Modern Optimization Theory:

Optimality Conditions and Perturbation Analysis

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Let us consider

(OP)

$$\begin{array}{ll} \min_{x \in X} & f(x) \\ \text{s.t.} & G(x) \in K\,, \end{array}$$

where $f : X \to \Re$ and $G : X \to Y$ are $\mathcal{C}^2$ (twice continuously differentiable), X, Y finite-dimensional real Hilbert vector spaces each equipped with a scalar product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, and K is a closed convex set in Y.

The Lagrangian function $L : X \times Y \to \Re$ for (OP) is defined by

$$L(x, \mu) := f(x) + \langle \mu, G(x) \rangle, \quad (x, \mu) \in X \times Y.$$

If $\bar{x}$ is a locally optimal solution to (OP) and the following Robinson's CQ holds at $\bar{x}$:

$$0 \in \text{int}\{G(\bar{x}) + \mathcal{J}G(\bar{x})X - K\},$$

$$(\text{or} \;\; \mathcal{J}G(\bar{x})X + \mathcal{T}_K(G(\bar{x})) = Y),$$

then there exists a Lagrangian multiplier $\bar{\mu} \in Y$, together with $\bar{x}$, satisfying the KKT condition:

$$\nabla_x L(\bar{x}, \bar{\mu}) = 0 \quad \text{and} \quad \bar{\mu} \in \mathcal{N}_K(G(\bar{x}))\,,$$

$$(\text{or} \;\; \nabla_x L(\bar{x}, \bar{\mu}) = 0 \quad \text{and} \quad G(\bar{x}) = \Pi_K(G(\bar{x}) + \bar{\mu}))$$

and equivalently if K is a closed convex cone

$$\nabla f(\bar{x}) + \nabla G(\bar{x})\bar{\mu} = 0 \quad \text{and} \quad K \ni G(\bar{x}) \perp (-\bar{\mu}) \in K^*\,.$$

Let $\mathcal{M}(\bar{x})$ denote the set of Lagrangian multipliers.

- Tremendous progress achieved in stability analysis in (OP) subject to data perturbation.

- K is a polyhedral set, the theory quite complete. Especially for

(NLP)

$$\begin{array}{ll} \min\limits_{x \in \Re^n} & f(x) \\ \text{s.t.} & h(x) = 0\,, \\ & g(x) \leq 0\,. \end{array}$$

For (NLP), Robinson's CQ reduces to the Mangasarian-Fromovitz constraint qualification (MFCQ):

$$\left\{\begin{array}{ll} \mathcal{J}h_i(\bar{x}), & i=1,\ldots,m, \text{ are linearly independent}, \\ \exists\, d \in X: & \mathcal{J}h_i(\bar{x})d = 0\,,\ i=1,\ldots,m\,,\ \mathcal{J}g_j(\bar{x})d < 0\,,\ j \in \mathcal{I}(\bar{x}), \end{array}\right.$$

where

$$\mathcal{I}(\bar{x}) := \{j \,:\, g_j(\bar{x}) = 0\,,\ j = 1,\ldots,p\}.$$

A stronger notion than the MFCQ in (NLP) is the linear independence constraint qualification (LICQ):

$$\{\mathcal{J}h_i(\bar{x})\}_{i=1}^m \text{ and } \{\mathcal{J}g_j(\bar{x})\}_{j\in\mathcal{I}(\bar{x})} \text{ are linearly independent.}$$

$\mathcal{M}(\bar{x})$ is nonempty and bounded if and only if the MFCQ holds at $\bar{x}$ while the LICQ implies that $\mathcal{M}(\bar{x})$ is a singleton.

In 1980, Robinson[a] introduced the far-reaching concept of strong regularity for generalized equations (KKT system is a special case) and the strong second order sufficient condition (SSOSC) for (NLP) (the later is also developed by Luenberger[b]).

Robinson proved for (NLP):

$$\text{SSOSC} + \text{LICQ} \Longrightarrow \text{Strong Regularity.}$$

[a] S.M. Robinson. Strongly regular generalized equations. *Mathematics of Operations Research* 5 (1980) 43–62.

[b] D.G. Luenberger. *Introduction to Linear and Nonlinear Programming*, Addison-Wesley (London, 1973.)

Jongen, Mobert, Rückmann, and Tammer[a]; Bonnans and Sulem[b]; Dontchev and Rockafellar[c] proved:

$$\text{SSOSC} + \text{LICQ} \Longleftarrow \text{Strong Regularity.}$$

[a] H.Th. Jongen, T. Mobert, J. Rückmann, and K. Tammer. On inertia and Schur complement in optimization. *Linear Algebra and its Applications* 95 (1987) 97–109.

[b] J.F. Bonnans and A. Sulem. Pseudopower expansion of solutions of generalized equations and constrained optimization problems. *Mathematical Programming* 70 (1995) 123–148.

[c] A.L. Dontchev and R.T. Rockafellar. Characterizations of strong regularity for variational inequalities over polyhedral convex sets. *SIAM Journal on Optimization* 6 (1996) 1087–1105.

In the above characterizations, K is a polyhedral set. Here we focus on the nonlinear semidefinite programming

$(NLSDP)$

$$\begin{array}{ll} \min\limits_{x \in X} & f(x) \\ \text{s.t.} & h(x) = 0\,, \\ & g(x) \in \mathcal{S}^p_+\,. \end{array}$$

Difficulty:

$\mathcal{S}^p_+$ is not a polyhedral set.

Let $A \in \mathcal{S}^p$ have the following spectral decomposition

$$A = P\Lambda P^T,$$

where Λ is the diagonal matrix of eigenvalues of A and P is a corresponding orthogonal matrix of orthonormal eigenvectors.

Then

$$A_+ := \Pi_{\mathcal{S}^p_+}(A) = P\Lambda_+ P^T.$$

Define

$$\alpha := \{i : \lambda_i > 0\},\ \beta := \{i : \lambda_i = 0\},\ \gamma := \{i : \lambda_i < 0\}.$$

Write

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_\gamma \end{bmatrix} \text{ and } P = [\ P_\alpha \quad P_\beta \quad P_\gamma\ ].$$

Define $U \in \mathcal{S}^p$:

$$U_{ij} := \frac{\max\{\lambda_i, 0\} + \max\{\lambda_j, 0\}}{|\lambda_i| + |\lambda_j|}, \quad i, j = 1, \ldots, p,$$

where 0/0 is defined to be 1.

The tangent cone of $\mathcal{S}^p_+$ at $A_+ = \Pi_{\mathcal{S}^p_+}(A)$ is [a]:

$$\mathcal{T}_{\mathcal{S}^p_+}(A_+) = \{B \in \mathcal{S}^p \,:\, P_{\bar{\alpha}}^T B P_{\bar{\alpha}} \succeq 0\}\,.$$

and the lineality space of $\mathcal{T}_{\mathcal{S}^p_+}(A_+)$, i.e., the largest linear space in $\mathcal{T}_{\mathcal{S}^p_+}(A_+)$,

$$\mathrm{lin}\left(\mathcal{T}_{\mathcal{S}^p_+}(A_+)\right) = \{B \in \mathcal{S}^n \,:\, P_{\bar{\alpha}}^T B P_{\bar{\alpha}} = 0\},$$

where $\bar{\alpha} := \{1, \ldots, p\} \backslash \alpha$ and $P_{\bar{\alpha}} := [P_\beta \; P_\gamma]$.

[a] V.I. Arnold. *Matrices depending on parameters.* Russian Mathematical Surveys, 26 (1971) 29–43.

The critical cone of $\mathcal{S}^p_+$ at $A \in \mathcal{S}^p$, is defined as

$$C(A;\mathcal{S}^p_+) := \mathcal{T}_{\mathcal{S}^p_+}(A_+) \cap (A_+ - A)^\perp,$$

$$= \left\{ B \in \mathcal{S}^p \, : \, P_\beta^T B P_\beta \succeq 0, \; P_\beta^T B P_\gamma = 0, \; P_\gamma^T B P_\gamma = 0 \right\}.$$

The affine hull of $C(A;\mathcal{S}^P_+)$, $\mathrm{aff}(C(A;\mathcal{S}^p_+))$, can be written as

$$\mathrm{aff}\left(C(A;\mathcal{S}^p_+)\right) = \left\{ B \in \mathcal{S}^p \, : \, P_\beta^T B P_\gamma = 0, \; P_\gamma^T B P_\gamma = 0 \right\}.$$

Definition 3.1 *For any $B \in \mathcal{S}^p$, define the linear-quadratic function $\Upsilon_B : S^p \times \mathcal{S}^p \to \Re$ by*

$$\Upsilon_B(\Gamma, A) := 2\left\langle \Gamma, AB^\dagger A \right\rangle, \quad (\Gamma, A) \in \mathcal{S}^p \times \mathcal{S}^p,$$

where $B^\dagger$ is the Moore-Penrose pseudo-inverse of B.

Proposition 3.1 *Suppose that $B \in \mathcal{S}^p_+$ and $\Gamma \in \mathcal{N}_{\mathcal{S}^p_+}(B)$, i.e.,*

$$B = \Pi_{\mathcal{S}^p_+}(B + \Gamma).$$

Then for any $V \in \partial\Pi_{\mathcal{S}^p_+}(B + \Gamma)$ and $\Delta B, \Delta\Gamma \in \mathcal{S}^p$ such that $\Delta B = V(\Delta B + \Delta\Gamma)$, it holds that

$$\langle \Delta B, \Delta\Gamma \rangle \geq -\Upsilon_B(\Gamma, \Delta B).$$

Let $\bar{x}$ be a stationary point of $(NLSDP)$. Let $(\bar{\zeta}, \overline{\Gamma}) \in \mathcal{M}(\bar{x})$ such that

$$\nabla_x L(\bar{x}, \bar{\zeta}, \overline{\Gamma}) = 0, \quad -h(\bar{x}) = 0, \quad \text{and} \quad \overline{\Gamma} \in \mathcal{N}_{\mathcal{S}^p_+}(g(\bar{x})).$$

Let $A := g(\bar{x}) + \bar{\Gamma}$ and[a]

$$g(\bar{x}) = P \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T, \quad \text{and} \quad \overline{\Gamma} = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_\gamma \end{bmatrix} P^T.$$

[a] Since $g(\bar{x})$ and $\overline{\Gamma}$ commute, we can simultaneously diagonalize them.

The critical cone $C(\bar{x})$ of $(NLSDP)$ at $\bar{x}$ is

$$
\begin{aligned}
C(\bar{x}) &= \Big\{ d : \mathcal{J}h(\bar{x})d = 0, \mathcal{J}g(\bar{x})d \in \mathcal{T}_{\mathcal{S}^p_+}(g(\bar{x})), \mathcal{J}f(\bar{x})d = 0 \Big\} \\
&= \Big\{ d : \mathcal{J}h(\bar{x})d = 0, \;\; P_\beta^T(\mathcal{J}g(\bar{x})d)P_\beta \succeq 0, \\
&\qquad P_\beta^T(\mathcal{J}g(\bar{x})d)P_\gamma = 0, \;\; P_\gamma^T(\mathcal{J}g(\bar{x})d)P_\gamma = 0 \Big\}.
\end{aligned}
$$

The difficulty is that the affine hull of $C(\bar{x})$, $\mathrm{aff}(C(\bar{x}))$, has no explicit formula.

Define the following outer approximation set to $\mathrm{aff}(C(\bar{x}))$ with respect to $(\bar{\zeta}, \overline{\Gamma})$ by

$$\mathrm{app}(\bar{\zeta}, \overline{\Gamma}) := \left\{ d \,:\, \mathcal{J}h(\bar{x})d = 0, \quad \mathcal{J}g(\bar{x})d \in \mathrm{aff}\,\big(C(A; \mathcal{S}^p_+)\big) \right\}.$$

It holds that

$$\begin{aligned} \mathrm{app}(\bar{\zeta}, \overline{\Gamma}) \quad = \quad & \Big\{ d \,:\, \mathcal{J}h(\bar{x})d = 0,\; P_\beta^T(\mathcal{J}g(\bar{x})d)P_\gamma = 0, \\ & P_\gamma^T(\mathcal{J}g(\bar{x})d)P_\gamma = 0 \Big\}. \end{aligned}$$

Then by the definition of $\mathrm{aff}(C(\bar{x}))$, we have for any $(\bar{\zeta}, \overline{\Gamma}) \in \mathcal{M}(\bar{x})$ that

$$\mathrm{aff}(C(\bar{x})) \subseteq \mathrm{app}(\bar{\zeta}, \overline{\Gamma}).$$

The two sets $\mathrm{aff}(C(\bar{x}))$ and $\mathrm{app}(\bar{\zeta}, \overline{\Gamma})$ coincide if the strict complementary condition holds at $(\bar{x}, \bar{\zeta}, \overline{\Gamma})$:

$$\mathrm{rank}(g(\bar{x})) + \mathrm{rank}(\overline{\Gamma}) = p,$$

where "rank" denotes the rank of a square matrix.

In general, these two sets may be different even if $\mathcal{M}(\bar{x})$ is a singleton as in the case for (NLP).

Proposition 3.2 *Suppose that* $(\bar{\zeta}, \overline{\Gamma})$ *satisfies the following strict constraint qualification:*

$$\begin{pmatrix} \mathcal{J}h(\bar{x}) \\ \mathcal{J}g(\bar{x}) \end{pmatrix} X + \begin{pmatrix} 0 \\ \mathcal{T}_{\mathcal{S}^p_+}(g(\bar{x})) \cap \overline{\Gamma}^{\perp} \end{pmatrix} = \begin{pmatrix} \Re^m \\ \mathcal{S}^p \end{pmatrix}.$$

Then $\mathcal{M}(\bar{x})$ *is a singleton, i.e.,* $\mathcal{M}(\bar{x}) = \{(\bar{\zeta}, \overline{\Gamma})\}$*, and*
$\mathrm{aff}(C(\bar{x})) = \mathrm{app}(\bar{\zeta}, \overline{\Gamma})$.

Recall that the "no-gap" second order necessary condition and the second order sufficient condition for $(NLSDP)$ can be stated as follows:

Theorem 3.1 *Let $K = \{0\} \times \mathcal{S}_+^p \subset \Re^m \times \mathcal{S}^p$. Suppose that $\bar{x}$ is a locally optimal solution to $(NLSDP)$ and Robinson's CQ holds at $\bar{x}$. Then*

$$\sup_{\mu \in \mathcal{M}(\bar{x})} \left\{ \left\langle d, \nabla^2_{xx} L(\bar{x}, \mu) d \right\rangle - \sigma \left(\mu, \mathcal{T}_K^2(G(\bar{x}), \mathcal{J}G(\bar{x})d) \right) \right\} \geq 0$$

for all $d \in C(\bar{x})$.

(continued)

Conversely, let $\bar{x}$ be a feasible solution to $(NLSDP)$ such that $\mathcal{M}(\bar{x})$ is nonempty. Suppose that Robinson's CQ holds at $\bar{x}$. Then the following condition

$$\sup_{\mu\in\mathcal{M}(\bar{x})} \left\{ \left\langle d, \nabla^2_{xx} L(\bar{x},\mu) d \right\rangle - \sigma\left(\mu, \mathcal{T}^2_K(G(\bar{x}), \mathcal{J}G(\bar{x})d)\right) \right\} > 0$$

for all $d \in C(\bar{x})\backslash\{0\}$ is necessary and sufficient for the quadratic growth condition at the point $\bar{x}$:

$$f(x) \geq f(\bar{x}) + c\|x - \bar{x}\|^2 \quad \forall\, x \in \widehat{N} \text{ such that } G(x) \in K$$

for some constant $c > 0$ and a neighborhood $\widehat{N}$ of $\bar{x}$ in X.

Proposition 3.3 *Let $\bar{x}$ be a feasible solution to $(NLSDP)$ such that $\mathcal{M}(\bar{x})$ is nonempty. Then for any $(\zeta, \Gamma) \in \mathcal{M}(\bar{x})$ with $\zeta \in \Re^m$ and $\Gamma \in \mathcal{S}^p$, one has*

$$\Upsilon_{g(\bar{x})}(\Gamma, \mathcal{J}g(\bar{x})d) = \sigma\left(\Gamma, \mathcal{T}^2_{\mathcal{S}^p_+}(g(\bar{x}), \mathcal{J}g(\bar{x})d)\right) \quad \forall\, d \in C(\bar{x}),$$

where

$$\Upsilon_B(\Gamma, A) = 2\left\langle \Gamma, AB^\dagger A \right\rangle, \quad (\Gamma, A) \in \mathcal{S}^p \times \mathcal{S}^p.$$

Definition 3.2 *Let $\bar{x}$ be a stationary point of $(NLSDP)$. We say that the strong second order sufficient condition (SSOSC) holds at $\bar{x}$ if*

$$\sup_{(\zeta,\Gamma)\in\mathcal{M}(\bar{x})} \left\{\left\langle d, \nabla^2_{xx}L(\bar{x},\zeta,\Gamma)d\right\rangle - \Upsilon_{g(\bar{x})}(\Gamma, \mathcal{J}g(\bar{x})d)\right\} > 0$$

for all $d \in \widehat{C}(\bar{x})\backslash\{0\}$, where for any $(\zeta,\Gamma) \in \mathcal{M}(\bar{x})$, $(\zeta,\Gamma) \in \Re^m \times \mathcal{S}^p$ and

$$\widehat{C}(\bar{x}) := \bigcap_{(\zeta,\Gamma)\in\mathcal{M}(\bar{x})} \mathrm{app}(\zeta,\Gamma).$$

Next, we define a nondegeneracy condition for $(NLSDP)$, which is an analogue of the LICQ for (NLP). The concept of nondegeneracy originally appeared in Robinson[a] for (OP).

Definition 3.3 *We say that a feasible point $\bar{x}$ to (OP) is constraint nondegenerate if*

$$\mathcal{J}G(\bar{x})X + \mathrm{lin}(\mathcal{T}_K(\bar{y})) = Y,$$

where $\bar{y} := G(\bar{x})$.

[a] S.M. Robinson. Local structure of feasible sets in nonlinear programming, Part II: Nondegeneracy. *Mathematical Programming Study* 22 (1984) 217–230.

Write down the KKT condition as

$$F(x, \zeta, \Gamma) : = \begin{bmatrix} \nabla L(x, \zeta, \Gamma) \\ -h(x) \\ -g(x) + \Pi_{\mathcal{S}^p_+}(g(x) + \Gamma) \end{bmatrix} = \begin{bmatrix} \nabla_x L(x, \zeta, \Gamma) \\ -h(x) \\ \Gamma - \Pi_{\mathcal{S}^p_-}(\Gamma + g(x)) \end{bmatrix} = 0,$$

which is equivalent to the following generalized equation:

$$0 \in \phi(z) + \mathcal{N}_D(z),$$

where ϕ is $\mathcal{C}^1$ and D is a closed convex set in Z.

Definition 3.4 *[Robinson'80] Let* $\bar{z}$ *be a solution of the generalized equation. We say that* $\bar{z}$ *is a strongly regular solution if there exist neighborhoods* $\mathcal{B}$ *of the origin* $0 \in Z$ *and* $\mathcal{V}$ *of* $\bar{z}$ *such that for every* $\delta \in \mathcal{B}$*, the following linearized generalized equation*

$$\delta \in \phi(\bar{z}) + \mathcal{J}\phi(\bar{z})(z - \bar{z}) + \mathcal{N}_D(z)$$

has a unique solution in $\mathcal{V}$*, denoted by* $z_{\mathcal{V}}(\delta)$*, and the mapping* $z_{\mathcal{V}} : \mathcal{B} \to \mathcal{V}$ *is Lipschitz continuous.*

Let U be a Banach space and $f : X \times U \to \Re$ and $G : X \times U \to Y$.

We say that $(f(x,u), G(x,u))$, with $u \in U$, is a *$\mathcal{C}^2$-smooth parameterization* of (OP) if $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are C^2 and there exists a $\bar{u} \in U$ such that $f(\cdot,\bar{u}) = f(\cdot)$ and $G(\cdot,\bar{u}) = G(\cdot)$. The corresponding parameterized problem takes the form:

(OP_u)

$$\begin{array}{ll} \min\limits_{x \in X} & f(x,u) \\ \text{s.t.} & G(x,u) \in K\,. \end{array}$$

We say that a parameterization is canonical if $U := X \times Y$, $\bar{u} = (0,0) \in X \times Y$, and

$$(f(x,u), G(x,u)) := (f(x) - \langle u_1, x\rangle, G(x) + u_2), \quad x \in X.$$

Definition 3.5 *[Bonnans and Shapiro'00] Let* $\bar{x}$ *be a stationary point of* (OP). *We say that the uniform second order (quadratic) growth condition holds at* $\bar{x}$ *with respect to a* $\mathcal{C}^2$*-smooth parameterization* $(f(x,u), G(x,u))$ *if there exist* $c > 0$ *and neighborhoods* $\mathcal{V}_X$ *of* $\bar{x}$ *and* $\mathcal{V}_U$ *of* $\bar{u}$ *such that for any* $u \in \mathcal{V}_U$ *and any stationary point* $x(u) \in \mathcal{V}_X$ *of* (OP_u), *the following holds:*

$$f(x,u) \geq f(x(u),u) + c\|x - x(u)\|^2 \quad \forall\, x \in \mathcal{V}_X \text{ such that } G(x,u) \in K.$$

We say that the uniform second order growth condition holds at $\bar{x}$ *if the above inequality holds for every* $\mathcal{C}^2$*-smooth parameterization of* (OP).

Definition 3.6 *[Kojima[a] and Bonnans and Shapiro'00]*

Let $\bar{x}$ be a stationary point of (OP). We say that $\bar{x}$ is strongly stable with respect to a $\mathcal{C}^2$-smooth parameterization $(f(x,u), G(x,u))$ if there exist neighborhoods $\mathcal{V}_X$ of $\bar{x}$ and $\mathcal{V}_U$ of $\bar{u}$ such that for any $u \in \mathcal{V}_U$, (OP_u) has a unique stationary point $x(u) \in \mathcal{V}_X$ and $x(\cdot)$ is continuous on $\mathcal{V}_U$.

If this holds for any $\mathcal{C}^2$-smooth parameterization, we say that $\bar{x}$ is strongly stable.

[a] M. Kojima. Strongly stable stationary solutions in nonlinear programs. In: S.M. Robinson, editor, *Analysis and Computation of Fixed Points*, Academic Press (New York, 1980), pp. 93-138.

Let

$$\Phi(\delta) := F'(\bar{x}, \bar{\zeta}, \overline{\Gamma}; \delta).$$

Let $\operatorname{ind}(\phi, \bar{z})$ denote the index of a continuous function $\phi : Z \to Z$ at an isolated zero $\bar{z} \in Z$ used in degree theory.

Based on several recent results of Bonnans and Shapiro'00; Gowda[a]; Pang, Sun and Sun[b]; Sun and Sun'02, we get

[a] M.S. Gowda. Inverse and implicit function theorems for H-differentiable and semismooth functions. *Optimization Methods and Software* 19 (2004) 443-461.

[b] J.S. Pang, D. Sun, and J. Sun. Semismooth homeomorphisms and strong stability of semidefinite and Lorentz complementarity problems. *Mathematics of Operations Research* 28 (2003) 39–63.

Theorem 2[a]**.** Let $\bar{x}$ be a locally optimal solution to $(NLSDP)$. Suppose that Robinson's CQ holds at $\bar{x}$ so that $\bar{x}$ is necessarily a stationary point of $(NLSDP)$. Let $(\bar{\zeta}, \overline{\Gamma}) \in \Re^m \times \mathcal{S}^p$ be such that $(\bar{x}, \bar{\zeta}, \overline{\Gamma})$ is a KKT point of $(NLSDP)$. Then the following TEN statements are equivalent:

(a) The SSOSC holds at $\bar{x}$ and $\bar{x}$ is constraint nondegenerate.

(b) Any element in $\partial F(\bar{x}, \bar{\zeta}, \overline{\Gamma})$ is nonsingular.

(c) The KKT point $(\bar{x}, \bar{\zeta}, \overline{\Gamma})$ is strongly regular.

(d) The uniform second order growth condition holds at $\bar{x}$ and $\bar{x}$ is constraint nondegenerate.

(e) The point $\bar{x}$ is strongly stable and $\bar{x}$ is constraint nondegenerate.

[a] D. Sun. The strong second order sufficient condition and constraint nondegeneracy in nonlinear semidefinite programming and their implications. *Mathematics of Operations Research* 31 (2006).

(continued)

(f) F is a locally Lipschitz homeomorphism near $(\bar{x}, \bar{\zeta}, \overline{\Gamma})$.

(g) For every $V \in \partial_B F(\bar{x}, \bar{\zeta}, \overline{\Gamma})$, $\operatorname{sgn}\det V = \operatorname{ind}(F, (\bar{x}, \bar{\zeta}, \overline{\Gamma})) = \pm 1$.

(h) Φ is a globally Lipschitz homeomorphism.

(i) For every $V \in \partial_B \Phi(0)$, $\operatorname{sgn}\det V = \operatorname{ind}(\Phi, 0) = \pm 1$.

(j) Any element in $\partial\Phi(0)$ is nonsingular.

Note that many more equivalent statements can be added by looking at statements (b) and (g).

Some unsolved problems:

(Q1) How far can we go beyond the SDP cone? Symmetric cone (SOC is fine)? Homogeneous cone? Hyperbolic cone?

(Q2) What can we say about the equivalent conditions in Theorem 2 if $\bar{x}$ is assumed to be a stationary point only? Or more generally

(Q3) How can we characterize the strong regularity for the conic complementarity problems?