A Proximal Point Method for Matrix Least Squares Problem with Nuclear Norm Regularization

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Let $S^n$ be the set of all real symmetric matrices and $S^n_+$ be the cone of all positive semidefinite matrices in $S^n$.

We consider the least squares SDP:

$$\min\left\{ \frac{1}{2} \|\mathcal{A}(X) - b\|^2 + \rho \langle I, X \rangle : \mathcal{B}(X) = d, \ X \in S^n_+ \right\},$$

where $\mathcal{A} : S^n \to \mathbb{R}^m$ and $\mathcal{B} : S^n \to \mathbb{R}^s$ are linear maps and $\rho$ is a given positive scalar.
An example — the regularized kernel estimation (RKE) problem in statistics:

we are given a set of $n$ objects and dissimilarity measures $d_{ij}$ for certain object pairs $(i, j) \in \mathcal{E}$.

The goal is to estimate a positive semidefinite kernel matrix $X \in \mathcal{S}^n_+$ such that the fitted squared distances between objects induced by $X$ satisfy

$$X_{ii} + X_{jj} - 2X_{ij} = \langle A_{ij}, X \rangle \approx d_{ij}^2 \quad \forall \ (i, j) \in \mathcal{E},$$

where $A_{ij} = (e_i - e_j)(e_i - e_j)^T$. 

One version of the RKE problem is to solve the following SDP:

\[
\min \left\{ \sum_{(i,j) \in \mathcal{E}} W_{ij} (\langle A_{ij}, X \rangle - d_{ij}^2)^2 + \rho \langle I, X \rangle : \langle E, X \rangle = 0, \ X \succeq 0 \right\},
\]

where \( W \in \mathcal{S}^n \) is a given weight matrix with positive entries.
Analogously, we consider the least squares problem with the nuclear norm regularization:

\[
\min \left\{ \frac{1}{2} \| \mathcal{A}(X) - b \|^2 + \rho \| X \|_* : \mathcal{B}(X) = d, \ X \in \mathbb{R}^{p \times q} \right\},
\]

where

\[
\| X \|_* = \sum_{i=1}^{k} \sigma_i(X)
\]

and \( \sigma_i(X) \) are the singular values of \( X \).
The matrix completion example:

$$\min \left\{ \operatorname{rank}(X) : X_{ij} \approx M_{ij} \quad \forall \ (i,j) \in \Omega \right\},$$

where

$$\Omega \in \{1, \ldots, p\} \times \{1, \ldots, q\} :$$

$$\begin{bmatrix}
* & * \\
* & * \\
* & * \\
* & * \\
* & *
\end{bmatrix}$$
get a relaxed convex problem:

$$\min \left\{ \|X\|_* : X_{ij} \approx M_{ij} \quad \forall (i, j) \in \Omega \right\}.$$ 

Further

$$\min \left\{ \frac{1}{2} \sum_{(i,j) \in \Omega} (X_{ij} - M_{ij})^2 + \rho \|X\|_* \right\}.$$ 

**The Netflix Prize problem:** the convex relaxation is pretty good.

http://www.netflixprize.com/index
For a random example:

- $p = q = 10^5$, $\text{rank}(X) = 10$, noise level $= 0.1$.
- $|\Omega| \approx 1.2 \times 10^7$.
- Proximal point method framework + gradient projection method.
- Need 416 seconds to achieve a relative accuracy 0.0453.
Consider the Moreau-Yosida regularization:

$$F_\sigma(X) = \min \frac{1}{2} \|u\|^2 + \rho \|Y\|_* + \frac{1}{2\sigma} \|Y - X\|^2$$

s.t. $A(Y) + u = b$

$B(Y) = d$

$Y \in \mathbb{R}^{p \times q}, \quad u \in \mathbb{R}^m.$
The Lagrangian dual problem of (1) is

\[
\max_{y \in \mathbb{R}^m, z \in \mathbb{R}^s} \left\{ \theta_\sigma^\rho (y, z; X) := \inf_{u \in \mathbb{R}^m, Y \in \mathbb{R}^{p \times q}} L_\sigma^\rho (Y, u; y, z, X) \right. \\
= -\frac{1}{2} \| y \|^2 + \langle b, y \rangle + \langle d, z \rangle \\
+ \frac{1}{2\sigma} \| X \|^2 - \frac{1}{2\sigma} \| D_\rho \sigma (W(y, z; X)) \|^2 \right\}, (2)
\]

where \( W(y, z; X) = X + \sigma (A^*y + B^*z) \).
For any $Y \in \mathbb{R}^{p \times q}$, $D_\rho(Y)$ is the unique optimal solution to the following strongly convex function

$$\min_X \|X\|_* + \frac{1}{2\rho} \|X - Y\|_F^2$$

It is well known that $D_\rho(\cdot)$ is globally Lipschitz continuous with modulus 1.
Let $Y \in \mathbb{R}^{p \times q}$ admit the following singular value decomposition:

$$Y = U[\Sigma 0]V^T,$$

where $U \in \mathbb{R}^{p \times p}$ and $V \in \mathbb{R}^{q \times q}$ are orthogonal matrices, $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_p)$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ are singular values of $Y$. For each $\rho > 0$, the operator $D_\rho$ is given by:

$$D_\rho(Y) = U[\Sigma_\rho 0]V^T,$$

where $\Sigma_\rho = \text{diag}((\sigma_1 - \rho)_+, \cdots, (\sigma_p - \rho)_+)$. 
Good news is: \( \|D_\rho(Y)\|^2 \) is continuously differentiable and

\[
\nabla \left( \frac{1}{2} \|D_\rho(Y)\|^2 \right) = D_\rho(Y).
\]

So we have a smooth convex optimization problem:

\[
\min_{y \in \mathbb{R}^m, z \in \mathbb{R}^s} \left\{ -\theta_\sigma^\rho(y, z; X) \right\}.
\]
Even better: $D_{\rho}(\cdot)$ is strongly semismooth everywhere.

A Lipschitz function $F : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be strongly semismooth at $x \in \mathcal{X}$ if

1) it is directionally differentiable at $x$; and 2) $F(x + \Delta x) - F(x) - F'(x + \Delta x)\Delta x = O(\|\Delta x\|^2)$ for all $x + \Delta x$ such that $F$ is Fréchet differentiable at $x + \Delta x$. 
One key issue:

\[ \theta^\rho_\sigma(\cdot, \cdot; X) \notin C^2. \]

This property allows \( \theta^\rho_\sigma(\cdot, \cdot; X) \) to possess nonsingular (generalized) Hessian, which is vital for an inexact second order method to be efficient.
We apply the **proximal point method** to solve the following unconstrained problem:

\[
\min_{X \in \mathbb{R}^{p \times q}} \Phi_{\sigma}(X) := \max\{\theta_{\sigma}(y, z; X) : y \in \mathbb{R}^{m}, z \in \mathbb{R}^{s}\}.
\]
PPA. Input $X^0 \in \mathbb{R}^{p \times q}$, $\sigma_0 > 0$, iterate:

1. Compute an approximate maximizer

$$(y^k, z^k) \approx \operatorname{argmax}\{ \theta_{\sigma_k}^\rho (y, z; X^k) : y \in \mathbb{R}^m, z \in \mathbb{R}^s \},$$

2. $X^{k+1} = D_{\rho \sigma_k} (W(y^k, z^k; X^k)), \quad Z^{k+1} = \frac{1}{\sigma_k} (D_{\rho \sigma_k} (W(y^k, z^k; X^k)) - W(y^k, z^k; X^k)),$

3. If $\| R_d^k := A^* y^k + B^* z^k + Z^{k+1} \|_F \leq \varepsilon$; stop; else, update $\sigma_k$. 
For the inner subproblem, the optimality condition is given by

\begin{align*}
\nabla_y \theta^\rho_{\sigma_k}(y, z; X^k) &= b - y - AD_{\rho\sigma}(W(y, z; X^k)) = 0 \\
\nabla_z \theta^\rho_{\sigma_k}(y, z; X^k) &= d - BD_{\rho\sigma}(W(y, z; X^k)) = 0
\end{align*}

(3)

We solve (3) by a **semismooth** Newton-CG method.
The inner problems can be solved by a \textbf{(fast) semismooth Newton-CG method}. The outer iteration

\[ X^{k+1} = D_{\rho \sigma_k} (W(y^k, z^k; X^k)) \]

only satisfies

\[ X^{k+1} = X^k - \sigma_k \nabla \Phi^\rho_{\sigma_k}(X^k), \]

\textbf{a gradient descent step}. The good news is that it can also be seen as \textbf{an approximate semismooth Newton method}, at least for the least squares SDP case.
Selected examples:

1. For each pair \((n, r)\), we generate a positive semidefinite matrix \(M \in \mathcal{S}^n\) of rank \(r\) by setting \(M = M_1M_1^T\) where \(M_1 \in \mathbb{R}^{n \times r}\) is a random matrix with i.i.d Gaussian entries. Then we sample a subset \(\Omega\) of \(m\) entries uniformly at random from the upper triangular part of \(M\). The observed data is set to be \(\tilde{M}_\Omega = M_\Omega + \alpha N_\Omega \|M_\Omega\|_F / \|N_\Omega\|_F\), where the random matrix \(N_\Omega \in \mathcal{S}^n\) is generated that has sparsity pattern \(\Omega\) and i.i.d Gaussian entries and \(\alpha\) is the noise level.
The minimization problem we solve is given by

$$\min \left\{ \frac{1}{2} \| X_{\Omega} - \tilde{M}_\Omega \|_F^2 + \rho \langle I, X \rangle : X \succeq 0 \right\}. \quad (4)$$

Numerical results: $n = 2000$, $r = 100$,
- for $\alpha = 0$, we need 15:00 and 8 (27) iterations; and
- for $\alpha = 0.05$, we need 39:15 and 18 (63) iterations
- The relative accuracy is below $10^{-6}$.
- The averaged CGs each step $\leq 10$.
- $|\Omega| \approx 975,000$. 
2. The nonsymmetric problem: similarly generated as in Example 1.

Numerical results: \( p = q = 1000, r = 50, \)

- for \( \alpha = 0, \) we need 4:07 and 12 (24) iterations; and
- for \( \alpha = 0.05, \) we need 16:01 and 26 (73) iterations.
- The averaged CGs each step \( \leq 5. \)
- The relative accuracy is below \( 10^{-6}. \)
- \( |\Omega| = 487,500. \)