An Introduction to Correlation Stress Testing

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The model

These problems can be modeled in the following way

$$
\begin{array}{ll}
\min & \|H \circ (X - G)\|_F \\
\text{s.t.} & X_{ii} = 1, \; i = 1, \ldots, n \\
& X_{ij} = e_{ij}, \; (i,j) \in \mathcal{B}_e \, , \\
& X_{ij} \geq l_{ij}, \; (i,j) \in \mathcal{B}_l \, , \\
& X_{ij} \leq u_{ij}, \; (i,j) \in \mathcal{B}_u \, , \\
& X \in \mathcal{S}^n_+ \, ,
\end{array}
\tag{1}
$$

where $\mathcal{B}_e$, $\mathcal{B}_l$, and $\mathcal{B}_u$ are three index subsets of $\{(i,j) \mid 1 \leq i < j \leq n\}$ satisfying $\mathcal{B}_e \cap \mathcal{B}_l = \emptyset$, $\mathcal{B}_e \cap \mathcal{B}_u = \emptyset$, and $l_{ij} < u_{ij}$ for any $(i,j) \in \mathcal{B}_l \cap \mathcal{B}_u$.

continued

Here $\mathcal{S}^n$ and $\mathcal{S}^n_+$ are, respectively, the space of $n \times n$ symmetric matrices and the cone of positive semidefinite matrices in $\mathcal{S}^n$.

$\| \cdot \|_F$ is the Frobenius norm defined in $\mathcal{S}^n$.

$H \geq 0$ is a weight matrix.

- H_{ij} is larger if G_{ij} is better estimated.
- $H_{ij} = 0$ if G_{ij} is missing.

A matrix $X \in \mathcal{S}^n_+$ is called a correlation matrix if $X \succeq 0$ (i.e., $X \in \mathcal{S}^n_+$) and $X_{ii} = 1$, $i = 1, \ldots, n$.

A simple correlation matrix model

$$\begin{aligned} \min \quad & \|H \circ (X - G)\|_F \\ \text{s.t.} \quad & X_{ii} = 1\,,\ i = 1, \ldots, n \\ & X \succeq 0\,, \end{aligned} \tag{2}$$

The simplest corr. matrix model

$$
\begin{array}{ll}
\min & \|(X - G)\|_F \\
\text{s.t.} & X_{ii} = 1\,,\ i = 1, \ldots, n \\
 & X \succeq 0\,,
\end{array}
\tag{3}
$$

In finance and statistics, correlation matrices are in many situations found to be inconsistent, i.e., $X \not\succeq 0$.

These include, but are not limited to,

- Structured statistical estimations; data come from different time frequencies
- Stress testing regulated by Basel II;
- Expert opinions in reinsurance, and etc.

One correlation matrix

Partial market data[1]

$$G = \begin{bmatrix} 1.0000 & 0.9872 & 0.9485 & 0.9216 & -0.0485 & -0.0424 \\ 0.9872 & 1.0000 & 0.9551 & 0.9272 & -0.0754 & -0.0612 \\ 0.9485 & 0.9551 & 1.0000 & 0.9583 & -0.0688 & -0.0536 \\ 0.9216 & 0.9272 & 0.9583 & 1.0000 & -0.1354 & -0.1229 \\ -0.0485 & -0.0754 & -0.0688 & -0.1354 & 1.0000 & 0.9869 \\ -0.0424 & -0.0612 & -0.0536 & -0.1229 & 0.9869 & 1.0000 \end{bmatrix}$$

The eigenvalues of G are: $0.0087, 0.0162, 0.0347, 0.1000$, 1.9669, and 3.8736.

[1] RiskMetrics (www.riskmetrics.com/stddownload_edu.html)

Stress tested

Let's change G to

[change $G(1,6) = G(6,1)$ from -0.0424 to -0.1000]

$$\begin{bmatrix} 1.0000 & 0.9872 & 0.9485 & 0.9216 & -0.0485 & -\mathbf{0.1000} \\ 0.9872 & 1.0000 & 0.9551 & 0.9272 & -0.0754 & -0.0612 \\ 0.9485 & 0.9551 & 1.0000 & 0.9583 & -0.0688 & -0.0536 \\ 0.9216 & 0.9272 & 0.9583 & 1.0000 & -0.1354 & -0.1229 \\ -0.0485 & -0.0754 & -0.0688 & -0.1354 & 1.0000 & 0.9869 \\ -\mathbf{0.1000} & -0.0612 & -0.0536 & -0.1229 & 0.9869 & 1.0000 \end{bmatrix}$$

The eigenvalues of G are: -0.0216, 0.0305, 0.0441, 0.1078, 1.9609, and 3.8783.

Missing data

On the other hand, some correlations may not be reliable or even missing:

$$G = \begin{bmatrix} 1.0000 & 0.9872 & 0.9485 & 0.9216 & -0.0485 & --- \\ 0.9872 & 1.0000 & 0.9551 & 0.9272 & -0.0754 & -0.0612 \\ 0.9485 & 0.9551 & 1.0000 & 0.9583 & -0.0688 & -0.0536 \\ 0.9216 & 0.9272 & 0.9583 & 1.0000 & -0.1354 & -0.1229 \\ -0.0485 & -0.0754 & -0.0688 & -0.1354 & 1.0000 & 0.9869 \\ --- & -0.0612 & -0.0536 & -0.1229 & 0.9869 & 1.0000 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \color{red}{0} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \color{red}{0} & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Let us rewrite the problem:

$$\begin{array}{ll} \min & \frac{1}{2}\|H \circ (X - G)\|_F^2 \\ \text{s.t.} & X_{ii} = 1\,,\; i = 1, \ldots, n \\ & X \succeq 0\,. \end{array} \tag{4}$$

When $H = E$, the matrix of ones, we get

$$\begin{array}{ll} \min & \frac{1}{2}\|X - G\|_F^2 \\ \text{s.t.} & X_{ii} = 1\,,\; i = 1, \ldots, n \\ & X \succeq 0\,. \end{array} \tag{5}$$

which is known as the nearest correlation matrix (NCM) problem, a terminology coined by Nick Higham (2002).

The story starts

The NCM problem is a special case of the best approximation problem

$$
\begin{array}{ll}
\min & \frac{1}{2}\|x-c\|^2 \\
\text{s.t.} & \mathcal{A}x \in b + Q\,, \\
 & x \in K\,,
\end{array}
$$

where $\mathcal{X}$ is a real Euclidean space equipped with a scalar product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$

$\mathcal{A} : \mathcal{X} \to \Re^m$ is a bounded linear operator

$Q = \{0\}^p \times \Re^q_+$ is a polyhedral convex cone, $1 \le p \le m$, $q = m - p$, and K is a closed convex cone in $\mathcal{X}$.

The KKT conditions

The Karush-Kuhn-Tucker conditions are

$$\begin{cases} (x+z) - c - \mathcal{A}^* y = 0 \\ Q^* \ni y \perp \mathcal{A}x - b \in Q \\ K^* \ni z \perp x \in K\,, \end{cases},$$

where "$\perp$" means the orthogonality. $Q^* = \Re^p \times \Re^q_+$ is the dual cone of Q and K^*[2] is the dual cone of K.

[2] $K^* := \{d \in \mathcal{X} \mid \langle d, x \rangle \geq 0 \; \forall\, x \in K\}$.

Equivalently,

$$\begin{cases} (x+z) - c - \mathcal{A}^* y = 0 \\ \\ Q^* \ni y \perp \mathcal{A}x - b \in Q \\ \\ x - \Pi_K(x+z) = 0 \end{cases},$$

where $\Pi_K(x)$ is the unique optimal solution to

$$\begin{array}{ll} \min & \frac{1}{2}\|u - x\|^2 \\ \text{s.t.} & u \in K\,. \end{array}$$

Consequently, by first eliminating $(x+z)$ and then x, we get

$$Q^* \ni y \perp \mathcal{A}\Pi_K(c+\mathcal{A}^* y) - b \in Q\,,$$

which is equivalent to

$$F(y) := y - \Pi_{Q^*}[y - (\mathcal{A}\Pi_K(c+\mathcal{A}^* y) - b)] = 0, \quad y \in \Re^m\,.$$

The dual formulation

The above is nothing but the first order optimality condition to the convex dual problem

$$\begin{array}{ll} \max & -\theta(y) := -\left[\frac{1}{2}\|\Pi_K(c+\mathcal{A}^*y)\|^2 - \langle b, y\rangle - \frac{1}{2}\|c\|^2\right] \\ \text{s.t.} & y \in Q^*\,. \end{array}$$

Then F can be written as

$$F(y) = y - \Pi_{Q^*}(y - \nabla\theta(y))\,.$$

Now, we only need to solve

$$F(y) = 0, \quad y \in \Re^m .$$

However, the difficulties are:

- F is not differentiable at y;
- F involves two metric projection operators;
- Even if F is differentiable at y, it is too costly to compute $F'(y)$.

The NCM problem

For the nearest correlation matrix problem,

- $\mathcal{A}(X) = \mathrm{diag}(X)$, a vector consisting of all diagonal entries of X.

.

- $\mathcal{A}^*(y) = \mathrm{diag}(y)$, the diagonal matrix.

- $b = e$, the vector of all ones in $\Re^n$ and $K = \mathcal{S}^n_+$.

Consequently, F can be written as

$$F(y) = \mathcal{A}\Pi_{\mathcal{S}^n_+}(G + \mathcal{A}^* y) - b.$$

The projector

For $n = 1$, we have

$$x_+ := \Pi_{\mathcal{S}^1_+}(x) = \max(0, x).$$

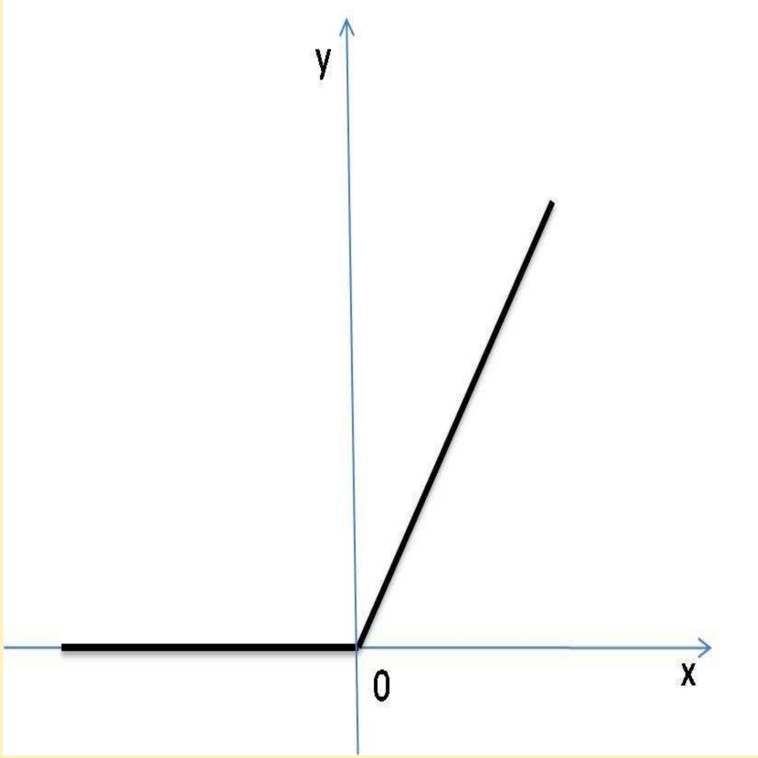
Note that

- x_+ is only piecewise linear, but not smooth.
- $(x_+)^2$ is continuously differentiable with

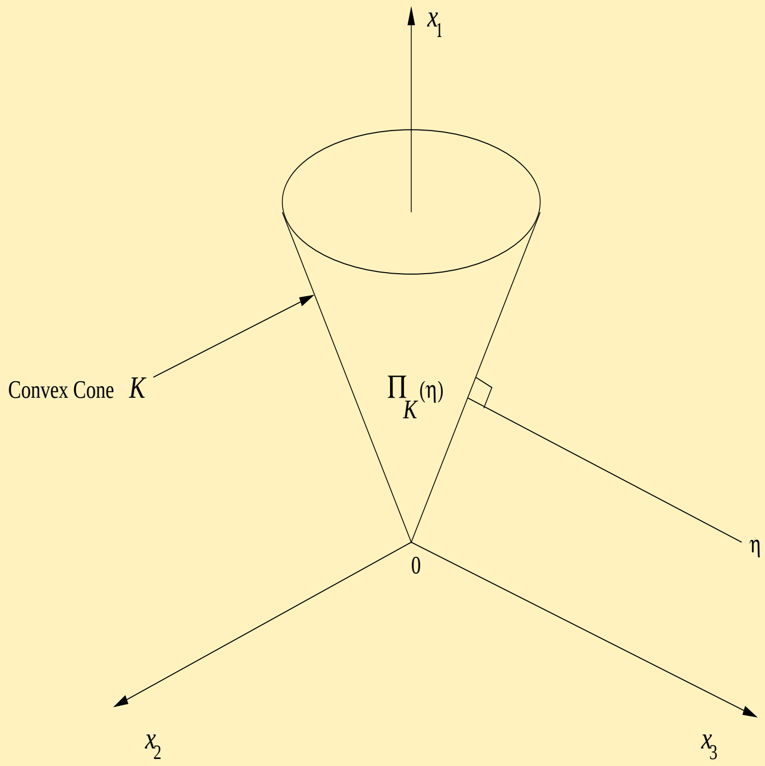
$$\nabla\left\{\frac{1}{2}(x_+)^2\right\} = x_+,$$

but is not twice continuously differentiable.

The one dimensional case



The projector for $K = \mathcal{S}^n_+$:



Let $X \in \mathcal{S}^n$ have the following spectral decomposition

$$X = P\Lambda P^T,$$

where Λ is the diagonal matrix of eigenvalues of X and P is a corresponding orthogonal matrix of orthonormal eigenvectors.

Then

$$X_+ := P_{\mathcal{S}^n_+}(X) = P\Lambda_+ P^T.$$

We have

- $\|X_+\|^2$ is continuously differentiable with

$$\nabla\left(\frac{1}{2}\|X_+\|^2\right) = X_+,$$

but is not twice continuously differentiable.

- X_+ is not piecewise smooth, but strongly semismooth[3].

[3] D.F. Sun and J. Sun. Semismooth matrix valued functions. *Mathematics of Operations Research* 27 (2002) 150–169.

A quadratically convergent Newton's method is then designed by Qi and Sun[4] The written code is called CorNewton.m.

`"This piece of research work is simply great and practical.  I enjoyed reading your paper."` – March 20, 2007, a home loan financial institution based in McLean, VA.

`"It's very impressive work and I've also run the Matlab code found in Defeng's home page.  It works very well."`– August 31, 2007, a major investment bank based in New York city.

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[4] H.D. Qi and D.F. Sun. A quadratically convergent Newton method for computing the nearest correlation matrix. *SIAM Journal on Matrix Analysis and Applications* 28 (2006) 360–385.

Inequality constraints

If we have lower and upper bounds on X, F takes the form

$$F(y) = y - \Pi_{Q^*}\left[y - (\mathcal{A}\Pi_{\mathcal{S}^n_+}(G + \mathcal{A}^* y) - b)\right],$$

which involves double layered projections over convex cones.

A quadratically convergent smoothing Newton method is designed by Gao and Sun[5].

Again, highly efficient.

[5] Y. Gao and D.F. Sun. Calibrating least squares covariance matrix problems with equality and inequality constraints, SIAM Journal on Matrix Analysis and Applications 31 (2009), 1432–1457.

Back to the original problem

$$\begin{array}{ll} \min & \frac{1}{2}\|H \circ (X - G)\|_F^2 \\ \text{s.t.} & \mathcal{A}(X) \in b + Q\,, \\ & X \in \mathcal{S}^n_+\,, \end{array}$$

The Majorization Method

Let $d \in \Re^n$ be a positive vector such that

$$H \circ H \leq dd^T.$$

For example, $d = \max(H_{ij})e$. Let $D^{1/2} = \operatorname{diag}(d_1^{0.5}, \ldots, d_n^{0.5})$.
Let

$$f(X) := \frac{1}{2}\|H \circ (X - G)\|_F^2.$$

Then g is majorized by

$$f^k(X) := f(X^k) + \langle H \circ H(X^k - G), X - X^k\rangle + \frac{1}{2}\|D^{1/2}(X - X^k)D^{1/2}\|_F^2,$$

i.e.,

$$f(X^k) = f^k(X^k) \quad \text{and} \quad f(X) \leq f^k(X).$$

The idea of majorization

The idea of the majorization is to solve, for given X^k, the following problem

$$\begin{array}{ll} \min & f^k(X) \\ \text{s.t.} & \mathcal{A}(X) \in b + Q\,, \\ & X \in \mathcal{S}^n_+\,, \end{array}$$

which is a diagonal weighted least squares correlation matrix problem

$$\begin{array}{ll} \min & \frac{1}{2}\|D^{1/2}(X - \overline{X}^k)D^{1/2}\|_F^2 \\ \text{s.t.} & \mathcal{A}(X) \in b + Q\,, \\ & X \in \mathcal{S}^n_+\,. \end{array}$$

Now, we can use the two Newton methods introduced earlier for the majorized subproblems!

$$f(X^{k+1}) < f(X^k) < \cdots < f(X^1).$$

A small example: $n = 4$

$$G = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0.5 \\ -1 & 1 & 0.5 & 1 \end{bmatrix}$$

Suppose that $G(1,2)$ and $G(2,1)$ are missing.

$$G = \begin{bmatrix} 1 & * & 1 & -1 \\ * & 1 & -1 & 1 \\ 1 & -1 & 1 & 0.5 \\ -1 & 1 & 0.5 & 1 \end{bmatrix}$$

A small example: $n = 4$ (continued)

We take

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

After 4 iterations, we get

$$X^* = \begin{bmatrix} 1.000 & -1.000 & 0.6894 & -0.6894 \\ -1.000 & 1.000 & -0.6894 & 0.6894 \\ 0.6894 & -0.6894 & 1.000 & 0.0495 \\ -0.6894 & 0.6894 & 0.0495 & 1.000 \end{bmatrix}$$

This is the same solution as the case with no-missing data.

Large examples

Example 1	CorrMajor		AugLag	
n	*time*	*residue*	*time*	*residue*
100	0.9	2.9006e1	1.1	2.9006e1
200	1.8	6.6451e1	3.2	6.6451e1
500	9.7	1.8815e2	23.5	1.8815e2
1000	51.3	4.0108e2	223.4	4.0108e2

Table 1: Numerical results for Example 1

Final remarks

- A code named CorrMajor.m can efficiently solve correlation matrix problems with all sorts of bound constraints.

- The techniques may be used to solve many other problems, e.g., low rank matrix problems with sparsity.

- The limitation is that it cannot solve problems for matrices exceeding the dimension $5,000$ by $5,000$ on a PC due to memory constraints.

End of talk

Thank you! :)