality such as

\[ \sum_j a_j x_j \geq b \]

holds or not by means of an indicator variable \( \delta \), the required constraint can easily be obtained by transforming the above constraint into a “≤” form. The corresponding constraint (12) and (17) above are

\[ \sum_j a_j x_j + m\delta \geq m + b, \quad (18) \]
\[ \sum_{j} a_j x_j - (M + \varepsilon)\delta \leq b - \varepsilon, \] (19)

where \( m \) and \( M \) are again lower and upper bounds respectively on the expression

\[ \sum_{j} a_j x_j - b. \]
Finally, to use an indicator variable $\delta$ for an “$=$” constraint such as

$$\sum_{j} a_j x_j = b$$

is slightly more complicated. We can use $\delta = 1$ to indicate the “$\leq$” and “$\geq$” cases to hold simultaneously. This is done by stating both (12) and (18) together.

If $\delta = 0$, we want to force either “$\leq$” or “$\geq$” constraint to be broken. This may be done by expressing (17) and (19) with two indicator variables $\delta'$ and $\delta''$ giving
\[ \sum_j a_j x_j - (m - \varepsilon)\delta' \geq b + \varepsilon, \quad (20) \]

\[ \sum_j a_j x_j - (M + \varepsilon)\delta'' \leq b - \varepsilon. \quad (21) \]

The indicator variable \( \delta \) forces the required condition by the extra constraint

\[ \delta' + \delta'' - \delta \leq 1. \quad (22) \]
Example 3. Use a $0 - 1$ variable $\delta$ to indicate whether or not the following constraint is satisfied:

$$2x_1 + 3x_2 \leq 1.$$ 

$[x_1, x_2 \geq 0$ and $x_1, x_2 \leq 1$, continuous.$]$ 

$$\delta = 1 \rightarrow 2x_1 + 3x_2 - 1 \leq 0,$$

$$2x_1 + 3x_2 - 1 \leq M(1 - \delta)$$

with $M = 4$.

$$\delta = 0 \rightarrow 2x_1 + 3x_2 - 1 \geq \varepsilon(= 0.01),$$

$$\delta = 0 \rightarrow 2x_1 + 3x_2 - 1 - \varepsilon \geq 0,$$
which can be represented by

$$2x_1 + 3x_2 - 1 - \varepsilon \geq \bar{m}\delta,$$

where $\bar{m}$ is an lower bound of $2x_1 + 3x_2 - 1 - \varepsilon$. By simple calculation, $\bar{m} = -1 - \varepsilon = -1.01$. Thus, we obtain

$$2x_1 + 3x_2 - 1.01\delta \geq 1.01.$$