First order optimality conditions for mathematical programs with semidefinite cone complementarity constraints

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Abstract

In this paper we consider a mathematical program with semidefinite cone complementarity constraints (SDCMPCC). Such a problem is a matrix analogue of the mathematical program with (vector) complementarity constraints (MPCC) and includes MPCC as a special case. We derive explicit expressions for the strong-, Mordukhovich- and Clarke- (S-, M- and C-)stationary conditions and give constraint qualifications under which a local solution of SDCMPCC is a S-, M- and C-stationary point.

Key words: mathematical program with semidefinite cone complementarity constraints, necessary optimality conditions, constraint qualifications, S-stationary conditions, M-stationary conditions, C-stationary conditions.

AMS subject classification:49K10, 49J52, 90C30, 90C22, 90C33

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1 Introduction

Let $\mathcal{S}^n$ be the linear space of all $n \times n$ real symmetric matrices equipped with the usual Frobenius inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{S}^n_+$ ($\mathcal{S}^n_-$) be the closed convex cone of all $n \times n$ positive (negative) semidefinite matrices in $\mathcal{S}^n$. In this paper we study first order necessary optimality conditions for the *mathematical program with (semidefinite) cone complementarity constraints* (MPSCCC or SDCMPCC):

$$
\begin{array}{lll}
\text{(SDCMPCC)} & \min & f(z) \\
& \text{s.t.} & h(z) = 0\,, \\
& & g(z) \leq 0\,, \\
& & \mathcal{S}^n_+ \ni G(z) \perp H(z) \in \mathcal{S}^n_-\,,
\end{array} \tag{1}
$$

where Z is a finite dimensional space, $f : Z \to \Re$, $G : Z \to \mathcal{S}^n$, $H : Z \to \mathcal{S}^n$, $h : Z \to \Re^p$, $g : Z \to \Re^q$ are continuously differentiable mappings, and "$G(z) \perp H(z)$" means that the matrices $G(z)$ and $H(z)$ are perpendicular to each other, i.e., $\langle G(z), H(z) \rangle = 0$.

SDCMPCC can be considered as a matrix analogue of the mathematical program with (vector) complementarity constraints (MPCC) since when the semidefinite cone complemenatrity constraint (1) is replaced by the vector complementarity constraint $\Re^n_+ \ni G(z) \perp H(z) \in \Re^n_-$, it becomes a MPCC. MPCC is a class of very important problems since they arise frequently in applications where the constraints come from equilibrium systems and hence is also known as the mathematical program with equilibrium constraints (MPEC); see [15, 21] for references. One of the main sources of MPCCs comes from bilevel programming problems which have numerous applications; see [6]. For simplicity in this paper we include only one semidefinite cone complementarity constraint. However all results can be generalized to the case of more than one semidefinite cone complementarity constraints in a straightforward manner. Therefore we may consider MPCC as a special case of SDCMPCC with the following n semidefinite cone complementarity constraints:

$$
\Re_+ \ni G_i(z) \perp H_i(z) \in \Re_-, \qquad i = 1, \ldots, n\,.
$$

The generalization from MPCC to SDCMPCC has very important applications. In practice it is more realistic to assume that an optimization problem involves uncertainty. A recent approach to optimization under uncertainty is robust optimization. For example, it makes sense to consider a robust bilevel programming problem where for a fixed upper level decision variable x, the lower level problem is replaced by its robust counterpart:

$$
P_x : \qquad \min_y \{ f(x, y, \zeta) : g(x, y, \zeta) \leq 0 \quad \forall\, \zeta \in \mathcal{U} \}\,,
$$

where $\mathcal{U}$ is some "uncertainty set" in the space of the data. It is well-known (see [1]) that if the uncertainty set $\mathcal{U}$ is given by a system of linear matrix inequalities, then the deterministic counterpart of the problem P_x is a semidefinite program. If this semidefinite program can be equivalently replaced by its Karush-Kuhn-Tucker (KKT) condition, then it yields a SDCMPCC.

MPCC is notoriously known as a difficult class of optimization problems since Mangasarian Fromovitz constraint qualification (MFCQ) fails to hold at each feasible point of the feasible region; see [41, Proposition 1.1]. One of the implications of the failure of MFCQ is that the classical KKT condition may not hold at a local optimizer. The classical KKT condition for MPCC is known to be equivalent to Strong (S-) stationary condition. Consequently weaker stationary conditions such as Mordukhovich stationary condition (M-stationary condition) and Clarke stationary condition (C-stationary condition) have been proposed and the constraint qualifications under which a local minimizer is a M-(C-)stationary point have been studied; see e.g. [39, 30] for a detailed discussion. For SDCMPCC, the usual constraint qualification is Robinson's CQ. In this paper we show that Robinson's CQ fails to hold at each feasible point of the SDCMPCC. Hence SDCMPCC is also a difficult class of optimization problems. One of the implications of the failure of Robinson's CQ is that the classical KKT condition may not hold at a local optimizer. In this paper we introduce the concepts of S-, M- and C-stationary conditions for SDCMPCC and derive exact expressions for S-, M- and C-stationary conditions. Under certain constraint qualifications we show that a local minimizer of SDCMPCC is a S-, M- and C-stationary point.

To the best of our knowledge, this is the first time explicit expressions for S-, M- and C-stationary conditions for SDCMPCC are given. In [36], a smoothing algorithm is given for mathematical program with symmetric cone complementarity constraints and the convergence to C-stationary points is shown. Although the problem studied in [36] may include our problem as a special case, there is no explicit expression for C-stationary condition given.

We organize our paper as following. In §2 we introduce the preliminaries and preliminary results on the background in variational analysis, first order conditions for a general problem and background in variational analysis in matrix spaces. In §3, we give the precise expressions for the proximal and limiting normal cones of the graph of the normal cone $N_{\mathcal{S}^n_+}$. In §4, we show that the Robinson's CQ fails at every feasible solution of SDCMPCC and derive the classical KKT condition under the Clarke calmness condition. Explicit expressions for S-stationary conditions are given in §5 where it is also shown that the classical KKT condition implies the S-stationary condition. Explicit expressions for M- and C-stationary conditions are given in §6 and §7 respectively.

2 Preliminaries and Preliminary Results

2.1 Background in variational analysis

In this subsection we summarize some background materials on variational analysis which will be used throughout the paper. Detailed discussions on these subjects can be found in [4, 5, 18, 19, 29]. In this subsection X is a finite dimensional space.

Definition 2.1 *(see e.g. [5, Proposition 1.5(a)] or [29, page 213]) Let Ω be a nonempty subset of X. Given $\bar{x} \in \operatorname{cl}\Omega$, the closure of set Ω, the following convex cone*

$$N^{\pi}_{\Omega}(\bar{x}) := \{\zeta \in X : \exists\, M > 0, \text{ such that } \langle \zeta, x - \bar{x}\rangle \leq M\|x - \bar{x}\|^2 \quad \forall\, x \in \Omega\} \tag{2}$$

is called the proximal normal cone to set Ω at point $\bar{x}$.

Definition 2.2 *(see e.g. [5, page 62 and Theorem 6.1(b)]) Let Ω be a nonempty subset of X. Given $\bar{x} \in \operatorname{cl}\Omega$, the following closed cone*

$$N_{\Omega}(\bar{x}) := \{\lim_{i\to\infty} \zeta_i : \zeta_i \in N^{\pi}_{\Omega}(x_i), \quad x_i \to \bar{x}, \quad x_i \in \Omega\} \tag{3}$$

is called the limiting normal cone (also known as Mordukhovich normal cone or basic normal cone) to set Ω *at point* $\bar{x}$ *and the closed convex hull of the limiting normal cone*

$$N^c_\Omega(\bar{x}) := \operatorname{clco} N_\Omega(\bar{x}) \,,$$

where $\operatorname{clco} C$ *denotes the closure of the convex hull of set* C*, is the Clarke normal cone ([4]) to set* Ω *at point* $\bar{x}$*.*

Alternatively in a finite dimensional space, the limiting normal cone can be also defined by the Fréchet (also called regular) normal cone instead of the proximal normal cone, see [18, Definition 1.1 (ii)]. In the case when Ω is convex, the proximal normal cone, the limiting normal cone and the Clarke normal cone coincide with the normal cone in the sense of the convex analysis [28], i.e.,

$$N_\Omega(\bar{x}) := \{\zeta \in X : \langle \zeta, x - \bar{x}\rangle \leq 0 \quad \forall\, x \in \Omega\} \;.$$

Definition 2.3 *Let* $f : X \to \Re \cup \{+\infty\}$ *be a lower semicontinuous function and finite at* $\bar{x} \in X$*. The proximal subdifferential ([29, Definition 8.45]) of* f *at* $\bar{x}$ *is defined as*

$$\begin{aligned} \partial^\pi f(\bar{x}) \;\; := \;\; & \{\zeta \in X : \exists\, \sigma > 0, \delta > 0 \text{ such that } f(x) \geq f(\bar{x}) + \langle \zeta, x - \bar{x}\rangle - \sigma\|x - \bar{x}\|^2 \\ & \forall\, x \in B(\bar{x}, \delta)\} \end{aligned}$$

and the limiting (Mordukhovich or basic [18]) subdifferential of f *at* $\bar{x}$ *is defined as*

$$\partial f(\bar{x}) := \{\lim_{k\to\infty} \zeta_k : \xi_k \in \partial^\pi f(x_k),\; x_k \to \bar{x},\; f(x_k) \to f(\bar{x})\} \,.$$

When f *is Lipschitz continuous near* $\bar{x}$*,*

$$\partial^c f(\bar{x}) := \operatorname{co} \partial f(\bar{x})$$

is the Clarke subdifferential [4] of f *at* $\bar{x}$*.*

Note that in a finite dimensional space, alternatively the limiting subgradient can be also constructed via Fréchet subgradients (also known as regular subgradients), see [18, Theorem 1.89]. The equivalence of the two definitions is well-known, see the commentary by Rockafellar and Wets [29, page 345]. In the case when f is convex and locally Lipschitz, the proximal subdifferential, the limiting subdifferential and the Clarke subdifferential coincide with the subdifferential in the sense of convex analysis [28]. In the case when f is strictly differentiable, the limiting subdifferenial and the Clarke subdifferential reduce to the classical derivative $f'(\bar{x})$, i.e., $\partial^c f(\bar{x}) = \partial f(\bar{x}) = \{f'(\bar{x})\}$.

2.2 First order optimality conditions for a general problem

In this subsection we discuss constraint qualifications and first order necessary optimality conditions for the following general optimization problem:

$$\begin{aligned} (GP) \qquad \min \quad & f(z) \\ \text{s.t.} \quad & h(z) = 0\,, \\ & g(z) \leq 0\,, \\ & G(z) \in K\,, \end{aligned}$$

where Y, Z are finite dimensional spaces, K is a closed subset of Y, $f : Z \to \Re$, $h : Z \to \Re^p$, $g : Z \to \Re^q$ and $G : Z \to K$ are locally Lipschitz mappings.

We denote the set of feasible solutions for (GP) by $\mathcal{F}$ and the perturbed feasible region by

$$\mathcal{F}(r, s, P) := \{z \in Z : h(z) + r = 0, \quad g(z) + s \leq 0, \quad G(z) + P \in K\}.$$

Then $\mathcal{F}(0, 0, 0) = \mathcal{F}$. The following definition is the Clarke calmness [4] adapted to our setting.

Definition 2.4 (Clarke calmness) *We say that problem (GP) is (Clarke) calm at a local optimal solution $\bar{z}$ if there exist positive ε and μ such that, for all (r, s, P) in εB, for all $z \in (\bar{z} + \varepsilon B) \cap \mathcal{F}(r, s, P)$, one has*

$$f(z) - f(\bar{z}) + \mu\|(r, s, P)\| \geq 0.$$

The following equivalence is obvious.

Proposition 2.1 *Problem (GP) is Clarke calm at a local optimal solution $\bar{z}$ if and only if $(\bar{z}, G(\bar{z}))$ is a local optimal solution to the penalized problem for some $\mu > 0$:*

$$\begin{array}{lll} (GP)_\mu & \min\limits_{z,X} & f(z) + \mu(\|h(z)\| + \|\max\{g(z), 0\}\| + \|G(z) - X\|) \\ & \text{s.t.} & X \in K. \end{array}$$

Theorem 2.1 *Let $\bar{z}$ be a local optimal solution of (GP). Suppose that (GP) is Clarke calm at $\bar{z}$. Then there exist $\lambda^h \in \Re^p$, $\lambda^g \in \Re^q$ and $\Omega^G \in \mathcal{S}^n$ such that*

$$\begin{aligned} & 0 \in \partial f(\bar{z}) + \partial\langle h, \lambda^h\rangle(\bar{z}) + \partial g(\bar{z})^* \lambda^g + \partial\langle G, \Omega^G\rangle(\bar{z}), \\ & \lambda^g \geq 0, \quad \langle g(\bar{z}), \lambda^g\rangle = 0, \\ & \Omega^G \in N_K(G(\bar{z})). \end{aligned}$$

Proof. The results follow from applying the limiting subdifferential version of the generalized Lagarange multiplier rule (see e.g. [19, Proposition 5.3]), calculus rules for limiting subdifferentials in particular the chain rule in [20, Proposition 2.5 and Corollary 6.3]). ■

The calmness condition involves both the constraint functions and the objective function. It is therefore not a constraint qualification in classical sense. Indeed it is a sufficient condition under which KKT type necessary optimality conditions hold. The calmness condition may hold even when the weakest constraint qualification does not hold. In practice one often uses some verifiable constraint qualifications sufficient to the calmness condition.

Definition 2.5 *(Calmness of a set-valued map) A set-valued map $\Phi : X \rightrightarrows Y$ is said to be calm at a point $(\bar{z}, \bar{v}) \in \operatorname{gph}\Phi$ if there exist a constant $M > 0$ and a neighborhood U of $\bar{z}$, a neighborhood V of $\bar{v}$ such that*

$$\Phi(z) \cap V \subseteq \Phi(\bar{z}) + M\|z - \bar{z}\|B \quad \forall\, z \in U.$$

Although the term "calmness" was coined in [29], the concept of calmness of a set-valued mapp was first introduced by Ye and Ye in [40] under the term "pseudo upper-Lipschitz continuity" which comes from the fact that it is a combination of Aubin's pseudo Lipschitz continuity [13] and Robinson's upper-Lipschitz continuity [24, 25]. For recent discussion on the properties and the criterion of calmness of a set-valued mapping, see Henrion and Outrata ([11, 12]). In what follows, we consider the calmness of the perturbed feasible region $\mathcal{F}(r,s,P)$ at $(r,s,P)=(0,0,0)$ to establish the Clarke calmness of the problem.

The proposition below is an easy consequence of Clarke's exact penalty principle [4, Proposition 2.4.3] and the calmness of the perturbed feasible region of the problem. See [38, Proposition 4.2] for a proof.

Proposition 2.2 *If the objective function of (GP) is Lipschitz near $\bar z \in Z$ and the perturbed feasible region of the constraint system $\mathcal{F}(r,s,P)$ defined as in (2.2) is calm at $(0,0,0,\bar z)$, then the problem (GP) is Clarke calm at $\bar z$.*

From the definition it is easy to verify that the set-valued mapping $\mathcal{F}(r,s,P)$ is calm at $(0,0,0,\bar z)$ if and only if there exists a constant $M>0$ and U, a neighborhood of $\bar z$, such that

$$\mathrm{dist}\,(z,\mathcal{F}) \le M\|(r,s,P)\| \quad \forall\, z \in U \cap \mathcal{F}(r,s,P)\,.$$

The above property is also referred to the existence of a local error bound for the feasible region $\mathcal{F}$. Hence any results on the existence of a local error bound of the constraint system may be used as a sufficient condition for calmness of the perturbed feasible region (see e.g. Wu and Ye [35] for such sufficient conditions).

By virtue of Proposition 2.2, the following four constraint qualifications are stronger than the Clarke calmness of (GP) at a local minimizer when the objective function of the problem (GP) is Lipschitz continuous.

Proposition 2.3 *Let $\mathcal{F}(r,s,P)$ be defined as in (2.2) and $\bar z \in Z$. Then the set-valued map $\mathcal{F}(r,s,P)$ is calm at $(0,0,0,\bar z)$ under one of the following constraint qualifications:*

(i) There is no singular Lagrange multimplier for problem (GP) at $\bar z$:

$$0 \in \partial\langle h,\lambda^h\rangle(\bar z) + \partial g(\bar z)^*\lambda^g + \partial\langle G,\Omega^G\rangle(\bar z), \quad \Omega^G \in N_K(G(\bar z)) \quad \Longrightarrow \quad (\lambda^h,\lambda^g,\Omega^G)=0\,,$$

where $\mathcal{A}^*$ denotes the adjoint of a linear operator $\mathcal{A}$.

(ii) Robinson's CQ ([26]) holds at $\bar z$: h, g and G are continuously differentiable at $\bar z$. K is a closed convex cone with a nonempty interior. The gradients $\nabla h_i(\bar z) := h_i'(\bar z)^*$ $(i=1,\ldots,p)$ are linearly independent and there exists a vector $d \in Z$ such that

$$\begin{aligned} &h_i'(\bar z)d = 0, \quad i=1,\ldots,p\,,\\ &g_i'(\bar z)d < 0, \quad i \in I_g(\bar z)\,,\\ &G(\bar z) + G'(\bar z)d \in \mathrm{int}\,K\,, \end{aligned}$$

where $I_g(\bar z) := \{i : g_i(\bar z)=0\}$ is the index of active inequality constraints.

(iii) Linear Independence Constraint Qualification (LICQ) holds at $\bar z$:

$$0 \in \partial\langle h,\lambda^h\rangle(\bar z) + \partial g(\bar z)^*\lambda^g + \partial\langle G,\Omega^G\rangle(\bar z) \quad \Longrightarrow \quad (\lambda^h,\lambda^g,\Omega^G)=0\,.$$

(iv) h, g and G are affine mapping and the set K is a union of finitely many polyhedral convex sets.

Proof. It is obvious that (iii) implies (i). By [3, Propositions 3.16 (ii) and 3.19 (iii)], Robinson's CQ (ii) is equivalent to (i) when all functions h, g, G are continuously differentiable and K is a closed convex cone with a nonempty interior. By Mordukhovich's criteria for pseudo-Lipschitz continuity, (i) implies that the set-valued map $\mathcal{F}(r,s,P)$ is pseudo-Lipschitz continuous around $(r,s,P) = (0,0,0)$ (see e.g. [20, Theorem 6.1]) and hence calm. By Robinson [27], (iv) implies the upper-Lipschitz continuity and hence the calmness of the set-valued map $\mathcal{F}(r,s,P)$ at $(0,0,0,\bar{z})$. ■

Combining Theorem 2.1 and Proposition 2.3, we have the following KKT conditions.

Theorem 2.2 *Let $\bar{z}$ be a local optimal solution of (GP). Suppose either the problem is Clarke calm at $\bar{z}$ or one of the constraint qualifications in Proposition 2.3 holds. Then the KKT condition in Theorem 2.1 holds at $\bar{z}$.*

2.3 Background in variational analysis in matrix spaces

Let $A \in \mathcal{S}^n$ be given. We use $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ to denote the real eigenvalues of A (counting multiplicity) being arranged in non-increasing order. Denote $\lambda(A) := (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))^T \in \Re^n$ and $\Lambda(A) := \text{diag}(\lambda(A))$, where for any $x \in \Re^n$, $\text{diag}(x)$ denotes the diagonal matrix whose i-th diagonal entry is x_i, $i = 1, \ldots, n$. Let $\mathcal{O}^n$ be the set of all $n \times n$ orthogonal matrices. Let $\overline{P} \in \mathcal{O}^n$ be such that

$$A = \overline{P}\Lambda(A)\overline{P}^T . \tag{4}$$

We denote the set of all such matrices $\overline{P}$ in the eigenvalue decomposition (4) by $\mathcal{O}^n(A)$. Define the three index sets of positive, zero, and negative eigenvalues of A, respectively, by

$$\alpha := \{i : \lambda_i(A) > 0\}, \quad \beta := \{i : \lambda_i(A) = 0\} \quad \text{and} \quad \gamma := \{i : \lambda_i(A) < 0\} . \tag{5}$$

For any matrix $P \in \mathcal{O}^n(A)$, we use p_j to represent the jth column of P, $j = 1, \ldots, n$. Let $\mathcal{J} \subseteq \{1, \ldots, n\}$ be an index set. We use $P_{\mathcal{J}}$ to denote the sub-matrix of P obtained by removing all columns of P not in $\mathcal{J}$. So for each j, we have $P_{\{j\}} = p_j$. Let $X \in \mathcal{S}^n$ and $\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, n\}$ be index sets. We use $X_{\mathcal{I}\mathcal{J}}$ to denote the sub-matrix of X obtained by removing all the rows of X not in $\mathcal{I}$ and all columns of X not in $\mathcal{J}$. For any $Z \in \mathcal{S}^n$, we use $Z \succeq 0$ and $Z \preceq 0$ to denote $Z \in \mathcal{S}^n_+$ and $Z \in \mathcal{S}^n_-$, respectively.

Proposition 2.4 *(see e.g., [9, Theorem 2.1]) For any $X \in \mathcal{S}^n_+$ and $Y \in \mathcal{S}^n_-$,*

$$\begin{aligned} N_{\mathcal{S}^n_+}(X) &= \{X^* \in \mathcal{S}^n_- : \langle X, X^* \rangle = 0\} = \{X^* \in \mathcal{S}^n_- : XX^* = 0\} , \\ N_{\mathcal{S}^n_-}(Y) &= \{Y^* \in \mathcal{S}^n_+ : \langle Y, Y^* \rangle = 0\} = \{Y^* \in \mathcal{S}^n_+ : YY^* = 0\} . \end{aligned}$$

We say that $X, Y \in \mathcal{S}^n$ have a simultaneous ordered eigenvalue decomposition provided that there exists $P \in \mathcal{O}^n$ such that $X = P\Lambda(X)P^T$ and $Y = P\Lambda(Y)P^T$. The following theorem is well-known and can be found in e.g. [13].

Theorem 2.3 *[von Neumann-Theobald] Any matrices X and Y in $\mathcal{S}^n$ satisfy the inequality*

$$\langle X, Y \rangle \leq \lambda(X)^\top \lambda(Y) ;$$

the equality holds if and only if X and Y admit a simultaneous ordered eigenvalue decomposition.

Proposition 2.5 *The graph of the set-valued map $N_{\mathcal{S}^n_+}$ can be written as*

$$\begin{aligned}\operatorname{gph} N_{\mathcal{S}^n_+} &= \{(X,Y)\in\mathcal{S}^n_+\times\mathcal{S}^n_- : \Pi_{\mathcal{S}^n_+}(X+Y)=X, \Pi_{\mathcal{S}^n_-}(X+Y)=Y\} \qquad (6)\\ &= \{(X,Y)\in\mathcal{S}^n_+\times\mathcal{S}^n_- : XY=YX=\langle X,Y\rangle=0\}\,, \qquad (7)\end{aligned}$$

where for any closed convex set $K\subseteq\mathcal{S}^n$, $\Pi_K(\cdot)$ denotes the metric projector operator over K.

Proof. Equation (6) is well-known (see [7]). Let $X\in\mathcal{S}^n_+$. Since $N_{\mathcal{S}^n_+}(X)=\partial\delta_{\mathcal{S}^n_+}(X)$, where δ_C is the indicate function of a set C, by [13, Theorem 3], since the function $\delta_{\mathcal{S}^n_+}(X)$ is an eigenvalue function, for any $Y\in N_{\mathcal{S}^n_+}(X)$, X and Y commute. Equation (7) then follows from the expression for the normal cone in Proposition 2.4. ■

From [32, Theorem 4.7] we know that the metric projection operator $\Pi_{\mathcal{S}^n_+}(\cdot)$ is directionally differentiable at any $A\in\mathcal{S}^n$ and the directional derivative of $\Pi_{\mathcal{S}^n_+}(\cdot)$ at A along direction $H\in\mathcal{S}^n$ is given by

$$\Pi'_{\mathcal{S}^n_+}(A;H)=\overline{P}\begin{bmatrix}\widetilde{H}_{\alpha\alpha} & \widetilde{H}_{\alpha\beta} & \Sigma_{\alpha\gamma}\circ\widetilde{H}_{\alpha\gamma}\\ \widetilde{H}^T_{\alpha\beta} & \Pi_{\mathcal{S}^{|\beta|}_+}(\widetilde{H}_{\beta\beta}) & 0\\ \Sigma^T_{\alpha\gamma}\circ\widetilde{H}^T_{\alpha\gamma} & 0 & 0\end{bmatrix}\overline{P}^T\,, \qquad (8)$$

where $\widetilde{H}:=\overline{P}^T H\overline{P}$ and

$$\Sigma_{ij}:=\frac{\max\{\lambda_i(A),0\}-\max\{\lambda_j(A),0\}}{\lambda_i(A)-\lambda_j(A)},\quad i,j=1,\ldots,n\,, \qquad (9)$$

where 0/0 is defined to be 1. Since $\Pi_{\mathcal{S}^n_+}(\cdot)$ is global Lipschitz continuous on $\mathcal{S}^n$, it is well-known that $\Pi_{\mathcal{S}^n_+}(\cdot)$ is *B(ouligand)-differentiable* (c.f. [8, Definition 3.1.2]) on $\mathcal{S}^n$. In the following proposition, we will show that $\Pi_{\mathcal{S}^n_+}(\cdot)$ is also *calmly B(ouligand)-differentiable* on $\mathcal{S}^n$. This result is not only of its own interest, but also is crucial for the study of the proximal and limiting normal cone of the normal cone mapping $N_{\mathcal{S}^n_+}$ in the next section.

Proposition 2.6 *The metric projection operator $\Pi_{\mathcal{S}^n_+}(\cdot)$ is calmly B-differentiable for any given $A\in\mathcal{S}^n$, i.e., for $\mathcal{S}^n\ni H\to 0$,*

$$\Pi_{\mathcal{S}^n_+}(A+H)-\Pi_{\mathcal{S}^n_+}(A)-\Pi'_{\mathcal{S}^n_+}(A;H)=O(\|H\|^2)\,. \qquad (10)$$

Proof. See the Appendix. ■

3 Expression of the proximal and limiting normal cones

In order to characterize the S-stationary and M-stationary conditions, we need to give the precise expressions for the proximal and limiting normal cones of the graph of the normal cone mapping $N_{\mathcal{S}^n_+}$ at any given point $(X,Y)\in\operatorname{gph} N_{\mathcal{S}^n_+}$. The purpose of this section is to provide such formulas. The result is also of independent interest.

3.1 Expression of the proximal normal cone

By using the directional derivative formula (8), Qi and Fusek [23] characterized the Fréchet normal cone of $\operatorname{gph} N_{\mathcal{S}^n_+}$. In this subsection, we will establish the representation of the desired proximal normal cone by using the same formula and the just proved calmly B-differentiability of the metric projection operator. The proximal normal cone is in general smaller than the Fréchet normal cone. For the set $\operatorname{gph} N_{\Re^n_+}$, however, it is well-known that the Fréchet normal cone coincides with the proximal normal cone. The natural question to ask is that whether this statement remains true for the set $\operatorname{gph} N_{\mathcal{S}^n_+}$. Our computations in this section give an affirmative answer, that is, the expression for the proximal normal cone coincides with the one for the Fréchet normal cone derived by Qi and Fusek in [23].

From Proposition 2.6, we know that for any given $X^* \in \mathcal{S}^n$ and any fixed $X \in \mathcal{S}^n$ there exist $M_1, M_2 > 0$ (depending on X and X^* only) such that for any $X' \in \mathcal{S}^n$ sufficiently close to X,

$$\langle X^*, \Pi_{\mathcal{S}^n_+}(X') - \Pi_{\mathcal{S}^n_+}(X)\rangle \le \langle X^*, \Pi'_{\mathcal{S}^n_+}(X; X' - X)\rangle + M_1\|X' - X\|^2 , \tag{11}$$

$$\langle X^*, \Pi_{\mathcal{S}^n_-}(X') - \Pi_{\mathcal{S}^n_-}(X)\rangle \le \langle X^*, \Pi'_{\mathcal{S}^n_-}(X; X' - X)\rangle + M_2\|X' - X\|^2 . \tag{12}$$

Proposition 3.1 *For any given $(X, Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$, $(X^*, Y^*) \in N^\pi_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)$ if and only if $(X^*, Y^*) \in \mathcal{S}^n \times \mathcal{S}^n$ satisfies*

$$\langle X^*, \Pi'_{\mathcal{S}^n_+}(X + Y; H)\rangle + \langle Y^*, \Pi'_{\mathcal{S}^n_-}(X + Y; H)\rangle \le 0 \quad \forall\, H \in \mathcal{S}^n. \tag{13}$$

Proof. " $\Longleftarrow$ " Suppose that $(X^*, Y^*) \in \mathcal{S}^n \times \mathcal{S}^n$ is given and satisfies the condition (13). Denote the set in the right-hand side by C. By Proposition 2.5, (11) and (12), we know that there exist a constant $\delta > 0$ and a constant $\widetilde{M} > 0$ such that for any $(X', Y') \in \operatorname{gph} N_{\mathcal{S}^n_+}$ and $\|(X', Y') - (X, Y)\| \le \delta$,

$$\begin{aligned} & \langle (X^*, Y^*), (X', Y') - (X, Y)\rangle \\ = \; & \langle (X^*, Y^*), (\Pi_{\mathcal{S}^n_+}(X' + Y'), \Pi_{\mathcal{S}^n_-}(X' + Y')) - (\Pi_{\mathcal{S}^n_+}(X + Y), \Pi_{\mathcal{S}^n_-}(X + Y))\rangle \\ \le \; & \widetilde{M}\|(X', Y') - (X, Y)\|^2 . \end{aligned}$$

By taking $M = \max\left\{\widetilde{M}, \|(X^*, Y^*)\|/\delta\right\}$, we know that for any $(X', Y') \in \operatorname{gph} N_{\mathcal{S}^n_+}$,

$$\langle (X^*, Y^*), (X', Y') - (X, Y)\rangle \le M\|(X', Y') - (X, Y)\|^2 ,$$

which implies, by the definition of the proximal normal cone, that $(X^*, Y^*) \in N^\pi_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)$.

" $\Longrightarrow$ " Let $(X^*, Y^*) \in N^\pi_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)$ be given. Then there exists $M > 0$ such that for any $(X', Y') \in \operatorname{gph} N_{\mathcal{S}^n_+}$,

$$\langle (X^*, Y^*), (X', Y') - (X, Y)\rangle \le M\|(X', Y') - (X, Y)\|^2 . \tag{14}$$

Let $H \in \mathcal{S}^n$ be arbitrary but fixed. For any $t \downarrow 0$, let

$$X'_t = \Pi_{\mathcal{S}^n_+}(X + Y + tH) \quad \text{and} \quad Y'_t = \Pi_{\mathcal{S}^n_-}(X + Y + tH) .$$

By noting that $(X_t', Y_t') \in \operatorname{gph} N_{\mathcal{S}^n_+}$ (c.f., (6) in Proposition 2.5) and $\Pi_{\mathcal{S}^n_+}(\cdot)$ and $\Pi_{\mathcal{S}^n_-}(\cdot)$ are globally Lipschitz continuous with modulus 1, we obtain from (14) that

$$\begin{aligned}
& \langle X^*, \Pi'_{\mathcal{S}^n_+}(X+Y;H)\rangle + \langle Y^*, \Pi'_{\mathcal{S}^n_-}(X+Y;H)\rangle \\
\leq\ & M \lim_{t\downarrow 0} \frac{1}{t}(\|X_t' - X\|^2 + \|Y_t' - Y\|^2) \leq M \lim_{t\downarrow 0} \frac{1}{t}(2t^2\|H\|^2) = 0\,.
\end{aligned}$$

Therefore, we know that $(X^*, Y^*) \in \mathcal{S}^n \times \mathcal{S}^n$ satisfies the condition (13). The proof is completed. ■

For any given $(X,Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$, let $A = X + Y$ have the eigenvalue decomposition (4). From (6), we know that $X = \Pi_{\mathcal{S}^n_+}(A)$ and $Y = \Pi_{\mathcal{S}^n_-}(A)$. It follows from the directional derivative formula (8) that for any $H \in \mathcal{S}^n$,

$$\Pi'_{\mathcal{S}^n_-}(A;H) = \overline{P} \begin{bmatrix} 0 & 0 & (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{H}_{\alpha\gamma} \\ 0 & \Pi_{\mathcal{S}^{|\beta|}_-}(\widetilde{H}_{\beta\beta}) & \widetilde{H}_{\beta\gamma} \\ \widetilde{H}^T_{\alpha\gamma} \circ (E_{\alpha\gamma} - \Sigma_{\alpha\gamma})^T & \widetilde{H}_{\beta\gamma} & \widetilde{H}_{\gamma\gamma} \end{bmatrix} \overline{P}^T\,, \tag{15}$$

where E is a $n \times n$ matrix whose entries are all ones. Denote

$$\Theta_1 := \begin{bmatrix} E_{\alpha\alpha} & E_{\alpha\beta} & \Sigma_{\alpha\gamma} \\ E^T_{\alpha\beta} & 0 & 0 \\ \Sigma^T_{\alpha\gamma} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Theta_2 := \begin{bmatrix} 0 & 0 & E_{\alpha\gamma} - \Sigma_{\alpha\gamma} \\ 0 & 0 & E_{\beta\gamma} \\ (E_{\alpha\gamma} - \Sigma_{\alpha\gamma})^T & E^T_{\beta\gamma} & E_{\gamma\gamma} \end{bmatrix}. \tag{16}$$

We are now in a position to derive the precise expression of the proximal normal cone to $\operatorname{gph} N_{\mathcal{S}^n_+}$.

Proposition 3.2 *For any $(X,Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$, let $A = X + Y$ have the eigenvalue decomposition (4). Then*

$$N^\pi_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X,Y) = \left\{ (X^*, Y^*) \in \mathcal{S}^n \times \mathcal{S}^n : \Theta_1 \circ \widetilde{X}^* + \Theta_2 \circ \widetilde{Y}^* = 0,\ \widetilde{X}^*_{\beta\beta} \preceq 0 \text{ and } \widetilde{Y}^*_{\beta\beta} \succeq 0 \right\},$$

where $\widetilde{X}^* := \overline{P}^T X^* \overline{P}$ and $\widetilde{Y}^* := \overline{P}^T Y^* \overline{P}$.

Proof. By Proposition 3.1, $(X^*, Y^*) \in N^\pi_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X,Y)$ if and only if

$$\langle X^*, \Pi'_{\mathcal{S}^n_+}(A;H)\rangle + \langle Y^*, \Pi'_{\mathcal{S}^n_-}(A;H)\rangle \leq 0 \quad \forall\ H \in \mathcal{S}^n\,,$$

which, together with the directional derivative formulas (8) and (15) implies that $(X^*, Y^*) \in N^\pi_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X,Y)$ if and only if

$$\langle \Theta_1 \circ \widetilde{X}^*, \widetilde{H}\rangle + \langle \Theta_2 \circ \widetilde{Y}^*, \widetilde{H}\rangle + \langle \widetilde{X}^*_{\beta\beta}, \Pi_{\mathcal{S}^{|\beta|}_+}(\widetilde{H}_{\beta\beta})\rangle + \langle \widetilde{Y}^*_{\beta\beta}, \Pi_{\mathcal{S}^{|\beta|}_-}(\widetilde{H}_{\beta\beta})\rangle \leq 0 \quad \forall\, H \in \mathcal{S}^n.$$

The conclusion of the proposition holds. ■

3.2 Expression of the limiting normal cone

In this subsection, we will use the formula of the proximal normal cone $N^{\pi}_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X,Y)$ obtained in Proposition 3.2 to characterize the limiting normal cone $N_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X,Y)$.

For any given $(X,Y) \in \mathrm{gph}\, N_{\mathcal{S}^n_+}$, let $A = X+Y$ have the eigenvalue decomposition (4). Firstly, we will characterize $N_{\mathrm{gph}\, N_{\mathcal{S}^{|\beta|}_+}}(0,0)$ for the case that $\beta \neq \emptyset$. Denote the set of all partitions of the index set β by $\mathscr{P}(\beta)$. Let $\Re^{|\beta|}_{\gtrsim}$ be the set of all vectors in $\Re^{|\beta|}$ whose components being arranged in non-increasing order, i.e.,

$$\Re^{|\beta|}_{\gtrsim} := \left\{ z \in \Re^{|\beta|} \, : \, z_1 \geq \ldots \geq z_{|\beta|} \right\}.$$

For any $z \in \Re^{|\beta|}_{\gtrsim}$, let $D(z)$ represent the generalized first divided difference matrix for $f(t) = \max\{t, 0\}$ at z, i.e.,

$$(D(z))_{ij} = \begin{cases} \dfrac{\max\{z_i,0\} - \max\{z_j,0\}}{z_i - z_j} \in [0,1] & \text{if } z_i \neq z_j\,, \\ 1 & \text{if } z_i = z_j > 0\,, \\ 0 & \text{if } z_i = z_j \leq 0\,, \end{cases} \quad i,j = 1,\ldots,|\beta|\,. \tag{17}$$

Denote

$$\mathcal{U}_{|\beta|} := \{ \overline{\Omega} \in \mathcal{S}^{|\beta|} \, : \, \overline{\Omega} = \lim_{k\to\infty} D(z^k), \quad z^k \to 0, \; z^k \in \Re^{|\beta|}_{\gtrsim} \}\,. \tag{18}$$

Let $\Xi_1 \in \mathcal{U}_{|\beta|}$. Then, from (17), it is easy to see that there exists a partition $\pi(\beta) := (\beta_+, \, \beta_0, \, \beta_-) \in \mathscr{P}(\beta)$ such that

$$\Xi_1 = \begin{bmatrix} E_{\beta_+\beta_+} & E_{\beta_+\beta_0} & (\Xi_1)_{\beta_+\beta_-} \\ E^T_{\beta_+\beta_0} & 0 & 0 \\ (\Xi_1)^T_{\beta_+\beta_-} & 0 & 0 \end{bmatrix}, \tag{19}$$

where each element of $(\Xi_1)_{\beta_+\beta_-}$ belongs to $[0,1]$. Let

$$\Xi_2 := \begin{bmatrix} 0 & 0 & E_{\beta_+\beta_-} - (\Xi_1)_{\beta_+\beta_-} \\ 0 & 0 & E_{\beta_0\beta_-} \\ (E_{\beta_+\beta_-} - (\Xi_1)_{\beta_+\beta_-})^T & E^T_{\beta_0\beta_-} & E_{\beta_-\beta_-} \end{bmatrix}. \tag{20}$$

Proposition 3.3 *The limiting norm cone to the graph of the normal cone mapping $N_{\mathcal{S}^{|\beta|}_+}$ at $(0,0)$ is given by*

$$N_{\mathrm{gph}\, N_{\mathcal{S}^{|\beta|}_+}}(0,0) = \bigcup_{\substack{Q \in \mathcal{O}^{|\beta|} \\ \Xi_1 \in \mathcal{U}_{|\beta|}}} \left\{ \begin{array}{ll} (U^*, V^*): & \Xi_1 \circ Q^T U^* Q + \Xi_2 \circ Q^T V^* Q = 0, \\ & Q^T_{\beta_0} U^* Q_{\beta_0} \preceq 0, \quad Q^T_{\beta_0} V^* Q_{\beta_0} \succeq 0 \end{array} \right\}. \tag{21}$$

Proof. See the Appendix. ■

We characterize the limiting normal cone $N_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X,Y)$ for any $(X,Y) \in \mathrm{gph}\, N_{\mathcal{S}^n_+}$ in the following theorem.

Theorem 3.1 *For any* $(X, Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$*, let* $A = X+Y$ *have the eigenvalue decomposition (4). Then,* $(X^*, Y^*) \in N_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)$ *if and only if*

$$X^* = \overline{P} \begin{bmatrix} 0 & 0 & \widetilde{X}^*_{\alpha\gamma} \\ 0 & \widetilde{X}^*_{\beta\beta} & \widetilde{X}^*_{\beta\gamma} \\ \widetilde{X}^*_{\gamma\alpha} & \widetilde{X}^*_{\gamma\beta} & \widetilde{X}^*_{\gamma\gamma} \end{bmatrix} \overline{P}^T \quad \text{and} \quad Y^* = \overline{P} \begin{bmatrix} \widetilde{Y}^*_{\alpha\alpha} & \widetilde{Y}^*_{\alpha\beta} & \widetilde{Y}^*_{\alpha\gamma} \\ \widetilde{Y}^*_{\beta\alpha} & \widetilde{Y}^*_{\beta\beta} & 0 \\ \widetilde{Y}^*_{\gamma\alpha} & 0 & 0 \end{bmatrix} \overline{P}^T \tag{22}$$

with

$$(\widetilde{X}^*_{\beta\beta}, \widetilde{Y}^*_{\beta\beta}) \in N_{\operatorname{gph} N_{\mathcal{S}^{|\beta|}_+}}(0,0) \quad \text{and} \quad \Sigma_{\alpha\gamma} \circ \widetilde{X}^*_{\alpha\gamma} + (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{Y}^*_{\alpha\gamma} = 0\,, \tag{23}$$

where Σ *is given by (9),* $\widetilde{X}^* = \overline{P}^T X^* \overline{P}$ *and* $\widetilde{Y}^* = \overline{P}^T Y^* \overline{P}$.

Proof. See the Appendix. ■

Remark 3.1 *For any given* $(X, Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$*, the (Mordukhovich) coderivative* $D^* N_{\mathcal{S}^n_+}(X, Y)$ *of the normal cone to the set* $\mathcal{S}^n_+$ *can be calculated by using Theorem 3.1 and the definition of coderivative, i.e., for given* $Y^* \in \mathcal{S}^n$,

$$X^* \in D^* N_{\mathcal{S}^n_+}(X, Y)(Y^*) \quad \Longleftrightarrow \quad (X^*, -Y^*) \in N_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)\,.$$

Furthermore, by (6) in Proposition 2.5, we know that

$$\operatorname{gph} N_{\mathcal{S}^n_+} = \{(X, Y) \in \mathcal{S}^n \times \mathcal{S}^n : L(X, Y) \in \operatorname{gph} \Pi_{\mathcal{S}^n_+}\}\,,$$

where $L : \mathcal{S}^n \times \mathcal{S}^n \to \mathcal{S}^n \times \mathcal{S}^n$ *is a linear function defined by*

$$L(X, Y) := (X + Y, X), \quad (X, Y) \in \mathcal{S}^n \times \mathcal{S}^n\,.$$

By noting that the derivative of L *is nonsingular and self-adjoint, we know from [17, Theorem 6.10] that for any given* $(X, Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$ *and* $Y^* \in \mathcal{S}^n$,

$$D^* N_{\mathcal{S}^n_+}(X, Y)(-Y^*) = \{X^* \in \mathcal{S}^n : (X^*, Y^*) \in L'(X, Y) N_{\operatorname{gph} \Pi_{\mathcal{S}^n_+}}(X + Y, X)\}\,.$$

Thus, for any given $U^* \in \mathcal{S}^n$, $V^* \in D^* \Pi_{\mathcal{S}^n_+}(X+Y)(U^*)$ *if and only if there exists* $(X^*, Y^*) \in N_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)$ *such that* $(X^*, Y^*) = L(V^*, -U^*)$*, that is,*

$$X^* = V^* - U^* \quad \text{and} \quad Y^* = V^*\,.$$

Note that for any given $Z \in \mathcal{S}^n$*, there exists a unique element* $(X, Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$ *such that* $Z = X + Y$*. Hence, the coderivative of the metric projector operator* $\Pi_{\mathcal{S}^n_+}(\cdot)$ *at any* $Z \in \mathcal{S}^n$ *can also be computed by Theorem 3.1.*

4 Failure of Robinson's CQ

Since for any $(G(z), H(z)) \in \mathcal{S}^n_+ \times \mathcal{S}^n_-$, by the von Neumann-Theobald theorem (Theorem 2.3), one always has

$$\langle G(z), H(z) \rangle \le \lambda(G(z))^T \lambda(H(z)) \le 0\,.$$

Consequently one can rewrite the problem SDCMPCC in the following form:

$$
\begin{array}{llll}
(CP-SDCMPCC) & \min & f(z) \\
 & s.t. & h(z)=0\,, \\
 & & g(z)\le 0\,, \\
 & & \langle G(z),H(z)\rangle \ge 0\,, \\
 & & (G(z),H(z))\in \mathcal{S}^n_+\times\mathcal{S}^n_-\,.
\end{array}
$$

The above problem belongs to the class of general optimization problems with a cone constraint (GP) with $K=\mathcal{S}^n_+\times\mathcal{S}^n_-$ as discussed in §2.2 and hence the necessary optimality condition stated in §2.2 can be applied to obtain the following classical KKT condition.

Definition 4.1 *Let $\bar z$ be a feasible solution of SDCMPCC. We call $\bar z$ a classical KKT point. If there exists $(\lambda^h,\lambda^g,\lambda^e,\Omega^G,\Omega^H)\in\Re^p\times\Re^q\times\Re\times\mathcal{S}^n\times\mathcal{S}^n$ with $\lambda^g\ge 0$, $\lambda^e\le 0$, $\Omega^G\preceq 0$ and $\Omega^H\succeq 0$ such that*

$$
\begin{array}{l}
0=\nabla f(\bar z)+h'(\bar z)^*\lambda^h+g'(\bar z)^*\lambda^g+\lambda^e[H'(\bar z)^*G(\bar z)+G'(\bar z)^*H(\bar z)]+G'(\bar z)^*\Omega^G+H'(\bar z)^*\Omega^H,\\
\langle g(\bar z),\lambda^g\rangle=0,\quad G(\bar z)\Omega^G=0,\quad H(\bar z)\Omega^H=0\,.
\end{array}
$$

Theorem 4.1 *Let $\bar z$ be a feasible solution of SDCMPCC. Suppose the problem CP-SDCMPCC is Clarke calm at $\bar z$. Then $\bar z$ is a classical KKT point.*

Proof. By Theorem 2.2, there exists a Lagrange multiplier $(\lambda^h,\lambda^g,\lambda^e,\Gamma^G,\Gamma^H)\in\Re^p\times\Re^q\times\Re\times\mathcal{S}^n\times\mathcal{S}^n$ with $\lambda^g\ge 0,\lambda^e\le 0$ such that

$$
\begin{array}{l}
0=\nabla f(\bar z)+h'(\bar z)^*\lambda^h+g'(\bar z)^*\lambda^g+\lambda^e[H'(\bar z)^*G(\bar z)+G'(\bar z)^*H(\bar z)]+G'(\bar z)^*\Gamma^G+H'(\bar z)^*\Gamma^H,\\
\langle g(\bar z),\lambda^g\rangle=0,\quad (\Gamma^G,\Gamma^H)\in N_{\mathcal{S}^n_+\times\mathcal{S}^n_-}(G(\bar z),H(\bar z))\,.
\end{array}
$$

The desired result follows from the normal cone expressions in Proposition 2.4. ■

Definition 4.2 *We say that $(\lambda^h,\lambda^g,\lambda^e,\Omega^G,\Omega^H)\in\Re^p\times\Re^q\times\Re\times\mathcal{S}^n\times\mathcal{S}^n$ with $\lambda^g\ge 0,\lambda^e\le 0,\Omega^G\preceq 0,\Omega^H\succeq 0$ is a singular Lagrange multiplier for CP-SDCMPCC if it is not equal to zero and*

$$
\begin{array}{l}
0=h'(\bar z)^*\lambda^h+g'(\bar z)^*\lambda^g+\lambda^e[H'(\bar z)^*G(\bar z)+G'(\bar z)^*H(\bar z)]+G'(\bar z)^*\Omega^G+H'(\bar z)^*\Omega^H,\\
\langle g(\bar z),\lambda^g\rangle=0,\quad G(\bar z)\Omega^G=0,\quad H(\bar z)\Omega^H=0\,.
\end{array}
$$

For a general optimization problem with a cone constraint such as CP-SDCMPCC, the following Robinson's CQ is considered to be a usual constraint qualification:

$$
h'(\bar z) \text{ is onto} \quad (\text{ equivalently } h_i'(\bar z)(i=1,\ldots,p) \text{ are linear independent})\,,
$$

$$
\exists\, d \text{ such that } \left\{
\begin{array}{ll}
h_i'(\bar z)d=0, & i=1,\ldots,p\,,\\
g_i'(\bar z)d<0, & i\in I_g(\bar z)\,,\\
(H'(\bar z)^*G(\bar z)+G'(\bar z)^*H(\bar z))d>0\,, & \\
G(\bar z)+G'(\bar z)d\in \operatorname{int}\mathcal{S}^n_+\,, & \\
H(\bar z)+H'(\bar z)d\in \operatorname{int}\mathcal{S}^n_-\,. &
\end{array}\right.
$$

It is well-known that the MFCQ never holds for MPCCs. We now show that Robinson's CQ will never hold for CP-SDCMPCC.

Proposition 4.1 *For CP-SDCMPCC, Robinson's constraint qualification fails to hold at every feasible solution of SDCMPCC.*

Proof. By the von Neumann-Theobald theorem, $G(z) \succeq 0, H(z) \preceq 0$ implies that $\langle G(z), H(z)\rangle \leq 0$. Hence any feasible solution $\bar{z}$ of SDCMPCC must be a solution to the following nonlinear semidefinite program:

$$\begin{array}{ll} \min & -\langle G(z), H(z)\rangle \\ \text{s.t.} & G(z) \succeq 0, \quad H(z) \preceq 0\,. \end{array}$$

Since for this problem, $f(z) = -\langle G(z), H(z)\rangle$, we have $\nabla f(z) = -H'(z)^*G(z) - G'(z)^*H(z)$. By the first order necessary optimality condition, there exist $\lambda^e = 1, \Omega^G \preceq 0, \Omega^H \succeq 0$ such that

$$\begin{aligned} & 0 = -\lambda^e[H'(\bar{z})^*G(\bar{z}) + G'(\bar{z})^*H(\bar{z})] + G'(\bar{z})^*\Omega^G + H'(\bar{z})^*\Omega^H\,, \\ & G(\bar{z})\Omega^G = 0, \quad H(\bar{z})\Omega^H = 0\,. \end{aligned}$$

Since $(-\lambda^e, \Omega^G, \Omega^H) \neq 0$, it is clear that $(0, 0, 0, -\lambda^e, \Omega^G, \Omega^H)$ is a singular Lagrange multiplier of CP-SDCMPCC. By [3, Propositions 3.16 (ii) and 3.19(iii)]), a singular Lagrange multiplier exists if and only if Robinson's CQ does not hold. Therefore we conclude that the Robinson's CQ does not hold at $\bar{z}$ for CP-SDCMPCC. ■

5 S-stationary conditions

In the MPCC literature it is well-known that S-stationary condition is equivalent to the classical KKT condition; see e.g. [10]. In this section we introduce the concept of S-stationary condition and show that the classical KKT condition implies the S-stationary condition.

For MPCC, the S-stationary condition is shown to be equivalent to the necessary optimality condition of a reformulated problem involving the proximal normal cone to the graph of the normal cone operator (see [37, Theorem 3.2]). Motivated by this fact and the precise expression for the proximal normal cone formula in Proposition 3.2, we introduce the concept of a S-stationary point for SDCMPCC.

Definition 5.1 *Let $\bar{z}$ be a feasible solution of SDCMPCC. Let $A := G(\bar{z}) + H(\bar{z})$ have the eigenvalue decomposition (4). We say that $\bar{z}$ is a S-stationary point of SDCMPCC if there exists $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \Re^p \times \Re^q \times \mathcal{S}^n \times \mathcal{S}^n$ such that*

$$0 = \nabla f(\bar{z}) + h'(\bar{z})^*\lambda^h + g'(\bar{z})^*\lambda^g + G'(\bar{z})^*\Gamma^G + H'(\bar{z})^*\Gamma^H\,, \tag{24}$$

$$\lambda^g \geq 0, \quad \langle \lambda^g, g(\bar{z})\rangle = 0\,, \tag{25}$$

$$\widetilde{\Gamma}^G_{\alpha\alpha} = 0, \quad \widetilde{\Gamma}^G_{\alpha\beta} = 0, \quad \widetilde{\Gamma}^G_{\beta\alpha} = 0\,, \tag{26}$$

$$\widetilde{\Gamma}^H_{\gamma\gamma} = 0, \quad \widetilde{\Gamma}^H_{\beta\gamma} = 0, \quad \widetilde{\Gamma}^H_{\gamma\beta} = 0\,, \tag{27}$$

$$\Sigma_{\alpha\gamma} \circ \widetilde{\Gamma}^G_{\alpha\gamma} + (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{\Gamma}^H_{\alpha\gamma} = 0\,, \tag{28}$$

$$\widetilde{\Gamma}^G_{\beta\beta} \preceq 0, \quad \widetilde{\Gamma}^H_{\beta\beta} \succeq 0\,, \tag{29}$$

where E is a $n \times n$ matrix whose entries are all ones and $\Sigma \in \mathcal{S}^n$ is defined by (9), and $\widetilde{\Gamma}^G = \overline{P}^T \Gamma^G \overline{P}$ and $\widetilde{\Gamma}^H = \overline{P}^T \Gamma^H \overline{P}$.

To see that the S-stationary condition for SDCMPCC coincides with the S-stationary condition in the MPCC case we consider the case when $n = 1$. In this case, $\lambda(A) = A$, $\overline{P} = 1$ and $\widetilde{\Gamma}^G = \Gamma^G$, $\widetilde{\Gamma}^H = \Gamma^H$. In this case SDCMPCC is a MPCC where there is only one complementarity constraint. If $G(\bar{z}) > 0, H(\bar{z}) = 0$. Then $\beta = \gamma = \emptyset$. So from (26), we know that $\Gamma^G = 0$ and Γ^H free. Similarly if $G(\bar{z}) = 0, H(\bar{z}) < 0$ we have $\Gamma^H = 0$ and Γ^G free. If $G(\bar{z}) = H(\bar{z}) = 0$. Then we have $\alpha = \gamma = \emptyset$. Consequently from (29), we know that $\Gamma^G \leq 0$ and $\Gamma^H \geq 0$.

It turns out that we can show that the classical KKT condition implies the S-stationary condition. However we are not able to show that the S-stationary condition implies the KKT condition for a general SDCMPCC unless it is a MPCC.

Proposition 5.1 *Let $\bar{z}$ be a feasible solution of SDCMPCC. If $\bar{z}$ is a classic KKT point, i.e., there exists a classical Lagrange multiplier $(\lambda^h, \lambda^g, \lambda^e, \Omega^G, \Omega^H) \in \Re^p \times \Re^q \times \Re \times \mathcal{S}^n \times \mathcal{S}^n$ with $\lambda^g \geq 0$, $\lambda^e \leq 0$, $\Omega^G \preceq 0$ and $\Omega^H \succeq 0$ such that*

$$0 = \nabla f(\bar{z}) + h'(\bar{z})^* \lambda^h + g'(\bar{z})^* \lambda^g + \lambda^e [H'(\bar{z})^* G(\bar{z}) + G'(\bar{z})^* H(\bar{z})] + G'(\bar{z})^* \Omega^G + H'(\bar{z})^* \Omega^H \,,$$
$$\langle \lambda^g, g(\bar{z}) \rangle = 0, \quad G(\bar{z})\Omega^G = 0, \quad H(\bar{z})\Omega^H = 0 \,,$$

then it is also a S-stationary point.

Proof. Denote $\overline{\Lambda} := \Lambda(A)$. Define $\Gamma^G := \Omega^G + \lambda^e H(\bar{z})$ and $\Gamma^H := \Omega^H + \lambda^e G(\bar{z})$. Then (24) and (25) hold. It remains to show (26)-(29). By the assumption we have

$$\mathcal{S}^n_+ \ni G(\bar{z}) \perp \Omega^G \in \mathcal{S}^n_- \quad \text{and} \quad \mathcal{S}^n_- \ni H(\bar{z}) \perp \Omega^H \in \mathcal{S}^n_+ \,.$$

By Theorem 2.3, we know that $G(\bar{z})$ and Ω^G ($H(\bar{z})$ and Ω^H) admit a simultaneous ordered eigenvalue decomposition, i.e., there exist two orthogonal matrices $\widetilde{P}, \widehat{P} \in \mathcal{O}^n$ such that

$$\Omega^G = \widetilde{P} \begin{bmatrix} 0 & 0 \\ 0 & \Lambda(\Omega^G)_{\gamma'\gamma'} \end{bmatrix} \widetilde{P}^T, \quad G(\bar{z}) = \widetilde{P} \begin{bmatrix} \overline{\Lambda}_{\alpha\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \widetilde{P}^T$$

and

$$\Omega^H = \widehat{P} \begin{bmatrix} \Lambda(\Omega^H)_{\alpha'\alpha'} & 0 \\ 0 & 0 \end{bmatrix} \widehat{P}^T, \quad H(\bar{z}) = \widehat{P} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{\Lambda}_{\gamma\gamma} \end{bmatrix} \widehat{P}^T \,,$$

where $\alpha' := \{i \,|\, \lambda_i(\Omega^H) > 0\}$ and $\gamma' := \{i \,|\, \lambda_i(\Omega^G) < 0\}$. Moreover, we have

$$\gamma' \subseteq \bar{\alpha} \quad \text{and} \quad \alpha' \subseteq \bar{\gamma}. \tag{30}$$

On the other hand, we know that

$$G(\bar{z}) = \Pi_{\mathcal{S}^n_+}(A) = \overline{P} \begin{bmatrix} \overline{\Lambda}_{\alpha\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overline{P}^T \quad \text{and} \quad H(\bar{z}) = \Pi_{\mathcal{S}^n_-}(A) = \overline{P} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{\Lambda}_{\gamma\gamma} \end{bmatrix} \overline{P}^T \,.$$

Therefore, it is easy to check that there exist two orthogonal matrices $S, T \in \mathcal{O}^n$ such that

$$\overline{P} = \widetilde{P} S \quad \text{and} \quad \overline{P} = \widehat{P} T \,,$$

with

$$S = \begin{bmatrix} S_{\alpha\alpha} & 0 \\ 0 & S_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_{\bar{\gamma}\bar{\gamma}} & 0 \\ 0 & T_{\gamma\gamma} \end{bmatrix} \,,$$

where $\bar{\alpha} := \beta \cup \gamma$, $\bar{\gamma} := \alpha \cup \beta$ and $S_{\alpha\alpha} \in \mathcal{O}^{|\alpha|}$, $S_{\bar{\alpha}\bar{\alpha}} \in \mathcal{O}^{|\bar{\alpha}|}$ and $T_{\bar{\gamma}\bar{\gamma}} \in \mathcal{O}^{|\bar{\gamma}|}$, $T_{\gamma\gamma} \in \mathcal{O}^{|\gamma|}$. Denote

$$S_{\bar{\alpha}\bar{\alpha}} = [S_1 \quad S_2] \quad \text{and} \quad T_{\bar{\gamma}\bar{\gamma}} = [T_1 \quad T_2]$$

with $S_1 \in \Re^{|\bar{\alpha}|\times|\beta|}$, $S_2 \in \Re^{|\bar{\alpha}|\times|\gamma|}$ and $T_1 \in \Re^{|\bar{\gamma}|\times|\alpha|}$ and $T_2 \in \Re^{|\bar{\gamma}|\times|\beta|}$. Then, we have

$$\begin{aligned}
\widetilde{\Gamma}^G &= \overline{P}^T(\Omega^G + \lambda^e H(\bar{z}))\overline{P} = S^T \widetilde{P}^T \Omega^G \widetilde{P} S + \lambda^e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{\Lambda}_{\gamma\gamma} \end{bmatrix} \\
&= \begin{bmatrix} S_{\alpha\alpha}^T & 0 \\ 0 & S_{\bar{\alpha}\bar{\alpha}}^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} S_{\alpha\alpha} & 0 \\ 0 & S_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} + \lambda^e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{\Lambda}_{\gamma\gamma} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_1^T \Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} S_1 & S_1^T \Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} S_2 \\ 0 & S_2^T \Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} S_1 & S_2^T \Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} S_2 + \lambda^e \overline{\Lambda}_{\gamma\gamma} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\Gamma}^H &= \overline{P}^T(\Omega^H + \lambda^e G(\bar{z}))\overline{P} = T^T \widetilde{P}^T \Omega^H \widetilde{P} T + \lambda^e \begin{bmatrix} \overline{\Lambda}_{\alpha\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} T_{\bar{\gamma}\bar{\gamma}}^T & 0 \\ 0 & T_{\gamma\gamma}^T \end{bmatrix} \begin{bmatrix} \Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{\bar{\gamma}\bar{\gamma}} & 0 \\ 0 & T_{\gamma\gamma} \end{bmatrix} + \lambda^e \begin{bmatrix} \overline{\Lambda}_{\alpha\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} T_1^T \Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} T_1 + \lambda^e \overline{\Lambda}_{\alpha\alpha} & T_1^T \Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} T_2 & 0 \\ T_2^T \Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} T_1 & T_2^T \Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} T_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Therefore it is easy to see that (26)-(28) hold. Since $\Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} \preceq 0$, $\Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} \succeq 0$ and $S_{\bar{\alpha}\bar{\alpha}}$, $T_{\bar{\gamma}\bar{\gamma}}$ are orthogonal, we know that

$$S_{\bar{\alpha}\bar{\alpha}}^T \Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} S_{\bar{\alpha}\bar{\alpha}} \preceq 0 \quad \text{and} \quad T_{\bar{\gamma}\bar{\gamma}}^T \Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} T_{\bar{\gamma}\bar{\gamma}} \succeq 0\,.$$

Hence, we have

$$\widetilde{\Gamma}_{\beta\beta}^G = S_1^T \Lambda(\Omega^G)_{\bar{\alpha}\bar{\alpha}} S_1 \preceq 0 \quad \text{and} \quad \widetilde{\Gamma}_{\beta\beta}^H = T_2^T \Lambda(\Omega^H)_{\bar{\gamma}\bar{\gamma}} T_2 \succeq 0\,,$$

which implies (29). Therefore $\bar{z}$ is also a S-stationary point. ■

Combining Theorem 4.1 and Proposition 5.1 we have the following necessary optimality condition in terms of S-stationary conditions.

Theorem 5.1 *Let $\bar{z}$ be a feasible solution of SDCMPCC. Suppose the problem CP-SDCMPCC is Clarke calm at $\bar{z}$. Then $\bar{z}$ is a S-stationary point.*

6 M-stationary conditions

In this section we study the M-stationary conditon for SDCMPCC. For this purpose rewrite the SDCMPCC as an optimization problem with a cone constraint:

$$\begin{array}{lll} \text{(GP-SDCMPCC)} & \min & f(z) \\ & \text{s.t.} & h(z) = 0\,, \\ & & g(z) \le 0\,, \\ & & (G(z), H(z)) \in \operatorname{gph} N_{\mathcal{S}^n_+}\,. \end{array}$$

Definition 6.1 *Let $\bar z$ be a feasible solution of SDCMPCC. Let $A = G(\bar z) + H(\bar z)$ have the eigenvalue decomposition (4). We say that $\bar z$ is a M-stationary point of SDCMPCC if there exists $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \Re^p \times \Re^q \times \mathcal{S}^n \times \mathcal{S}^n$ such that (24)-(28) hold and there exist $Q \in \mathcal{O}^{|\beta|}$ and $\Xi_1 \in \mathcal{U}_{|\beta|}$ (with a partition $\pi(\beta) = (\beta_+, \beta_0, \beta_-)$ of β and the form (19)) such that*

$$\Xi_1 \circ Q^T \widetilde{\Gamma}^G Q + \Xi_2 \circ Q^T \widetilde{\Gamma}^H Q = 0\,, \tag{31}$$

$$Q^T_{\beta_0} \widetilde{\Gamma}^G_{\beta\beta} Q_{\beta_0} \preceq 0, \quad Q^T_{\beta_0} \widetilde{\Gamma}^H_{\beta\beta} Q_{\beta_0} \succeq 0\,, \tag{32}$$

where $\widetilde{\Gamma}^G = \overline{P}^T \Gamma^G \overline{P}$, $\widetilde{\Gamma}^H = \overline{P}^T \Gamma^H \overline{P}$ and

$$\Xi_2 = \begin{bmatrix} 0 & 0 & E_{\beta_+\beta_-} - (\Xi_1)_{\beta_+\beta_-} \\ 0 & 0 & E_{\beta_0\beta_-} \\ (E_{\beta_+\beta_-} - (\Xi_1)_{\beta_+\beta_-})^T & E^T_{\beta_0\beta_-} & E_{\beta_-\beta_-} \end{bmatrix}.$$

We say that $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \Re^p \times \Re^q \times \mathcal{S}^n \times \mathcal{S}^n$ is a singular M-multiplier for SDCMPCC if it is not equal to zero and all conditions above hold except the term $\nabla f(\bar z)$ vanishes in (24).

To see that the M-stationary condition for SDCMPCC coincides with the M-stationary condition in the MPCC case we consider the case when $n = 1$. In this case $\lambda(A) = A$, $\overline{P} = 1$ and $\widetilde{\Gamma}^G = \Gamma^G$, $\widetilde{\Gamma}^H = \Gamma^H$. We only need to consider the case $G(\bar z) = H(\bar z) = 0$ since the other cases are the same as the S-stationary condition. Then we have $\alpha = \gamma = \emptyset$. Let $\pi(\beta) = (\beta_+, \beta_0, \beta_-)$ be a partition of β. We know that there are only three cases:

- **Case 1:** $\beta = \beta_+ \neq \emptyset$. From (31), we know that $\Gamma^G = 0$.
- **Case 2:** $\beta = \beta_- \neq \emptyset$. From (31), we know that $\Gamma^H = 0$.
- **Case 3:** $\beta = \beta_0 \neq \emptyset$. From (32), we know that $\Gamma^G \le 0 \quad \text{and} \quad \Gamma^H \ge 0\,.$

Therefore, we may conclude that if $G(\bar z) = H(\bar z) = 0$, either $\Gamma^G < 0$, $\Gamma^H > 0$ or $\Gamma^G \Gamma^H = 0$.

Theorem 6.1 *Let $\bar z$ be a local optimal solution of SDCMPCC. Suppose that either the problem GP-SDCMPCC is Clarke calm at $\bar z$ or one of the following constraint qualifications holds. Then $\bar z$ is a M-stationary point of SDCMPCC.*

(i) There is no singular M-multiplier for problem SDCMPCC at $\bar z$.

(ii) *SDCMPCC LICQ holds at* $\bar z$*: there is no nonzero* $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \Re^p \times \Re^q \times \mathcal{S}^n \times \mathcal{S}^n$ *such that*

$$\begin{aligned} & h'(\bar z)^*\lambda^h + g'(\bar z)^*\lambda^g + G'(\bar z)^*\Gamma^G + H'(\bar z)^*\Gamma^H = 0\,, \quad (33)\\ & \widetilde{\Gamma}^G_{\alpha\alpha} = 0, \quad \widetilde{\Gamma}^G_{\alpha\beta} = 0, \quad \widetilde{\Gamma}^G_{\beta\alpha} = 0\,,\\ & \widetilde{\Gamma}^H_{\gamma\gamma} = 0, \quad \widetilde{\Gamma}^H_{\beta\gamma} = 0, \quad \widetilde{\Gamma}^H_{\gamma\beta} = 0\,,\\ & \Sigma_{\alpha\gamma} \circ \widetilde{\Gamma}^G_{\alpha\gamma} + (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{\Gamma}^H_{\alpha\gamma} = 0\,. \end{aligned}$$

(iii) *Assume that there is no inequality constraint* $g(z) \le 0$*. Assume also that* $Z = X \times \mathcal{S}^n$ *where* X *is a finite dimensional space and* $G(x,u) = u$*. The following generalized equation is strongly regular in the sense of Robinson:*

$$0 \in -F(x,u) + N_{\Re^q \times \mathcal{S}^n_+}(x,u)\,,$$

where $F(x,u) = (h(x,u), H(x,u))$*.*

(iv) *Assume that there is no inequality constraint* $g(z) \le 0$*. Assume also that* $Z = X \times \mathcal{S}^n$*,* $G(z) = u$ *and* $F(x,u) = (h(x,u), H(x,u))$*.* $-F$ *is locally strongly monotone in* u *uniformly in* x *with modulus* $\delta > 0$*, i.e., there exist neighborhood* U_1 *of* $\bar x$ *and* U_2 *of* $\bar u$ *such that*

$$\langle -F(x,u) + F(x,v), u - v\rangle \ge \delta \|u - v\|^2 \quad \forall\, u \in U_2 \cap \mathcal{S}^n_+,\ v \in \mathcal{S}^n_+,\ x \in U_1\,.$$

Proof. Condition (ii) is obviously stronger than Part (i). Condition (i) is a necessary and sufficient condition for the perturbed feasible region of the constraint system to be pseudo Lipschitz continuous. See [38, Theorem 4.7] for the proof of the implication of (iii) to (i). (iv) is a sufficient condition for (iii) and the direct proof can be found in [40, Theorem 3.2(b)]. The desired result follows from Theorem 2.2 and the expression of the limiting normal cone in Theorem 3.1. ■

Remark 6.1 *SDCMPCC LICQ is the analogue of the well-known MPCC LICQ (also called MPEC LICQ). To see this we consider the case of SDCMPCC with* $n = 1$*. Suppose that* $G(\bar z) = H(\bar z) = 0$*. Then we have* $\alpha = \gamma = \emptyset$ *and SDCMPCC LICQ means that (33) implies that* $\lambda^h = 0$*,* $\lambda^g = 0$*,* $\Gamma^G = 0$*,* $\Gamma^H = 0$*. The other two cases* $G(\bar z) > 0$*,* $H(\bar z) = 0$ *and* $G(\bar z) = 0$*,* $H(\bar z) < 0$ *are also easy to see. We would like to remark that unlike in MPCC case, we can only show that SDCMPCC LICQ is a constraint qualification for M-stationary condition instead of S-stationary condition.*

7 C-stationary conditions

In this section, we consider the C-stationary condition by reformulating SDCMPCC as a nonsmooth problem:

$$\begin{aligned} (\mathrm{NS-SDCMPCC}) \qquad \min \quad & f(z)\\ \text{s.t.} \quad & h(z) = 0,\\ & g(z) \le 0,\\ & G(z) - \Pi_{\mathcal{S}^n_+}(G(z) + H(z)) = 0\,. \end{aligned}$$

From (6), we know that the reformulation NS-SDCMPCC is equivalent to SDCMPCC. As in the MPCC case, C-stationary condition introdued below is the nonsmooth KKT condition of NS-SDCMPCC by using the Clarke subdifferential.

Definition 7.1 *Let $\bar{z}$ be a feasible solution of SDCMPCC. Let $A = G(\bar{z}) + H(\bar{z})$ have the eigenvalue decomposition (4). We say that $\bar{z}$ is a C-stationary point of SDCMPCC if there exists $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \Re^p \times \Re^q \times \mathcal{S}^n \times \mathcal{S}^n$ such that (24)-(28) hold and*

$$\langle \widetilde{\Gamma}^G_{\beta\beta}, \widetilde{\Gamma}^H_{\beta\beta} \rangle \le 0\,, \tag{34}$$

where $\widetilde{\Gamma}^G = \overline{P}^T \Gamma^G \overline{P}$ and $\widetilde{\Gamma}^H = \overline{P}^T \Gamma^H \overline{P}$. We say that $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \Re^p \times \Re^q \times \mathcal{S}^n \times \mathcal{S}^n$ is a singular C-multiplier for SDCMPCC if it is not equal to zero and all conditions above hold except the term $\nabla f(\bar{z})$ vanishes in (24).

It is easy to see that as in MPCC case,

S-stationary condition $\Longrightarrow$ M-stationary condition $\Longrightarrow$ C-stationary condition.

Also, we know that the C-stationary condition coincides with the C-stationary condition in the MPCC case. To show this we consider the case $n = 1$. We only need to consider the case $G(\bar{z}) = H(\bar{z}) = 0$ since the other cases are the same as the S- and M-stationary conditions. In this case we know from (34) that $\Gamma^G \Gamma^H \le 0$.

Theorem 7.1 *Let $\bar{z}$ be a local optimal solution of SDCMPCC. Suppose that the problem SDCMPCC is Clarke calm at $\bar{z}$ or there is no singular C-multiplier for problem SDCMPCC at $\bar{z}$. Then $\bar{z}$ is a C-stationary point of SDCMPCC.*

Proof. By Theorem 2.1 with $K = \{0\}$, we know that there exist $\lambda^h \in \Re^p$, $\lambda^g \in \Re^q$ and $\Gamma \in S^n$ such that

$$0 \in \partial^c_z L(\bar{z}, \lambda^h, \lambda^g, \Gamma), \quad \lambda^g \ge 0 \quad \text{and} \quad \langle \lambda^g, g(\bar{z}) \rangle = 0\,, \tag{35}$$

where $L(z, \lambda^h, \lambda^g, \Gamma) := f(z) + \langle \lambda^h, h(z) \rangle + \langle \lambda^g, g(z) \rangle + \langle \Gamma, G(z) - \Pi_{\mathcal{S}^n_+}(G(z) + H(z)) \rangle$.

Consider the Clarke subdifferential of the nonsmooth part $S(z) := \langle \Gamma, \Pi_{\mathcal{S}^n_+}(G(z)+H(z)) \rangle$ of L. By the chain rule [4, Corollary pp.75], for any $v \in Z$, we have

$$\partial^c S(\bar{z}) v = \langle \Gamma, \partial^c \Pi_{\mathcal{S}^n_+}(A)(G'(\bar{z})v + H'(\bar{z})v) \rangle\,.$$

Therefore, since V is self-adjoint (see e.g., [16, Proposition 1(a)]), we know from (35) that there exists $V \in \partial^c \Pi_{\mathcal{S}^n_+}(A)$ such that

$$\nabla f(\bar{z}) + h'(\bar{z})^* \lambda^h + g'(\bar{z})^* \lambda^g + G'(\bar{z})^* \Gamma - (G'(\bar{z})^* + H'(\bar{z})^*) V(\Gamma) = 0\,. \tag{36}$$

Define $\Gamma^G := \Gamma - V(\Gamma)$ and $\Gamma^H := -V(\Gamma)$. Then (24)-(25) follow from (35) and (36) immediately. By [31, Proposition 2.2], we know that there exists a $W \in \partial^c \Pi_{\mathcal{S}^{|\beta|}_+}(0)$ such that

$$V(\Gamma) = \overline{P} \begin{bmatrix} \widetilde{\Gamma}_{\alpha\alpha} & \widetilde{\Gamma}_{\alpha\beta} & \Sigma_{\alpha\gamma} \circ \widetilde{\Gamma}_{\alpha\gamma} \\ \widetilde{\Gamma}^T_{\alpha\beta} & W(\widetilde{\Gamma}_{\beta\beta}) & 0 \\ \widetilde{\Gamma}^T_{\alpha\gamma} \circ \Sigma^T_{\alpha\gamma} & 0 & 0 \end{bmatrix} \overline{P}^T,$$

where $\Sigma \in \mathcal{S}^n$ is defined by (9). Therefore, it is easy to see that (26)-(28) hold. Moreover, from [16, Proposition 1(c)], we know that

$$\langle W(\widetilde{\Gamma}_{\beta\beta}), \widetilde{\Gamma}_{\beta\beta} - W(\widetilde{\Gamma}_{\beta\beta}) \rangle \ge 0\,,$$

which implies $\langle \widetilde{\Gamma}^G_{\beta\beta}, \widetilde{\Gamma}^H_{\beta\beta} \rangle \le 0$. Hence, we know $\bar{z}$ is a C-stationary point of SDCMPCC. ■

References

[1] A. Ben-Tal and A. Nemirovski, *Robust convex optimization-methodology and applications*, Mathematical Programming, **92** (2002), pp. 453–480.

[2] R. Bhatia. *Matrix Analysis*, Springer-Verlag, New York, 1997.

[3] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[5] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.

[6] S. Dempe, *Foundations of Bilevel Programming*, Kluwer Academic Publishers, 2002.

[7] B. C. Eaves, *On the basic theorem for complementarity*, Mathematical Programming, **1** (1971), pp. 68–75.

[8] F. Faccchinei and J. S. Pang, *Finite-dimensional Variational Inequalities and Complementarity Problem*, Springer-Verlag, New York, 2003.

[9] R. Fletcher, *Semi-definite matrix constraints in optimization*, SIAM Journal on Control and Optimization, **23** (1985), pp. 493–513.

[10] M. L. Flegel and C. Kanzow, *On the Guignard constraint qualification for mathematical programs with equilibrium constraints*, Optimization, **54** (2005), pp. 517–534.

[11] R. Henrion and J. Outrata, *On the calmness of a class of multifunctions*, SIAM Journal on Optimization, **13** (2002), pp. 603–618.

[12] R. Henrion and J. Outrata, *Calmness of constraint systems with applications*, Mathematical Programming, Seies. B, **104** (2005), pp. 437–464.

[13] A. S. Lewis, *Nonsmooth analysis of eigenvalues*, Mathematical Programming, **84** (1999), pp. 1–24.

[14] K. Löwner, *Über monotone matrixfunktionen*, Mathematische Zeitschrift, **38** (1934), pp. 177–216.

[15] Z. Q. Luo, J. S. Pang and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.

[16] F. Meng, D. F. Sun and G. Y. Zhao, *Semismoothness of solutions to generalized equations and Moreau-Yosida regularization*, Mathematical Programming, **104** (2005), pp. 561–581.

[17] B. S. Mordukhovich , *Generalized differential calculus for nonsmooth and set-valued mappings*, Journal of Mathematical Analysis and Applications, **183** (1994), pp.250–288.

[18] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory*, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 330, Springer, 2006.

[19] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, II: Applications*, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 331, Springer, 2006.

[20] B. S. Mordukhovich and Y. Shao, *Nonsmooth sequential analysis in Asplund space*, Transactions of the American Mathematical Society, **348** (1996), pp. 215–220.

[21] J. V. Outrata, M. Kočvara and J. Zowe, *Nonsmooth Approach to Optimization Problem with Equilibrium Constraints: Theory, Application and Numerical Results*, Kluwer, Dordrecht, The Netherlands, 1998.

[22] J. S. Pang, D. F. Sun and J. Sun, *Semismooth homeomorphisms and strong stability of semidefinite and Lorentz cone complementarity problems*, Mathematics of Operations Research **28** (2003), pp. 39–63.

[23] H. D. Qi and P. Fusek, *Metric regularity and strong regularity in linear and nonlinear semidefinite programming*, Technical Report, School of Mathematics, the University of Southampton (2007).

[24] S. M. Robinson, *Stability theory for systems of inequalities, part I: linear systems*, SIAM Journal on Numerical Analysis, **12** (1975), pp. 754–769.

[25] S. M. Robinson, *Stability theory for systems of inequalities, part II: nonlinear systems*, SIAM Journal on Numerical Analysis, **13** (1976), pp. 473–513.

[26] S. M. Robinson, *First order conditions for general nonlinear optimization*, SIAM Journal on Applied Mathematics, **30** (1976), pp. 597–607.

[27] S. M. Robinson, *Some continuity properties of polyhedral multifunctions*, Mathematical Programming Studies, **14** (1981), pp. 206–214.

[28] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[29] R. T. Rockafellar and R. J-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

[30] H. Scheel and S. Scholtes, *Mathematical programs with complementarity constraints: stationarity, optimality and sensitivity*, Mathematics of Operations Research, **25** (2000), pp. 1–22.

[31] D. F. Sun, *The strong second-order sufficient condition and constraint nondegeneracy in nonlinear semidefinite programming and their implications*, Mathematics of Operations Research, **31** (2006), pp. 761–776.

[32] D. F. Sun and J. Sun, *Semismooth matrix valued functions.* Mathematics of Operations Research, **27** (2002), pp. 150–169.

[33] D. F. Sun and J. Sun, *Strong semismoothness of eigenvalues of symmetric matrices and its applications in inverse eigenvalue problems*, SIAM Journal on Numerical Analysis, **40** (2003), pp. 2352–2367.

[34] N. K. Tsing, M. K. H. Fan, and E.I. Verriest, *On analyticity of functions involving eigenvalues*, Linear Algebra and its Applications, **207** (1994), pp. 159–180.

[35] Z. Wu and J. J. Ye, *First and second order condition for error bounds*, SIAM Journal on Optimization, **14** (2003), pp. 621–645.

[36] T. Yan and M. Fukushima, *Smoothing method for mathematical programs with symmetric cone complementarity constraints*, Technical Report 2009-017, Department of Applied Mathematics and Physics, Kyoto University (September 2009).

[37] J. J. Ye, *Optimality conditions for optimization problems with complementarity constraints*, SIAM Journal on Optimization, **9** (1999), pp. 374–387.

[38] J. J. Ye, *Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints*, SIAM Journal on Optimization, **10** (2000), pp. 943–962.

[39] J. J. Ye, *Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints*, Journal of Mathematical Analysis and Applications, **307** (2005), pp. 305–369.

[40] J. J. Ye and X. Y. Ye, *Necessary optimality conditions for optimization problems with variational inequality constraints*, Mathematics of Operations Research, **22** (1997), pp. 977–977.

[41] J. J. Ye, D. L. Zhu and Q. J. Zhu, *Exact penalization and necessary optimality conditions for generalized bilevel programming problems*, SIAM Journal on Optimization, **7** (1997), pp. 481–507.

Appendix

Proof of Proposition 2.6: Firstly, we will show that (10) holds for the case that $A = \Lambda(A)$. For any $H \in \mathcal{S}^n$, denote $Y := A + H$. Let $P \in \mathcal{O}^n$ (depending on H) be such that

$$\Lambda(A) + H = P\Lambda(Y)P^T. \tag{37}$$

Let $\delta > 0$ be any fixed number such that $0 < \delta < \frac{\lambda_{|\alpha|}}{2}$ if $\alpha \neq \emptyset$ and be any fixed positive number otherwise. Then, define the following continuous scalar function

$$f(t) := \begin{cases} t & \text{if} \quad t > \delta\,, \\ 2t - \delta & \text{if} \quad \frac{\delta}{2} < t < \delta\,, \\ 0 & \text{if} \quad t < \frac{\delta}{2}\,. \end{cases}$$

Therefore, we have

$$\{\lambda_1(A), \ldots, \lambda_{|\alpha|}(A)\} \in (\delta, +\infty) \quad \text{and} \quad \{\lambda_{|\alpha|+1}(A), \ldots, \lambda_n(A)\} \in (-\infty, \frac{\delta}{2})\,.$$

For the scalar function f, let $F : \mathcal{S}^n \to \mathcal{S}^n$ be the corresponding Löwner's operator [14], i.e.,

$$F(Z) := \sum_{i=1}^{n} f(\lambda_i(Z))u_i u_i^T \quad \forall\, Z \in \mathcal{S}^n\,, \tag{38}$$

where $U \in \mathcal{O}^n(Z)$. Since f is real analytic on the open set $(-\infty, \frac{\delta}{2}) \cup (\delta, +\infty)$, we know from [34, Theorem 3.1] that F is analytic at A. Therefore, it is well-known (see e.g., [2, Theorem V.3.3]) that for H sufficiently close to zero,

$$F(A+H) - F(A) - F'(A)H = O(\|H\|^2) \tag{39}$$

and

$$F'(A)H = \begin{bmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & \Sigma_{\alpha\gamma} \circ H_{\alpha\gamma} \\ H_{\alpha\beta}^T & 0 & 0 \\ \Sigma_{\alpha\gamma}^T \circ H_{\alpha\gamma}^T & 0 & 0 \end{bmatrix},$$

where $\Sigma \in \mathcal{S}^n$ is given by (9) . Let $R(\cdot) := \Pi_{\mathcal{S}^n_+}(\cdot) - F(\cdot)$. By the definition of f, we know that $F(A) = A_+ := \Pi_{\mathcal{S}^n_+}(A)$, which implies that $R(A) = 0$. Meanwhile, it is clear that the matrix valued function R is directionally differentiable at A, and from (8), the directional derivative of R for any given direction $H \in \mathcal{S}^n$, is given by

$$R'(A;H) = \Pi'_{\mathcal{S}^n_+}(A;H) - F'(A)H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Pi_{\mathcal{S}^{|\beta|}_+}(H_{\beta\beta}) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{40}$$

By the Lipschitz continuity of $\lambda(\cdot)$, we know that for H sufficiently close to zero,

$$\{\lambda_1(Y), \ldots, \lambda_{|\alpha|}(Y)\} \in (\delta, +\infty), \quad \{\lambda_{|\alpha|+1}(Y), \ldots, \lambda_{|\beta|}(Y)\} \in (-\infty, \frac{\delta}{2})$$

and

$$\{\lambda_{|\beta|+1}(Y), \ldots, \lambda_n(Y)\} \in (-\infty, 0).$$

Therefore, by the definition of F, we know that for H sufficiently close to zero,

$$R(A+H) = \Pi_{\mathcal{S}^n_+}(A+H) - F(A+H) = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\Lambda(Y)_{\beta\beta})_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T. \tag{41}$$

Since P satisfies (37), we know that for any $\mathcal{S}^n \ni H \to 0$, there exists an orthogonal matrix $Q \in \mathcal{O}^{|\beta|}$ such that

$$P_\beta = \begin{bmatrix} O(\|H\|) \\ P_{\beta\beta} \\ O(\|H\|) \end{bmatrix} \quad \text{and} \quad P_{\beta\beta} = Q + O(\|H\|^2), \tag{42}$$

which was stated in [33] and was essentially proved in the derivation of Lemma 4.12 in [32]. Therefore, by noting that $(\Lambda(Y)_{\beta\beta})_+ = O(\|H\|)$, we obtain from (40), (41) and (42) that

$$\begin{aligned} R(A+H) - R(A) - R'(A;H) &= \begin{bmatrix} O(\|H\|^3) & O(\|H\|^2) & O(\|H\|^3) \\ O(\|H\|^2) & P_{\beta\beta}(\Lambda(Y)_{\beta\beta})_+ P_{\beta\beta}^T - \Pi_{\mathcal{S}^{|\beta|}_+}(H_{\beta\beta}) & O(\|H\|^2) \\ O(\|H\|^3) & O(\|H\|^2) & O(\|H\|^3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q(\Lambda(Y)_{\beta\beta})_+ Q^T - \Pi_{\mathcal{S}^{|\beta|}_+}(H_{\beta\beta}) & 0 \\ 0 & 0 & 0 \end{bmatrix} + O(\|H\|^2). \end{aligned}$$

By (37) and (42), we know that

$$\Lambda(Y)_{\beta\beta} = P_\beta^T \Lambda(A) P_\beta + P_\beta^T H P_\beta = P_{\beta\beta}^T H_{\beta\beta} P_{\beta\beta} + O(\|H\|^2) = Q^T H_{\beta\beta} Q + O(\|H\|^2)\,.$$

Since $Q \in \mathcal{O}^{|\beta|}$, we have

$$H_{\beta\beta} = Q\Lambda(Y)_{\beta\beta} Q^T + O(\|H\|^2)\,.$$

By noting that $\Pi_{\mathcal{S}_+^{|\beta|}}(\cdot)$ is globally Lipschitz continuous, we obtain that

$$Q(\Lambda(Y)_{\beta\beta})_+ Q^T - \Pi_{\mathcal{S}_+^{|\beta|}}(H_{\beta\beta}) = Q(\Lambda(Y)_{\beta\beta})_+ Q^T - \Pi_{\mathcal{S}_+^{|\beta|}}(Q\Lambda(Y)_{\beta\beta}Q^T) + O(\|H\|^2) = O(\|H\|^2)\,.$$

Therefore,

$$R(A+H) - R(A) - R'(A;H) = O(\|H\|^2)\,. \tag{43}$$

By combining (39) and (43), we know that for any $\mathcal{S}^n \ni H \to 0$,

$$\Pi_{\mathcal{S}_+^n}(\Lambda(A)+H) - \Pi_{\mathcal{S}_+^n}(\Lambda(A)) - \Pi'_{\mathcal{S}_+^n}(\Lambda(A);H) = O(\|H\|^2)\,. \tag{44}$$

Next, consider the case that $A = \overline{P}^T \Lambda(A) \overline{P}$. Re-write (37) as

$$\Lambda(A) + \overline{P}^T H \overline{P} = \overline{P}^T P \Lambda(Y) P^T \overline{P}\,.$$

Let $\widetilde{H} := \overline{P}^T H \overline{P}$. Then, we have

$$\Pi_{\mathcal{S}_+^n}(A+H) = \overline{P}\, \Pi_{\mathcal{S}_+^n}(\Lambda(A) + \widetilde{H}) \overline{P}^T\,.$$

Therefore, since $\overline{P} \in \mathcal{O}^n$, we know from (44) and (8) that for any $\mathcal{S}^n \ni H \to 0$, (10) holds. ■

Proof of Proposition 3.3: Denote the set in the righthand side of (21) by $\mathcal{N}$. We first show that $N_{\text{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(0,0) \subseteq \mathcal{N}$. By the definition of the limiting normal cone in (3), we know that $(U^*, V^*) \in N_{\text{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(0,0)$ if and only if there exist two sequences $\{(U^{k^*}, V^{k^*})\}$ converging to (U^*, V^*) and $\{(U^k, V^k)\}$ converging to $(0,0)$ with $(U^{k^*}, V^{k^*}) \in N^\pi_{\text{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(U^k, V^k)$ and $(U^k, V^k) \in \text{gph}\, N_{\mathcal{S}_+^{|\beta|}}$ for each k.

For each k, denote $A^k := U^k + V^k \in \mathcal{S}^{|\beta|}$ and let $A^k = P^k \Lambda(A^k)(P^k)^T$ with $P^k \in \mathcal{O}^{|\beta|}$ be the eigenvalue decomposition of A^k. Then for any $i \in \{1, \cdots, |\beta|\}$, we have

$$\lim_{k\to\infty} \lambda_i(A^k) = 0.$$

Since $\{P^k\}_{k=1}^\infty$ is uniformly bounded, by taking a subsequence if necessary, we may assume that $\{P^k\}_{k=1}^\infty$ converges to an orthogonal matrix $Q := \lim_{k\to\infty} P^k \in \mathcal{O}^{|\beta|}$. For each k, we know that the vector $\lambda(A^k)$ is an element of $\Re_{\gtrsim}^{|\beta|}$. By taking a subsequence if necessary, we may assume that for each k, $\Lambda(A^k)$ has the same form, i.e.,

$$\Lambda(A^k) = \begin{bmatrix} \Lambda(A^k)_{\beta_+\beta_+} & 0 & 0 \\ 0 & \Lambda(A^k)_{\beta_0\beta_0} & 0 \\ 0 & 0 & \Lambda(A^k)_{\beta_-\beta_-} \end{bmatrix},$$

where β_+, β_0 and β_- are the three index sets defined by

$$\beta_+ := \{i : \lambda_i(A^k) > 0\}, \quad \beta_0 := \{i : \lambda_i(A^k) = 0\} \quad \text{and} \quad \beta_- := \{i : \lambda_i(A^k) < 0\}\,.$$

Since $(U^{k*}, V^{k*}) \in N^\pi_{\mathrm{gph}\, N_{\mathcal{S}^{|\beta|}_+}}(U^k, V^k)$, we know from Proposition 3.2 that for each k, there exist

$$\Theta_1^k = \begin{bmatrix} E_{\beta_+\beta_+} & E_{\beta_+\beta_0} & \Sigma^k_{\beta_+\beta_-} \\ E^T_{\beta_+\beta_0} & 0 & 0 \\ (\Sigma^k_{\beta_+\beta_-})^T & 0 & 0 \end{bmatrix}$$

and

$$\Theta_2^k = \begin{bmatrix} 0 & 0 & E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-} \\ 0 & 0 & E_{\beta_0\beta_-} \\ (E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-})^T & (E_{\beta_0\beta_-})^T & E_{\beta_-\beta_-} \end{bmatrix}$$

such that

$$\Theta_1^k \circ \widetilde{U^*}^k + \Theta_2^k \circ \widetilde{V^k}^* = 0, \quad \widetilde{U^k}^*_{\beta_0\beta_0} \preceq 0 \quad \text{and} \quad \widetilde{V^k}^*_{\beta_0\beta_0} \succeq 0\,, \tag{45}$$

where $\widetilde{U^k}^* = (P^k)^T U^{k*} P^k$, $\widetilde{V^k}^* = (P^k)^T V^{k*} P^k$ and

$$(\Sigma^k)_{i,j} = \frac{\max\{\lambda_i(A^k), 0\} - \max\{\lambda_j(A^k), 0\}}{\lambda_i(A^k) - \lambda_j(A^k)} \quad \forall\, (i,j) \in \beta_+ \times \beta_-\,. \tag{46}$$

Since for each k, each element of $\Sigma^k_{\beta_+\beta_-}$ belongs to the interval $[0,1]$, by further taking a subsequence if necessary, we may assume that the limit of $\{\Sigma^k_{\beta_+\beta_-}\}_{k=1}^\infty$ exists. Therefore, by the definition of $\mathcal{U}_{|\beta|}$ in (18), we know that

$$\lim_{k\to\infty} \Theta_1^k = \Xi_1 \in \mathcal{U}_{|\beta|} \quad \text{and} \quad \lim_{k\to\infty} \Theta_2^k = \Xi_2\,,$$

where Ξ_1 and Ξ_2 are given by (20). Therefore, we obtain from (45) that $(U^*, V^*) \in \mathcal{N}$.

Conversely, let $(U^*, V^*) \in \mathcal{N}$. By the definition of $\mathcal{N}$, we know that there exist $\Xi_1 \in \mathcal{U}_{|\beta|}$ (with a partition $\pi(\beta) = (\beta_+, \beta_0, \beta_-) \in \mathscr{P}(\beta)$) and $Q \in \mathcal{O}_{|\beta|}$ such that

$$\Xi_1 \circ Q^T U^* Q + \Xi_2 \circ Q^T V^* Q = 0, \quad Q^T_{\beta_0} U^* Q_{\beta_0} \preceq 0 \quad \text{and} \quad Q^T_{\beta_0} V^* Q_{\beta_0} \succeq 0\,. \tag{47}$$

Since $\Xi_1 \in \mathcal{U}_{|\beta|}$, there exists a sequence $\{z^k\} \in \Re^{|\beta|}_{\gtrsim}$, $z^k \to 0$ such that $\Xi_1 = \lim_{k\to\infty} D(z^k)$. Without loss of generality, we can assume that for all k sufficiently large,

$$z_i^k > 0 \quad \forall\, i \in \beta_+, \quad z_i^k = 0 \quad \forall\, i \in \beta_0 \quad \text{and} \quad z_i^k < 0 \quad \forall\, i \in \beta_-\,.$$

For each k, define

$$U^k := Q \begin{bmatrix} z^k_{\beta_+} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^T, \quad V^k := Q \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z^k_{\beta_-} \end{bmatrix} Q^T \quad \text{and} \quad A^k := Q\,\mathrm{diag}(z^k)\,Q^T\,.$$

From Proposition 2.5, it is clear that for each k, $(U^k, V^k) \in \mathrm{gph}\, N_{\mathcal{S}^{|\beta|}_+}$ and $U^k + V^k = A^k$. Since $z^k \to 0$ as $k \to \infty$, we know that

$$A^k \to 0 \quad \text{and} \quad (U^k, V^k) \to (0,0) \quad \text{as} \quad k \to \infty\,.$$

For each k, define

$$\Theta_1^k := \begin{bmatrix} E_{\beta_+\beta_+} & E_{\beta_+\beta_0} & \Sigma^k_{\beta_+\beta_-} \\ E^T_{\beta_+\beta_0} & 0 & 0 \\ (\Sigma^k_{\beta_+\beta_-})^T & 0 & 0 \end{bmatrix}$$

and

$$\Theta_2^k := \begin{bmatrix} 0 & 0 & E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-} \\ 0 & 0 & E_{\beta_0\beta_-} \\ (E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-})^T & (E_{\beta_0\beta_-})^T & E_{\beta_-\beta_-} \end{bmatrix},$$

where $\Sigma^k_{\beta_+\beta_-} = (D(z^k))_{\beta_+\beta_-}$.

Next, we define a sequence $\{(U^{k^*}, V^{k^*})\}_{k=1}^{\infty}$ such that for each k,

$$(U^{k^*}, V^{k^*}) \in N^{\pi}_{\mathrm{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(U^k, V^k) \quad \text{and} \quad (U^*, V^*) = \lim_{k\to\infty} (U^{k^*}, V^{k^*})\,.$$

Let $i, j \in \{1, \ldots, |\beta|\}$. If (i,j) and $(j,i) \notin \beta_+ \times \beta_-$, then by observing that

$$(\Theta_1^k)_{i,j} = (\Xi_1)_{i,j} \quad \text{and} \quad (\Theta_2^k)_{i,j} = (\Xi_2)_{i,j}\,,$$

we define

$$\widetilde{U^k}^*_{i,j} \equiv \widetilde{U}^*_{i,j} \quad \text{and} \quad \widetilde{V^k}^*_{i,j} \equiv \widetilde{V}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{48}$$

Now, suppose that (i,j) or $(j,i) \in \beta_+ \times \beta_-$. For each k, denote $c^k := (\Sigma^k_{\beta_+\beta_-})_{i,j}$. Then, from (46), we have

$$c^k \in (0,1) \quad \forall\, k \quad \text{and} \quad c^k \to c := (\Xi_1)_{i,j} \in [0,1] \quad \text{as } k \to \infty\,.$$

Consider the following two cases:

Case 1: $c \neq 1$. Since $c^k \neq 1$ for all k, we define

$$\widetilde{U^k}^*_{i,j} \equiv \widetilde{U}^*_{i,j} \quad \text{and} \quad \widetilde{V^k}^*_{i,j} = \frac{c^k}{c^k - 1}\widetilde{U^k}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{49}$$

By (47), we know that $c\widetilde{U}^*_{i,j} + (1-c)\widetilde{V}^*_{i,j} = 0$, which implies $\widetilde{V}^*_{i,j} = \dfrac{c}{c-1}\widetilde{U}^*_{i,j} = \lim\limits_{k\to\infty} \widetilde{V^k}^*_{i,j}$. Therefore, we obtain that

$$c_k\widetilde{U^k}^*_{i,j} + (1-c_k)\widetilde{V^k}^*_{i,j} = 0 \quad \forall\, k \quad \text{and} \quad (\widetilde{U^k}^*_{i,j}, \widetilde{V^k}^*_{i,j}) \to (\widetilde{U}^*_{i,j}, \widetilde{V}^*_{i,j}) \quad \text{as } k \to \infty\,.$$

Case 2: $c = 1$. Since $c^k \neq 0$ for all k, in this case we define

$$\widetilde{V^k}^*_{i,j} \equiv \widetilde{V}^*_{i,j} \quad \text{and} \quad \widetilde{U^k}^*_{i,j} = \frac{c^k - 1}{c^k}\widetilde{V^k}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{50}$$

Again, by (47), we know that $\widetilde{U}^*_{i,j} = \dfrac{c-1}{c}\widetilde{V}^*_{i,j} = \lim\limits_{k\to\infty} \widetilde{U^k}^*_{i,j}$. Therefore, we have

$$c_k\widetilde{U^k}^*_{i,j} + (1-c_k)\widetilde{V^k}^*_{i,j} = 0 \quad \forall\, k \quad \text{and} \quad (\widetilde{U^k}^*_{i,j}, \widetilde{V^k}^*_{i,j}) \to (\widetilde{U}^*_{i,j}, \widetilde{V}^*_{i,j}) \quad \text{as } k \to \infty\,.$$

Finally, by (47), we know that the sequence $\{(U^{k^*}, V^{k^*})\}_{k=1}^{\infty}$ defined by (48), (49) and (50) satisfies

$$\Theta_1^k \circ \widetilde{U^k}^* + \Theta_2^k \circ \widetilde{V^k}^* = 0, \quad \widetilde{U^k}^*_{\beta_0\beta_0} = \widetilde{U}^*_{\beta_0\beta_0} \preceq 0 \quad \text{and} \quad \widetilde{V^k}^*_{\beta_0\beta_0} = \widetilde{V}^*_{\beta_0\beta_0} \succeq 0\,.$$

This implies that for each k, $(U^{k^*}, V^{k^*}) \in N^\pi_{\mathrm{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(U^k, V^k)$ and

$$(U^*, V^*) = \lim_{k\to\infty} (U^{k^*}, V^{k^*}) .$$

Therefore, $(U^*, V^*) \in N_{\mathrm{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(0,0)$. Hence, the assertion of the proposition follows. ■

Proof of Theorem 3.1: " $\Longrightarrow$ " Suppose that $(X^*, Y^*) \in N_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X, Y)$. By the definition of the limiting normal cone in (3), we know that $(X^*, Y^*) = \lim_{k\to\infty} (X^{k^*}, Y^{k^*})$ with

$$(X^{k^*}, Y^{k^*}) \in N^\pi_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X^k, Y^k) \quad k = 1, 2, \dots ,$$

where $(X^k, Y^k) \to (X, Y)$ and $(X^k, Y^k) \in \mathrm{gph}\, N_{\mathcal{S}^n_+}$. For each k, denote $A^k := X^k + Y^k$ and let $A^k = P^k \Lambda(A^k)(P^k)^T$ be the eigenvalue decomposition of A^k. Since $\Lambda(A) = \lim_{k\to\infty} \Lambda(A^k)$, we know that $\Lambda(A^k)_{\alpha\alpha} \succ 0$, $\Lambda(A^k)_{\gamma\gamma} \prec 0$ for k sufficiently large and $\lim_{k\to\infty} \Lambda(A^k)_{\beta\beta} = 0$. Since $\{P^k\}_{k=1}^\infty$ is uniformly bounded, by taking a subsequence if necessary, we may assume that $\{P^k\}_{k=1}^\infty$ converges to an orthogonal matrix $\widehat{P} \in \mathcal{O}^n(A)$. We can write $\widehat{P} = \begin{bmatrix} \overline{P}_\alpha & \overline{P}_\beta Q & \overline{P}_\gamma \end{bmatrix}$, where $Q \in \mathcal{O}^{|\beta|}$ can be any $|\beta| \times |\beta|$ orthogonal matrix. By further taking a subsequence if necessary, we may also assume that there exists a partition $\pi(\beta) = (\beta_+, \beta_0, \beta_-)$ of β such that for each k,

$$\lambda_i(A^k) > 0 \quad \forall\, i \in \beta_+, \quad \lambda_i(A^k) = 0 \quad \forall\, i \in \beta_0 \quad \text{and} \quad \lambda_i(A^k) < 0 \quad \forall\, i \in \beta_- .$$

This implies that for each k,

$$\{i : \lambda_i(A^k) > 0\} = \alpha \cup \beta_+, \quad \{i : \lambda_i(A^k) = 0\} = \beta_0 \quad \text{and} \quad \{i : \lambda_i(A^k) < 0\} = \beta_- \cup \gamma .$$

Then, for each k, since $(X^{k^*}, Y^{k^*}) \in N^\pi_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X^k, Y^k)$, we know from Proposition 3.2 that there exist

$$\Theta_1^k = \begin{bmatrix} E_{\alpha\alpha} & E_{\alpha\beta_+} & E_{\alpha\beta_0} & \Sigma^k_{\alpha\beta_-} & \Sigma^k_{\alpha\gamma} \\ E^T_{\alpha\beta_+} & E_{\beta_+\beta_+} & E_{\beta_+\beta_0} & \Sigma^k_{\beta_+\beta_-} & \Sigma^k_{\beta_+\gamma} \\ E^T_{\alpha\beta_0} & E^T_{\beta_+\beta_0} & 0 & 0 & 0 \\ {\Sigma^k_{\alpha\beta_-}}^T & {\Sigma^k_{\beta_+\beta_-}}^T & 0 & 0 & 0 \\ {\Sigma^k_{\alpha\gamma}}^T & {\Sigma^k_{\beta_+\gamma}}^T & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Theta_2^k = \begin{bmatrix} 0 & 0 & 0 & E_{\alpha\beta_-} - \Sigma^k_{\alpha\beta_-} & E_{\alpha\gamma} - \Sigma^k_{\alpha\gamma} \\ 0 & 0 & 0 & E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-} & E_{\beta_+\gamma} - \Sigma^k_{\beta_+\gamma} \\ 0 & 0 & 0 & E_{\beta_0\beta_-} & E_{\beta_0\gamma} \\ (E_{\alpha\beta_-} - \Sigma^k_{\alpha\beta_-})^T & (E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-})^T & E^T_{\beta_0\beta_-} & E_{\beta_-\beta_-} & E_{\beta_-\gamma} \\ (E_{\alpha\gamma} - \Sigma^k_{\alpha\gamma})^T & (E_{\beta_+\gamma} - \Sigma^k_{\beta_+\gamma})^T & E^T_{\beta_0\gamma} & E^T_{\beta_-\gamma} & E_{\gamma\gamma} \end{bmatrix}$$

such that

$$\Theta_1^k \circ \widetilde{X^k}^* + \Theta_2^k \circ \widetilde{Y^k}^* = 0, \quad \widetilde{X^k}^*_{\beta_0\beta_0} \preceq 0 \quad \text{and} \quad \widetilde{Y^k}^*_{\beta_0\beta_0} \succeq 0 , \tag{51}$$

where $\widetilde{X^k}^* = (P^k)^T X^{k^*} P^k, \widetilde{Y^k}^* = (P^k)^T Y^{k^*} P^k$ and

$$(\Sigma^k)_{i,j} = \frac{\max\{\lambda_i(A^k), 0\} - \max\{\lambda_j(A^k), 0\}}{\lambda_i(A^k) - \lambda_j(A^k)} \quad \forall\, (i,j) \in (\alpha \cup \beta_+) \times (\beta_- \cup \gamma) . \tag{52}$$

By taking limits as $k \to \infty$, we obtain that

$$\widetilde{X^k}^* \to \widehat{P}^T X^* \widehat{P} = \begin{bmatrix} \widetilde{X}^*_{\alpha\alpha} & \widetilde{X}^*_{\alpha\beta}Q & \widetilde{X}^*_{\alpha\gamma} \\ (\widetilde{X}^*_{\alpha\beta}Q)^T & Q^T\widetilde{X}^*_{\beta\beta}Q & Q^T\widetilde{X}^*_{\beta\gamma} \\ (\widetilde{X}^*_{\alpha\gamma})^T & (Q^T\widetilde{X}^*_{\beta\gamma})^T & \widetilde{X}_{\gamma\gamma} \end{bmatrix}$$

and

$$\widetilde{Y^k}^* \to \widehat{P}^T Y^* \widehat{P} = \begin{bmatrix} \widetilde{Y}^*_{\alpha\alpha} & \widetilde{Y}^*_{\alpha\beta}Q & \widetilde{Y}^*_{\alpha\gamma} \\ (\widetilde{Y}^*_{\alpha\beta}Q)^T & Q^T\widetilde{Y}^*_{\beta\beta}Q & Q^T\widetilde{Y}^*_{\beta\gamma} \\ (\widetilde{Y}^*_{\alpha\gamma})^T & (Q^T\widetilde{Y}^*_{\beta\gamma})^T & \widetilde{Y}_{\gamma\gamma} \end{bmatrix}.$$

By simple calculations, we obtain from (52) that

$$\lim_{k\to\infty} \Sigma^k_{\alpha\beta_-} = E_{\alpha\beta_-}, \quad \lim_{k\to\infty} \Sigma^k_{\beta_+\gamma} = 0 \quad \text{and} \quad \lim_{k\to\infty} \Sigma^k_{\alpha\gamma} = \Sigma_{\alpha\gamma}.$$

This, together with the definition of $\mathcal{U}_{|\beta|}$, shows that there exist $\Xi_1 \in \mathcal{U}_{|\beta|}$ and the corresponding Ξ_2 such that

$$\lim_{k\to\infty} \Theta^k_1 = \begin{bmatrix} E_{\alpha\alpha} & E_{\alpha\beta} & \Sigma_{\alpha\gamma} \\ E_{\beta\alpha} & \Xi_1 & 0 \\ \Sigma^T_{\alpha\gamma} & 0 & 0 \end{bmatrix} = \Theta_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Xi_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\lim_{k\to\infty} \Theta^k_2 = \begin{bmatrix} 0 & 0 & E_{\alpha\gamma} - \Sigma_{\alpha\gamma} \\ 0 & \Xi_2 & E_{\beta\gamma} \\ (E_{\alpha\gamma} - \Sigma_{\alpha\gamma})^T & E_{\gamma\beta} & E_{\gamma\gamma} \end{bmatrix} = \Theta_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Xi_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where Θ_1 and Θ_2 are given by (16). Meanwhile, since $Q \in \mathcal{O}^{|\beta|}$, by taking limits in (51) as $k \to \infty$, we obtain that

$$\Theta_1 \circ \widetilde{X}^* + \Theta_2 \circ \widetilde{Y}^* = 0, \quad \Xi_1 \circ Q^T\widetilde{X}^*_{\beta\beta}Q + \Xi_2 \circ Q^T\widetilde{Y}^*_{\beta\beta}Q = 0 \tag{53}$$

and

$$Q^T_{\beta_0}\widetilde{X}^*_{\beta\beta}Q_{\beta_0} \preceq 0 \quad \text{and} \quad Q^T_{\beta_0}\widetilde{Y}^*_{\beta\beta}Q_{\beta_0} \succeq 0.$$

Hence, by Proposition 3.3, we conclude that $(\widetilde{X}^*_{\beta\beta}, \widetilde{Y}^*_{\beta\beta}) \in N_{\mathrm{gph}\, N_{\mathcal{S}^{|\beta|}_+}}(0,0)$. From (53), it is easy to check that (X^*, Y^*) satisfies the conditions (22) and (23).

" $\Longleftarrow$ " Let (X^*, Y^*) satisfies (22) and (23). We shall show that there exist two sequences $\{(X^k, Y^k)\}$ converging to (X, Y) and $\{(X^{k^*}, Y^{k^*})\}$ converging to (X^*, Y^*) with $(X^k, Y^k) \in \mathrm{gph}\, N_{\mathcal{S}^n_+}$ and $(X^{k^*}, Y^{k^*}) \in N^\pi_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X^k, Y^k)$ for each k.

Since $(\widetilde{X}^*_{\beta\beta}, \widetilde{Y}^*_{\beta\beta}) \in N_{\mathrm{gph}\, N_{\mathcal{S}^{|\beta|}_+}}(0,0)$, by Proposition 3.3, we know that there exist an orthogonal matrix $Q \in \mathcal{O}^{|\beta|}$ and $\Xi_1 \in \mathcal{U}_{|\beta|}$ such that

$$\Xi_1 \circ Q^T\widetilde{X}^*_{\beta\beta}Q + \Xi_2 \circ Q^T\widetilde{Y}^*_{\beta\beta}Q = 0, \quad Q^T_{\beta_0}\widetilde{X}^*_{\beta\beta}Q_{\beta_0} \preceq 0 \quad \text{and} \quad Q^T_{\beta_0}\widetilde{Y}^*_{\beta\beta}Q_{\beta_0} \succeq 0. \tag{54}$$

Since $\Xi_1 \in \mathcal{U}_{|\beta|}$, we know that there exists a sequence $\{z^k\} \in \Re^{|\beta|}_{\gtrsim}$ converging to 0 such that $\Xi_1 = \lim_{k\to\infty} D(z^k)$. Without loss of generality, we can assume that there exists a partition $\pi(\beta) = (\beta_+, \beta_0, \beta_-) \in \mathscr{P}(\beta)$ such that for all k,

$$z^k_i > 0 \quad \forall\, i \in \beta_+, \quad z^k_i = 0 \quad \forall\, i \in \beta_0 \quad \text{and} \quad z^k_i < 0 \quad \forall\, i \in \beta_-.$$

For each k, let

$$X^k = \widehat{P}\begin{bmatrix} \Lambda(A)_{\alpha\alpha} & 0 & 0 & 0 & 0 \\ 0 & (z^k)_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\widehat{P}^T \quad \text{and} \quad Y^k = \widehat{P}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (z^k)_- & 0 \\ 0 & 0 & 0 & 0 & \Lambda(A)_{\gamma\gamma} \end{bmatrix}\widehat{P}^T,$$

where $\widehat{P} = \begin{bmatrix}\overline{P}_\alpha & \overline{P}_\beta Q & \overline{P}_\gamma\end{bmatrix} \in \mathcal{O}^n(A)$. Then, it is clear that $\{(X^k, Y^k)\} \in \operatorname{gph} N_{\mathcal{S}^n_+}$ converging to (X, Y). For each k, denote

$$A^k = X^k + Y^k, \quad \Theta_1^k = \begin{bmatrix} E_{\alpha\alpha} & E_{\alpha\beta_+} & E_{\alpha\beta_0} & \Sigma^k_{\alpha\beta_-} & \Sigma_{\alpha\gamma} \\ E^T_{\alpha\beta_+} & E_{\beta_+\beta_+} & E_{\beta_+\beta_0} & \Sigma^k_{\beta_+\beta_-} & \Sigma^k_{\beta_+\gamma} \\ E^T_{\alpha\beta_0} & E^T_{\beta_+\beta_0} & 0 & 0 & 0 \\ (\Sigma^k_{\alpha\beta_-})^T & (\Sigma^k_{\beta_+\beta_-})^T & 0 & 0 & 0 \\ (\Sigma_{\alpha\gamma})^T & ({\Sigma^k}_{\beta_+\gamma})^T & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Theta_2^k = \begin{bmatrix} 0 & 0 & 0 & E_{\alpha\beta_-} - \Sigma^k_{\alpha\beta_-} & E_{\alpha\gamma} - \Sigma_{\alpha\gamma} \\ 0 & 0 & 0 & E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-} & E_{\beta_+\gamma} - \Sigma^k_{\beta_+\gamma} \\ 0 & 0 & 0 & E_{\beta_0\beta_-} & E_{\beta_0\gamma} \\ (E_{\alpha\beta_-} - \Sigma^k_{\alpha\beta_-})^T & (E_{\beta_+\beta_-} - \Sigma^k_{\beta_+\beta_-})^T & E^T_{\beta_0\beta_-} & E_{\beta_-\beta_-} & E_{\beta_-\gamma} \\ (E_{\alpha\gamma} - \Sigma_{\alpha\gamma})^T & (E_{\beta_+\gamma} - \Sigma^k_{\beta_+\gamma})^T & E^T_{\beta_0\gamma} & E^T_{\beta_-\gamma} & E_{\gamma\gamma} \end{bmatrix},$$

where

$$(\Sigma^k)_{i,j} = \frac{\max\{\lambda_i(A^k)), 0\} - \max\{\lambda_j(A^k)), 0\}}{\lambda_i(A^k) - \lambda_j(A^k)} \quad \forall\, (i,j) \in (\alpha \cup \beta_+) \times (\beta_- \cup \gamma)\,.$$

Next, for each k, we define two matrices $\widehat{X^k}^*, \widehat{Y^k}^* \in \mathcal{S}^n$. Let $i, j \in \{1, \ldots, n\}$. If (i,j) and $(j,i) \notin (\alpha \times \beta_-) \cup (\beta_+ \times \gamma) \cup (\beta \times \beta)$. We define

$$\widehat{X^k}^*_{i,j} \equiv \widetilde{X}^*_{i,j}, \quad \widehat{Y^k}^*_{i,j} \equiv \widetilde{Y}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{55}$$

Otherwise, denote $c^k := (\Sigma^k)_{i,j}$, $k = 1, 2, \ldots$. We consider the following four cases.

Case 1: (i,j) or $(j,i) \in \alpha \times \beta_-$. In this case, we know from (22) that $\widetilde{X}^*_{i,j} = 0$. Since $c_k \neq 0$ for all k and $c^k \to 1$ as $k \to \infty$, we define

$$\widehat{Y^k}^*_{i,j} \equiv \widetilde{Y}^*_{i,j} \quad \text{and} \quad \widehat{X^k}^*_{i,j} = \frac{c^k - 1}{c^k}\widehat{Y^k}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{56}$$

Then, we have

$$c_k\widehat{X^k}^*_{i,j} + (1 - c_k)\widehat{Y^k}^*_{i,j} = 0 \quad \forall\, k \quad \text{and} \quad (\widehat{X^k}^*_{i,j}, \widehat{Y^k}^*_{i,j}) \to (\widetilde{X}^*_{i,j}, \widetilde{Y}^*_{i,j}) \quad \text{as } k \to \infty\,.$$

Case 2: (i,j) or $(j,i) \in \beta_+ \times \gamma$. In this case, we know from (22) that $\widetilde{Y}^*_{i,j} = 0$. Since $c_k \neq 1$ for all k and $c^k \to 0$ as $k \to \infty$, we define

$$\widehat{X^k}^*_{i,j} \equiv \widetilde{X}^*_{i,j} \quad \text{and} \quad \widehat{Y^k}^*_{i,j} = \frac{c^k}{c^k - 1}\widehat{X^k}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{57}$$

Then, we know that

$$c_k\widehat{X^k}^*_{i,j} + (1-c_k)\widehat{Y^k}^*_{i,j} = 0 \quad \forall\, k \quad \text{and} \quad (\widehat{X^k}^*_{i,j}, \widehat{Y^k}^*_{i,j}) \to (\widetilde{X}^*_{i,j}, \widetilde{Y}^*_{i,j}) \quad \text{as } k \to \infty\,.$$

Case 3: (i,j) or $(j,i) \in (\beta \times \beta) \setminus (\beta_+ \times \beta_-)$. In this case, we define

$$\widehat{X^k}^*_{i,j} \equiv q_i^T \widetilde{X}^*_{\beta\beta} q_j, \quad \widehat{Y^k}^*_{i,j} \equiv q_i^T \widetilde{Y}^*_{\beta\beta} q_j, \quad k = 1, 2, \ldots\,. \tag{58}$$

Case 4: (i,j) or $(j,i) \in \beta_+ \times \beta_-$. Since $c \in [0,1]$, we consider the following two sub-cases:

Case 4.1: $c \neq 1$. Since $c_k \neq 1$ for all k, we define

$$\widehat{X^k}^*_{i,j} \equiv q_i^T \widetilde{X}^*_{\beta\beta} q_j \quad \text{and} \quad \widehat{Y^k}^*_{i,j} = \frac{c^k}{c^k - 1}\widehat{X^k}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{59}$$

Then, from (54), we know that

$$\widehat{Y^k}^*_{i,j} \to \frac{c}{c-1} q_i^T \widetilde{X}^*_{\beta\beta} q_j = q_i^T \widetilde{Y}^*_{\beta\beta} q_j \quad \text{as } k \to \infty\,.$$

Case 4.2: $c = 1$. Since $c_k \neq 0$ for all k, we define

$$\widehat{Y^k}^*_{i,j} \equiv q_i^T \widetilde{Y}^*_{\beta\beta} q_j \quad \text{and} \quad \widehat{X^k}^*_{i,j} = \frac{c^k - 1}{c^k}\widehat{Y^k}^*_{i,j}, \quad k = 1, 2, \ldots\,. \tag{60}$$

Then, again from (54), we know that

$$\widehat{X^k}^*_{i,j} \to \frac{c-1}{c} q_i^T \widetilde{Y}^*_{\beta\beta} q_j = q_i^T \widetilde{X}^*_{\beta\beta} q_j \quad \text{as } k \to \infty\,.$$

For each k, define $X^{k^*} = \widehat{P}\widehat{X^k}^* \widehat{P}^T$ and $Y^{k^*} = \widehat{P}\widehat{Y^k}^* \widehat{P}^T$. Then, from (55)-(60) we obtain that

$$\Theta_1^k \circ \widehat{P}^T X^{k^*} \widehat{P} + \Theta_2^k \circ \widehat{P}^T Y^{k^*} \widehat{P} = 0, \quad k = 1, 2, \ldots\,.$$

and

$$(\widehat{P}^T X^{k^*} \widehat{P}, \widehat{P}^T Y^{k^*} \widehat{P}) \to (\widehat{P}^T X^* \widehat{P}, \widehat{P}^T Y^* \widehat{P}) \quad \text{as } k \to \infty\,. \tag{61}$$

Moreover, from (58) and (54), we have

$$Q_{\beta_0}^T \widetilde{X^k}^*_{\beta\beta} Q_{\beta_0} \equiv Q_{\beta_0}^T \widetilde{X}^*_{\beta\beta} Q_{\beta_0} \preceq 0 \quad \text{and} \quad Q_{\beta_0}^T \widetilde{Y^k}^*_{\beta\beta} Q_{\beta_0} \equiv Q_{\beta_0}^T \widetilde{Y}^*_{\beta\beta} Q_{\beta_0} \succeq 0, \quad k = 1, 2, \ldots\,.$$

From Proposition 3.2 and (61), we know that

$$(X^{k^*}, Y^{k^*}) \in N^\pi_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X^k, Y^k) \quad \text{and} \quad (X^*, Y^*) = \lim_{k\to\infty} (X^{k^*}, Y^{k^*})\,.$$

Hence, the assertion of the theorem follows. ∎