Sub-Quadratic Convergence of a Smoothing Newton Algorithm for the P_0- and Monotone LCP

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Abstract

Given $M \in \Re^{n\times n}$ and $q \in \Re^n$, the linear complementarity problem (LCP) is to find $(x, s) \in \Re^n \times \Re^n$ such that $(x, s) \geq 0$, $s = Mx + q$, $x^T s = 0$. By using the Chen-Harker-Kanzow-Smale (CHKS) smoothing function, the LCP is reformulated as a system of parameterized smooth-nonsmooth equations. As a result, a smoothing Newton algorithm, which is a modified version of the Qi-Sun-Zhou algorithm [Mathematical Programming, Vol. 87, 2000, pp. 1–35], is proposed to solve the LCP with M being assumed to be a P_0-matrix (P_0-LCP). The proposed algorithm needs only to solve one system of linear equations and to do one line search at each iteration. It is proved in this paper that the proposed algorithm has the following convergence properties: (i) it is well-defined and any accumulation point of the iteration sequence is a solution of the P_0-LCP; (ii) it generates a bounded sequence if the P_0-LCP has a nonempty and bounded solution set; (iii) if an accumulation point of the iteration sequence satisfies a nonsingularity condition, which implies the P_0-LCP has a unique solution, then the whole iteration sequence converges to this accumulation point sub-quadratically with a Q-rate $2 - t$, where $t \in (0, 1)$ is a parameter; and (iv) if M is positive semidefinite and an accumulation point of the iteration sequence satisfies a strict complementarity condition, then the whole sequence converges to the accumulation point quadratically.

Keywords Linear complementarity problem, smoothing Newton method, global convergence, sub-quadratic convergence.

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1 Introduction

Recently there has been much interest in smoothing (non-interior continuation) Newton-type methods for solving some mathematical programming problems, such as linear programming problems [41, 6, 17, 18], linear complementarity problems (LCPs) [1, 2, 3, 7, 10, 26], nonlinear complementarity problems (NCPs) [4, 5, 8, 9, 11, 14, 15, 21, 23, 27, 29, 33, 34, 36, 38, 42, 45, 46, 49, 47], variational inequality problems [37, 43], semidefinite complementarity problems [12, 13, 28, 44], and so on. The main idea of this class of methods is to reformulate the problem concerned as a family of parameterized smooth equations and then to solve the smooth equations approximately by using Newton's method at each iteration. By driving the parameter to converge to zero, one can expect to find a solution to the original problem.

In [41], Smale initiated the study on smoothing (non-interior continuation) Newton-type methods for solving linear programming problems and LCPs. Independent of Smale's work [41], Chen and Harker [7] introduced a non-interior continuation method for solving the LCP with a P_0 and R_0 matrix. They concentrated on establishing properties of smoothing paths. Later, the smoothing function used in [7], was refined and generalized by Kanzow [26], Chen and Mangasarian [11], and Gabriel and Moré [20]. In [1], Burke and Xu introduced the concept of neighborhood of smoothing paths into their continuation method. This allowed them to establish a global linear convergence result for LCPs. Chen and Xiu [9] improved the method of Burke and Xu by simplifying the definition of neighborhood and adding an approximate Newton step to obtain a local quadratic convergence result. In [2, 3], Burke and Xu further proposed two predictor-corrector-type non-interior continuation methods for LCPs. They also obtained a local quadratic convergence result. Qi and Sun [36] analyzed the local superlinear convergence of the non-interior point method of Hotta and Yoshise for NCPs [23]. It should be noted that in order to obtain the local superlinear convergence one needs to assume that the strict complementarity condition holds at a solution point and that the iteration matrices are uniformly nonsingular [2, 3, 9, 36]. It is well-known that the latter assumption implies that the solution set is a singleton. To relax this relatively restrictive assumption, Tseng [46] developed a new approach to the analysis of local quadratic convergence of general predictor-corrector-type path-following methods for solving monotone complementarity problems. By using the error bound theory, Tseng discussed the local quadratic convergence under a strict complementarity condition. The assumptions made in [46] do not imply (explicitly or implicitly) that the solution set is a singleton. Very recently, Engelke and Kanzow [17, 18] further investigated the methods developed in [46] and proposed two specific predictor-corrector smoothing methods for solving linear programming problems. Under the assumption that the iteration sequence converges to a strict complementary solution, they proved the local quadratic convergence of their algorithms without assuming the uniqueness of the solution set. Very encouraging numerical results were also reported in [17, 18]. Just as the algorithms developed in [2, 3, 5, 9], the algorithms given in [46, 17, 18] usually need to solve two systems of linear equations and to do two or three line searches at each iteration. It should also be pointed out that the methods given in [46, 17, 18] depend strongly on the strict complementarity condition.

By exploiting a so-called Jacobian consistency property for smoothing functions, Chen, Qi, and Sun [14] designed a class of globally and locally superlinearly convergent smoothing

Newton methods for NCPs with a nonsingularity condition, but without any strict complementarity condition. Some modifications were made in [15] and [29]. Deviating from [14], Qi, Sun, and Zhou [38] proposed a class of new smoothing methods for solving NCPs and box constrained variational inequality problems. The Qi-Sun-Zhou (QSZ) method treats the smoothing parameter as a free variable and solves one system of linear equations at each iteration. Based on the semismoothness of smoothing functions, the QSZ method was proved to possess fast local convergence under a nonsingularity assumption [38]. Very encouraging numerical results of this class of methods were reported in [49]. Due to its simplicity and weaker assumptions imposed on smoothing functions, the QSZ method has also been used to deal with other problems [33, 34, 42, 44, 49]. It is worth mentioning that the assumptions used in [14, 15, 33, 34, 38, 42, 49] imply that the solution set is a singleton, but do not imply that the strict complementarity condition holds.

In this paper, we focus on LCPs. Given $M \in \Re^{n\times n}$ and $q \in \Re^n$, the LCP is to find a vector $(x, s) \in \Re^n \times \Re^n$ such that

$$(x, s) \geq 0, \quad s = Mx + q, \quad x^T s = 0\,. \tag{1.1}$$

We shall present a smoothing Newton method for solving (1.1) by assuming M to be a P_0-matrix, i.e., all of its principal minors are nonnegative. This smoothing Newton method is a modified version of the QSZ method [38]. Just as the QSZ method, the new method needs only to solve one system of linear equations and to do one line search at each iteration. By using the regularization technique [19, 33, 42] and the upper semicontinuity property of the inverse of a weakly univalent function [40, Theorem 2.5], we investigate the boundedness of the generated iteration sequence under the assumption that the solution set of (1.1) is nonempty and bounded. Compared to previous smoothing (non-interior continuation) Newton-type methods, the method presented in this paper possesses the following stronger local convergence properties:

- If an accumulation point of the iteration sequence satisfies a nonsingularity condition, which implies that the P_0-LCP has a unique solution, then the whole iteration sequence converges to this accumulation point sub-quadratically with a Q-rate $2 - t$, where $t \in (0, 1)$ is a parameter and can be close to zero as much as wanted.
- If M is a positive semidefinite matrix and an accumulation point of the iteration sequence satisfies a strict complementarity condition, then the whole iteration sequence converges to this accumulation point quadratically. It is worth noting that here only one accumulation point is assumed to satisfy the strict complementarity condition.

To the best of our knowledge, this is the first smoothing (non-interior continuation) method to have the above local convergence properties simultaneously.

The rest of this paper is organized as follows. In the next section, we present a modified smoothing Newton algorithm. We prove its global convergence in Section 3. In Section 4, we show the local sub-quadratic convergence of the algorithm with a nonsingularity condition, but without the strict complementarity condition. In Section 5, we prove the local quadratic convergence of the algorithm without the nonsingularity condition, but with the strict complementarity condition. Some final remarks are made in Section 6.

To help the later discussion, we introduce some notation here. All vectors are column vectors, the supscript T denotes transpose, $\Re^n$ denotes the space of $n-$dimensional real column vectors, and $\Re^n_+$ (respectively, $\Re^n_{++}$) denotes the nonnegative (respectively, positive) orthant in $\Re^n$. We denote $\mathcal{I} = \{1, 2, \ldots, n\}$. For any vector u, we denote by u_i the ith component of u and, for any $\mathcal{K} \subset \mathcal{I}$, by $u_\mathcal{K}$ the vector obtained after removing from u those u_i with $i \notin \mathcal{K}$. We also write u as $\mathrm{vec}\{u_i : i \in \mathcal{I}\}$. We denote by $\mathrm{diag}\{u_i : i \in \mathcal{I}\}$ the diagonal matrix whose ith diagonal element is u_i. We denote by $\|u\|$ the 2-norm of u. For any vectors $u, v \in \Re^n$, we write $(u^T, v^T)^T$ as (u, v) for simplicity, and denote by $\min\{u, v\}$ the vector whose ith component is $\min\{u_i, v_i\}$. We denote by $\Re^{n\times n}$ the space of $n\times n$ real matrices. For any $A \in \Re^{n\times n}$ and $\mathcal{K}, \mathcal{L} \in \mathcal{I}$, we denote $A_{\mathcal{K}\mathcal{L}}$ the submatrix of A obtained by removing all rows of A with indices outside of $\mathcal{K}$ and removing all columns of A with indices outside of $\mathcal{L}$. Also, we denote $\|A\| = \max_{u\in\Re^n, \|u\|=1} \|Au\|$. For any continuously differentiable function $g = (g_1, g_2, \ldots, g_m)^T : \Re^m \to \Re^m$, we denote its Jacobian by $g' = (\bigtriangledown g_1, \bigtriangledown g_2, \ldots, \bigtriangledown g_m)^T$, where $\bigtriangledown g_i$ denotes the gradient of g_i for $i = 1, 2, \ldots, m$. We denote by $\mathcal{F}$ and $\mathcal{S}$ the feasible set and the solution set of (1.1), respectively, i.e.,

$$\mathcal{F} := \{(x, s) \in \Re^{2n} : s = Mx + q\}, \quad \mathcal{S} := \{(x, s) \in \mathcal{F} : (x, s) \geq 0, x^T s = 0\}.$$

We denote by $\mathrm{dist}((u, v), \mathcal{S})$ the Euclidean distance of the vector $(u, v) \in \Re^{2n}$ to the solution set $\mathcal{S}$ of (1.1), i.e., $\mathrm{dist}((u, v), \mathcal{S}) = \inf_{(x,s)\in\mathcal{S}} \|(u, v) - (x, s)\|$. For any $\alpha, \beta \in \Re_{++}$, we write $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$) to mean α/β is uniformly bounded (respectively, tends to zero) as $\beta \to 0$. Let $k \geq 0$ denote the iteration index. For any $(\mu, x, s), (\mu_k, x^k, s^k) \in \Re_+ \times \Re^{2n}$, we always use the following notation throughout this paper unless stated otherwise:

$$w := (x, s), \quad w^k := (x^k, s^k), \quad z := (\mu, w) := (\mu, x, s), \quad z^k := (\mu_k, w^k) := (\mu_k, x^k, s^k).$$

2 A Smoothing Newton Algorithm

Let $\phi : \Re^3 \to \Re$ denote the Chen-Harker-Kanzow-Smale (CHKS) [7, 26, 41] smoothing function

$$\phi(\mu, a, b) = a + b - \sqrt{(a - b)^2 + 4\mu^2}$$

and let $\Phi : \Re^{2n+1} \to \Re^n$ be defined by

$$\Phi(z) := \begin{pmatrix} \phi(\mu, x_1, s_1) \\ \vdots \\ \phi(\mu, x_n, s_n) \end{pmatrix}.$$

Then, solving the LCP (1.1) is equivalent to finding a root of the following equation:

$$H(z) := \begin{pmatrix} \mu \\ s - Mx - q \\ \Phi(z) + p(\mu)x \end{pmatrix} = 0\,, \tag{2.1}$$

where $p : \Re \to \Re_+$ is a twice continuously differentiable function which satisfies $p(\mu) > 0$ for $\mu \neq 0$, and

$$p(0) = 0, \quad |p(\mu)| = O(\mu^3), \quad \text{and} \quad |p'(\mu)| = O(\mu^2). \tag{2.2}$$

From (2.1), for any $\mu \neq 0$ a straightforward calculation yields

$$H'(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -M & I \\ d(z) + p'(\mu)x & D(z) + p(\mu)I & E(z) \end{bmatrix}, \tag{2.3}$$

where I denotes the $n \times n$ identity matrix,

$$d(z) : = \text{vec}\left\{-4\mu/\sqrt{(x_i - s_i)^2 + 4\mu^2} : i \in \mathcal{I}\right\}, \tag{2.4}$$

$$D(z) : = \text{diag}\left\{1 - (x_i - s_i)/\sqrt{(x_i - s_i)^2 + 4\mu^2} : i \in \mathcal{I}\right\}, \tag{2.5}$$

$$E(z) : = \text{diag}\left\{1 + (x_i - s_i)/\sqrt{(x_i - s_i)^2 + 4\mu^2} : i \in \mathcal{I}\right\}. \tag{2.6}$$

For any $z := (\mu, w) = (\mu, x, s) \in \Re \times \Re^{2n}$, let

$$\Phi_0(w) := 2\min\{x, s\} \quad \text{and} \quad \xi(w) := \min\{|x_i - s_i| : i \in \mathcal{I}\}. \tag{2.7}$$

The following result on Φ_0 and Φ can be checked easily.

Lemma 2.1 *For any $z = (\mu, w) = (\mu, x, s) \in \Re_+ \times \Re^{2n}$, we have*

$$\|\Phi_0(w) - \Phi(z)\| \leq 2\sqrt{n}\mu. \tag{2.8}$$

Moreover, if $\xi(w) \geq \varepsilon$ for some $\varepsilon > 0$, then there exists a constant $C_1(\varepsilon) > 0$ such that

$$\|\Phi_0(w) - \Phi(z)\| \leq C_1(\varepsilon)\mu^2; \tag{2.9}$$

and if $\xi(w) \geq \kappa\mu^t$ for two constants $t \in (0, 1)$ and $\kappa > 0$, then

$$\|\Phi_0(w) - \Phi(z)\| \leq \frac{2n}{\kappa}\mu^{2-t}. \tag{2.10}$$

Suppose that $\bar{\mu} \in \Re_{++}$ and $\gamma \in (0, 1)$. Define $\psi : \Re^{2n+1} \to \Re_+$ and $\beta : \Re^{2n+1} \to \Re_+$ by

$$\psi(z) := \|H(z)\|^2 \quad \text{and} \quad \beta(z) := \gamma \min\{1, \psi(z)\}, \tag{2.11}$$

respectively. Let $\tau \in (0, 1)$ and $d(\cdot)$ be defined by (2.4). Denote $u : \Re^{2n+1} \to \Re^n$ by

$$u(z) := \begin{cases} \Phi_0(w) - \Phi(z) + \bar{\mu}\beta(z)d(z) & \text{if } \mu \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

and $v : \Re^{2n+1} \to \Re^n$ by

$$v(z) := \begin{cases} \tau\mu e & \text{if } \tau\sqrt{n}\mu \leq \|u(z)\| \\ u(z) & \text{otherwise,} \end{cases} \tag{2.13}$$

where e denotes the n-vector of all ones. Suppose that $t \in (0,1)$ and $\kappa > 0$ are two constants, and that $\xi : \Re^{2n} \to \Re_+$ is defined by (2.7). We now define $\Upsilon : \Re^{2n+1} \to \Re \times \Re^n \times \Re^n$ as follows:

$$\Upsilon(z) := \begin{pmatrix} \bar{\mu}\beta(z) \\ 0 \\ v(z) \end{pmatrix} \text{ if } \xi(w) > \kappa\mu^t; \quad \text{and} \quad \Upsilon(z) := \begin{pmatrix} \bar{\mu}\beta(z) \\ 0 \\ 0 \end{pmatrix} \text{ if } \xi(w) \leq \kappa\mu^t. \tag{2.14}$$

Algorithm 2.1 *(A Smoothing Newton Algorithm)*

Step 0 *Choose $t, \delta, \sigma \in (0,1)$ and $\kappa, \bar{\mu} \in (0, \infty)$. Let $x^0 \in \Re^n$ be an arbitrary vector. Set $\mu_0 := \bar{\mu}$, $s^0 := Mx^0 + q$, and $z^0 := (\mu_0, x^0, s^0)$. Choose $\gamma \in (0,1)$ and $\tau \in (0,1)$ such that $\gamma\mu_0 + \tau\sqrt{n} < 1$. Set $\eta := \gamma\mu_0 + \tau\sqrt{n}$ and $k := 0$.*

Step 1 *If $\|H(z^k)\| = 0$, stop.*

Step 2 *Compute $\Delta z^k := (\Delta\mu_k, \Delta x^k, \Delta y^k) \in \Re \times \Re^n \times \Re^n$ by*

$$H(z^k) + H'(z^k)\Delta z^k = \Upsilon(z^k). \tag{2.15}$$

Step 3 *Let λ_k be the maximum of the values $1, \delta, \delta^2, \cdots$ such that*

$$\psi(z^k + \lambda_k \Delta z^k) \leq [1 - 2\sigma(1-\eta)\lambda_k]\psi(z^k). \tag{2.16}$$

Step 4 *Set $z^{k+1} := z^k + \lambda_k \Delta z^k$ and $k := k+1$. Go to Step 1.*

Remark 2.1 Algorithm 2.1 is a modified version of the QSZ smoothing Newton algorithm developed in [38]. The main feature of Algorithm 2.1 is that we add a term $v(z^k)$ into the perturbed Newton equation (see (2.15) and (2.14)). This modification allows us to prove stronger local convergence properties for Algorithm 2.1. Just as the QSZ algorithm, Algorithm 2.1 needs only to solve one system of linear equations and to do one line search at each iteration.

Lemma 2.2 *Suppose that M is a P_0-matrix. If $\mu_k > 0$ for some k, then Algorithm 2.1 is well-defined at the k-th step.*

Proof. Since M is a P_0-matrix, it is not difficult to show from (2.3) that the Jacobian matrix $H'(z^k)$ is nonsingular if $\mu_k > 0$. This implies that equation (2.15) is solvable. Thus, to show that Algorithm 2.1 is well-defined at the k-th step, it suffices to verify that Step 3 of Algorithm 2.1 is well-defined. Since (2.11) implies $\beta(z^k) \leq \gamma\psi(z^k)^{1/2}$ and (2.13) implies $\|v(z^k)\| \leq \tau\sqrt{n}\mu_k$, from (2.14) we have either $\|\Upsilon(z^k)\| \leq \mu_0\gamma\psi(z^k)^{1/2} + \tau\sqrt{n}\mu_k \leq \eta\psi(z^k)^{1/2}$ or $\|\Upsilon(z^k)\| \leq \mu_0\gamma\psi(z^k)^{1/2} < \eta\psi(z^k)^{1/2}$. Thus, similar to Lemma 5 in [38] we can obtain that there exists a constant $\bar{\alpha} \in (0,1)$ such that

$$\psi(z^k + \alpha\Delta z^k) \leq [1 - 2\sigma(1-\eta)\alpha]\psi(z^k)$$

holds for any $\alpha \in (0, \bar{\alpha}]$. This demonstrates that (2.16) is well-defined. The proof is completed. □

3 Global Convergence

Denote $\Omega := \{z \in \Re \times \Re^{2n} : w \in \mathcal{F}, \mu \geq \mu_0 \beta(z)\}$.

Lemma 3.1 *Suppose that M is a P_0-matrix. Then Algorithm 2.1 generates an infinite iteration sequence $\{z^k\}$ with $\mu_k > 0$ and $z^k \in \Omega$ for any $k \geq 0$.*

Proof. We only need to show that $w^k \in \mathcal{F}$ holds for all $k \geq 0$ because other results can be obtained as in Lemma 5 and Proposition 6 of [38]. Obviously, $w^0 \in \mathcal{F}$. Assume that $w^{k-1} \in \mathcal{F}$ for some $k \geq 1$, i.e., $s^{k-1} = Mx^{k-1} + q$. By equation (2.15), we have

$$-M\Delta x^{k-1} + \Delta s^{k-1} = -(s^{k-1} - Mx^{k-1} - q)$$

which, implies $\Delta s^{k-1} = M\Delta x^{k-1}$. Hence,

$$\begin{aligned} s^k &= s^{k-1} + \lambda_{k-1}\Delta s^{k-1} = Mx^{k-1} + q + \lambda_{k-1} M\Delta x^{k-1} \\ &= M(x^{k-1} + \lambda_{k-1}\Delta x^{k-1}) + q = Mx^k + q\,. \end{aligned}$$

This proves $w^k \in \mathcal{F}$. □

Theorem 3.1 *Suppose that M is a P_0-matrix. Then Algorithm 2.1 generates an infinite iteration sequence $\{z^k\}$ with $\lim_{k\to\infty} \psi(z^k) = 0$. In particular, any accumulation point of $\{z^k\}$ is a solution of $H(z) = 0$.*

Proof. By using Lemma 3.1, we can prove the above theorem similarly as in Theorem 4.1 of [42]. We omit the details here for brevity. □

Theorem 3.1 shows that if there exists an accumulation point z^* of $\{z^k\}$, then z^* is a solution of (1.1). This does not mean that $\{z^k\}$ has an accumulation point. In order to assure that $\{z^k\}$ has an accumulation point, we need the following assumption.

Assumption 3.1 *The solution set of* (1.1) *is nonempty and bounded.*

It is well known that for the monotone nonlinear complementarity problem, Assumption 3.1 is equivalent to that the problem has a strictly feasible solution [30, 31]. The latter has been used extensively in interior point methods for the monotone complementarity problem. It is also known that Assumption 3.1 is weaker than those required by most existing smoothing (non-interior continuation) methods [24]. In addition, Assumption 3.1 has been used in regularized smoothing algorithms [25, 33, 42].

Theorem 3.2 *Suppose that M is a P_0-matrix and Assumption 3.1 is satisfied. Then the infinite sequence $\{z^k\}$ generated by Algorithm 2.1 is bounded and any accumulation point of $\{z^k\}$ is a solution of $H(z) = 0$.*

Proof. It is not difficult to show that the function $H : \Re^{2n+1} \to \Re^{2n+1}$ defined by (2.1) is a weakly univalent function (see [21]). Since Assumption 3.1 implies that the inverse image $H^{-1}(0)$ is nonempty and bounded, by using Theorem 2.5 in [40] we obtain that the sequence $\{z^k\}$ is bounded, and hence, by Theorem 3.1, any accumulation point of $\{z^k\}$ is a solution of $H(z) = 0$. □

4 Sub-Quadratic Convergence under Nonsingularity

Let $z^* := (\mu_*, w^*) := (\mu_*, x^*, s^*) \in \Re_+ \times \Re^{2n}$ be an accumulation point of the iteration sequence generated by Algorithm 2.1. Then Theorem 3.1 implies that $\mu_* = 0$ and (x^*, s^*) is a solution of (1.1). In this section, we consider the case that (x^*, s^*) satisfies a nonsingularity condition but may not satisfy the strict complementarity condition. In order to discuss the local superlinear convergence of the algorithm, we need the concept of semismoothness, which was originally introduced by Mifflin [32] for functionals. Qi and Sun [39] extended the definition of semismoothness to vector valued functions. A locally Lipschitz function $F : \Re^{m_1} \to \Re^{m_2}$, which has the generalized Jacobian $\partial F(x)$ in the sense of Clarke [16], is said to be semismooth at $x \in \Re^{m_1}$, if

$$\lim_{\substack{V \in \partial F(x+th') \\ h' \to h, t \downarrow 0}} \{Vh'\}$$

exists for any $h \in \Re^{m_1}$. F is said to be strongly semismooth at x if F is semismooth at x and for any $V \in \partial F(x+h), h \to 0$, it follows that

$$F(x+h) - F(x) - Vh = O(\|h\|^2).$$

Lemma 4.1 *Suppose that M is a P_0-matrix. Let $t \in (0,1)$ be given as in Algorithm 2.1 and the sequence $\{z^k\}$ be generated by Algorithm 2.1. Then, for all k sufficiently large,*

$$\|\Upsilon(z^k)\| = O(\|H(z^k)\|^{2-t}).$$

Proof. Since the infinite sequence $\{z^k\}$ is generated by Algorithm 2.1, it follows from Theorem 3.1 that $\lim_{k\to\infty} \psi(z^k) = 0$. This, together with the definition of $\beta(z^k)$, implies that $\beta(z^k) = \gamma\psi(z^k)$ holds for all k sufficiently large. For any k, we have either $\xi(w^k) \le \kappa(\mu_k)^t$ or $\xi(w^k) > \kappa(\mu_k)^t$. For the former case, we have for all k sufficiently large that

$$\|\Upsilon(z^k)\| = \mu_0\beta(z^k) = \mu_0\gamma\psi(z^k) = O(\|H(z^k)\|^2)\,.$$

For the latter case, since it follows from (2.10), (2.4), and Lemma 3.1 that

$$\begin{aligned}
\|u(z^k)\| &\le \|\Phi_0(w^k) - \Phi(z^k)\| + \mu_0\beta(z^k)\|d(z^k)\| \\
&\le \frac{2n}{\kappa}(\mu_k)^{2-t} + \mu_0\beta(z^k)\|d(z^k)\| \\
&= \frac{2n}{\kappa}(\mu_k)^{2-t} + \left\|\mathrm{vec}\left\{-4\mu_k/\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} : i \in \mathcal{I}\right\}\right\| \mu_0\beta(z^k) \\
&= O((\mu_k)^{2-t}),
\end{aligned}$$

by (2.13) we obtain that $v(z^k) = u(z^k)$. Hence, for all k sufficiently large,

$$\|\Upsilon(z^k)\| = \sqrt{(\mu_0\beta(z^k))^2 + \|u(z^k)\|^2} = O(\|H(z^k)\|^{2-t}).$$

The proof is completed. $\square$

By using Lemma 4.1, we can obtain the following sub-quadratic convergence result. Its proof can be done in a similar way as Theorem 8 in [38].

Theorem 4.1 *Suppose that M is a P_0-matrix and Assumption 3.1 is satisfied. Suppose that z^* is an accumulation point of the sequence $\{z^k\}$ generated by Algorithm 2.1. If all $V \in \partial H(z^*)$ are nonsingular, then the whole iteration sequence $\{z^k\}$ converges to z^*,*

$$\|z^{k+1} - z^*\| = O(\|z^k - z^*\|^{2-t}) \quad \text{and} \quad \mu_{k+1} = O((\mu_k)^{2-t}).$$

In Theorem 4.1, for the sub-quadratic convergence, all $V \in \partial H(z^*)$ are assumed to be nonsingular. It is noticed here that this nonsingularity condition implies that the solution set is a singleton, but not necessarily a strict complementary solution. For conditions to guarantee this nonsingularity assumption, see [38]. In the next section, under a strict complementarity condition, we shall discuss the quadratic convergence of our method with multiple solutions.

5 Quadratic Convergence under Strict Complementarity

In this section, we shall discuss the rate of convergence of Algorithm 2.1 without assuming the nonsingularity condition, but with a strict complementarity condition and M being a positive semidefinite matrix. Let $z^* := (\mu_*, w^*) := (\mu_*, x^*, s^*) \in \Re_+ \times \Re^{2n}$ be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. Then Theorem 3.1 says that $\mu_* = 0$ and (x^*, s^*) is a solution of (1.1). In this section, we consider the case that (x^*, s^*) satisfies the strict complementarity condition $x^* + s^* > 0$. Let

$$\mathcal{B} := \{i \in \mathcal{I} : x_i^* > 0\} \quad \text{and} \quad \mathcal{N} := \{i \in \mathcal{I} : s_i^* > 0\}.$$

Since (x^*, s^*) is a strict complementary solution of (1.1), it follows that $\mathcal{B} \cup \mathcal{N} = \mathcal{I}$ and $\mathcal{B} \cap \mathcal{N} = \emptyset$.

For any $w = (x, s) \in \Re^n \times \Re^n$, let

$$G(w) := \begin{pmatrix} s - (Mx + q) \\ 2s_{\mathcal{B}} \\ 2x_{\mathcal{N}} \end{pmatrix} \quad \text{and} \quad \mathcal{S}_0 := \{w \in \Re^{2n} : G(w) = 0\}.$$

Lemma 5.1 *Denote $\varepsilon := \min\{\min_{i\in\mathcal{B}} x_i^*, \min_{i\in\mathcal{N}} s_i^*\}$ and*

$$\Delta(w^*) := \{w = (x, s) \in \Re^n \times \Re^n : \ |x_i - x_i^*| \le \varepsilon/3, \ |s_i - s_i^*| \le \varepsilon/3, \quad i \in \mathcal{I}\}.$$

Then, for any $w \in \Delta(w^*) \cap \mathcal{F}$, there exists a constant $\lambda > 0$ such that

$$\|\Phi_0(w)\| = \|G(w)\| \geq \lambda \mathrm{dist}(w, \mathcal{S}_0), \tag{5.1}$$

where $\Phi_0(\cdot)$ is defined by (2.7).

Proof. It is obvious that $G(w) = 0$ is solvable since $G(w^*) = 0$. Hence $\mathcal{S}_0 \neq \emptyset$. By Hoffman's error bound result for a linear system [22], there exists a positive number $\lambda > 0$ such that for any $w \in \Re^{2n}$,

$$||G(w)|| \geq \lambda \mathrm{dist}(w, \mathcal{S}_0). \tag{5.2}$$

For any $w \in \Delta(w^*)$, we have

$$\begin{aligned} x_i &= x_i^* + x_i - x_i^* \geq \varepsilon - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon, \quad |s_i| \leq \frac{1}{3}\varepsilon \quad \forall\, i \in \mathcal{B}, \\ s_i &= s_i^* + s_i - s_i^* \geq \varepsilon - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon, \quad |x_i| \leq \frac{1}{3}\varepsilon \quad \forall\, i \in \mathcal{N}. \end{aligned}$$

Hence, $\|\Phi_0(w)\| = \|G(w)\|$ holds for all $w \in \Delta(w^*) \cap \mathcal{F}$. This and (5.2) imply (5.1). □

Lemma 5.1 indicates that if $w \in \mathcal{S}_0$ with w sufficiently close to $w^* := (x^*, s^*)$, then w solves (1.1), i.e., $w \in \mathcal{S}$. Furthermore, from (5.1), there exists a constant $\bar{\rho} > 0$ such that

$$\mathrm{dist}(w, \mathcal{S}_0) \leq \bar{\rho}\|\Phi_0(w)\| \tag{5.3}$$

for all $w \in \mathcal{F}$ sufficiently close to w^*. Noting that $\max\{\mu_k, \|\Phi(z^k)\|\} \leq \|H(z^k)\|$, we can obtain from (2.8), (5.3) and Lemma 3.1 that for all z^k sufficiently close to z^*,

$$\mathrm{dist}(w^k, \mathcal{S}_0) \leq \bar{\rho}(\|\Phi(z^k)\| + \|\Phi_0(w^k) - \Phi(z^k)\|) \leq C_2\|H(z^k)\|, \tag{5.4}$$

where $C_2 := \bar{\rho}(1 + 2\sqrt{n})$.

Lemma 5.2 *Suppose that M is a P_0-matrix and Assumption 3.1 is satisfied. Let z^* be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. If (x^*, s^*) satisfies the strict complementarity condition $x^* + s^* > 0$, then $v(z^k) = u(z^k)$ holds for all z^k sufficiently close to z^*, where $v(\cdot)$ and $u(\cdot)$ are defined by* (2.13) *and* (2.12), *respectively.*

Proof. Since the strict complementarity condition holds, i.e., $x^* + s^* > 0$, it is easy to see that there exists a constant $\varepsilon > 0$ such that $|x_i^k - s_i^k| \geq \varepsilon$ holds for all $i \in \mathcal{I}$ and all z^k sufficiently close to z^*. Thus, by combining (2.12) with (2.9), (2.4), and Lemma 3.1, we have for all z^k sufficiently close to z^* that

$$\begin{aligned} \|u(z^k)\| &\leq \|\Phi_0(w^k) - \Phi(z^k)\| + \|d(z^k)\mu_0\beta(z^k)\| \\ &\leq C_1(\varepsilon)(\mu_k)^2 + \left\|\mathrm{vec}\left\{-4\mu_k/\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} : i \in \mathcal{I}\right\}\right\| \mu_0\beta(z^k) \\ &\leq \gamma\sqrt{n}\mu_k, \end{aligned} \tag{5.5}$$

where the last inequality used the fact that μ_k is sufficiently close to 0. Therefore, (2.13) and (5.5) yield the desired result. □

In the following lemma, by using Lemma 5.1 and the techniques in [46, 17], we shall prove for monotone LCPs that the term $\|\Delta z^k\|$ is of the same order as $\|H(z^k)\|$ for all z^k sufficiently close to a strict complementary solution.

Lemma 5.3 *Suppose that M is a positive semidefinite matrix and Assumption 3.1 is satisfied. Let z^* be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. If (x^*, s^*) satisfies the strict complementarity condition $x^* + s^* > 0$, then there exists a constant $C_3 > 0$ such that for all z^k sufficiently close to z^*,*

$$\|\Delta z^k\| \leq C_3 \|H(z^k)\|. \tag{5.6}$$

Proof. By making use of the fact that $v(z^k) = u(z^k)$ (by Lemma 5.2) and $w^k \in \mathcal{F}$ (by Lemma 3.1), we obtain from (2.15) that for all z^k sufficiently close to z^*,

$$\Delta\mu_k = -\mu_k + \mu_0\beta(z^k), \tag{5.7}$$

$$-M\Delta x^k + \Delta s^k = 0, \tag{5.8}$$

$$\begin{aligned}(d(z^k) + p'(\mu_k)x^k)\Delta\mu_k + (D(z^k) + p(\mu_k)I)\Delta x^k + E(z^k)\Delta s^k \\ = -\Phi(z^k) - p(\mu_k)x^k + u(z^k),\end{aligned} \tag{5.9}$$

where the functions $d(\cdot)$, $D(\cdot)$, $E(\cdot)$, and $u(\cdot)$ are defined by (2.4), (2.5), (2.6), and (2.12), respectively. We prove (5.6) by investigating the following three issues:

Firstly, we estimate $|\Delta\mu_k|$. From (5.7) it follows that for all z^k sufficiently close to z^*,

$$|\Delta\mu_k| \leq \mu_k + \mu_0\beta(z^k) \leq (1 + \mu_0\gamma)\|H(z^k)\|. \tag{5.10}$$

Secondly, we estimate $\|\Delta x^k\|$. For any z^k sufficiently close to z^*, there exists a vector $w^{k*} := (x^{k*}, s^{k*}) \in \mathcal{S}_0$ (and hence $w^{k*} \in \mathcal{S}$ by Lemma 5.1) such that

$$\text{dist}(w^k, \mathcal{S}_0) = \|w^k - w^{k*}\|. \tag{5.11}$$

By using the fact that $w^k, w^{k*} \in \mathcal{F}$, we get from (5.8) and (5.9) that

$$s^k + \Delta s^k - s^{k*} = M(x^k + \Delta x^k - x^{k*}) \tag{5.12}$$

and

$$(D(z^k) + p(\mu_k)I)(x^k + \Delta x^k - x^{k*}) + E(z^k)(s^k + \Delta s^k - s^{k*}) = \alpha(z^k), \tag{5.13}$$

where $\alpha(z^k) := \varpi(z^k) + \varrho(z^k)$ with

$$\varpi(z^k) := (D(z^k) + p(\mu_k)I)(x^k - x^{k*}) + E(z^k)(s^k - s^{k*}) - \Phi_0(w^k) \tag{5.14}$$

and

$$\begin{aligned}\varrho(z^k) :&= \Phi_0(w^k) - \Phi(z^k) + u(z^k) - d(z^k)\Delta\mu_k - \hat{p}(z^k) \\ &= 2(\Phi_0(w^k) - \Phi(z^k)) + d(z^k)\mu_0\beta(z^k) - d(z^k)\Delta\mu_k - \hat{p}(z^k) \\ &= 2(\Phi_0(w^k) - \Phi(z^k)) + d(z^k)\mu_k - \hat{p}(z^k),\end{aligned} \tag{5.15}$$

where $\hat{p}(z^k) := p'(\mu_k)\Delta\mu_k x^k + p(\mu_k)x^k$. From the positive semidefiniteness of M and (5.12), it follows that $(x^k + \Delta x^k - x^{k*})^T(s^k + \Delta s^k - s^{k*}) \geq 0$. Hence, multiplying (5.13) on the left side by $(x^k + \Delta x^k - x^{k*})^T E(z^k)^{-1}$ yields

$$\begin{aligned}
&\min_{i\in\mathcal{I}}(D(z^k)_{ii} + p(\mu_k))E(z^k)_{ii}\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\|^2 \\
&\quad\le (x^k + \Delta x^k - x^{k*})^T E(z^k)^{-1}(D(z^k) + p(\mu_k)I)E(z^k)E(z^k)^{-1}(x^k + \Delta x^k - x^{k*}) \\
&\quad\le (x^k + \Delta x^k - x^{k*})^T E(z^k)^{-1}\alpha(z^k) \\
&\quad\le \|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\|\|\alpha(z^k)\|,
\end{aligned}$$

and so,

$$\min_{i\in\mathcal{I}}(D(z^k)_{ii} + p(\mu_k))E(z^k)_{ii}\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \quad\le\quad \|\alpha(z^k)\|. \tag{5.16}$$

It can be seen easily that there exists a constant $\varepsilon > 0$ such that

$$x_i^k - s_i^k > \varepsilon \text{ for all } i \in \mathcal{B} \quad \text{and} \quad s_i^k - x_i^k > \varepsilon \text{ for all } i \in \mathcal{N} \tag{5.17}$$

hold for all z^k sufficiently close to z^*. Since $w^{k*} \in \mathcal{S}_0$, we have

$$s_i^{k*} = 0 \text{ for all } i \in \mathcal{B} \quad \text{and} \quad x_i^{k*} = 0 \text{ for all } i \in \mathcal{N}. \tag{5.18}$$

By using (5.17), (5.18), and the fact that $D(z^k) + E(z^k) = 2I$ for all k (by (2.5) and (2.6)), we obtain from (5.14) that for all z^k sufficiently close to z^*,

$$\begin{aligned}
\varpi(z^k)_{\mathcal{B}} &= (D(z^k)_{\mathcal{B}\mathcal{B}} + p(\mu_k)I_{\mathcal{B}\mathcal{B}})(x_{\mathcal{B}}^k - x_{\mathcal{B}}^{k*}) + E(z^k)_{\mathcal{B}\mathcal{B}}(s_{\mathcal{B}}^k - s_{\mathcal{B}}^{k*}) - 2s_{\mathcal{B}}^k \\
&= D(z^k)_{\mathcal{B}\mathcal{B}}(x_{\mathcal{B}}^k - x_{\mathcal{B}}^{k*} - s_{\mathcal{B}}^k + s_{\mathcal{B}}^{k*}) + p(\mu_k)(x_{\mathcal{B}}^k - x_{\mathcal{B}}^{k*}),
\end{aligned}$$

and similarly,

$$\varpi(z^k)_{\mathcal{N}} = E(z^k)_{\mathcal{N}\mathcal{N}}(s_{\mathcal{N}}^k - s_{\mathcal{N}}^{k*} - x_{\mathcal{N}}^k + x_{\mathcal{N}}^{k*}) + p(\mu_k)(x_{\mathcal{N}}^k - x_{\mathcal{N}}^{k*}).$$

Hence, for all z^k sufficiently close to z^*,

$$\|\varpi(z^k)\| \le [\max\{\|D(z^k)_{\mathcal{B}\mathcal{B}}\|, \|E(z^k)_{\mathcal{N}\mathcal{N}}\|\} + p(\mu_k)](\|x^k - x^{k*}\| + \|s^k - s^{k*}\|).$$

For any $k \ge 0$, let $r_k = \min_{i\in\mathcal{I}}(D(z^k)_{ii} + p(\mu_k))E(z^k)_{ii}$. By (2.2), (2.6), and (5.17) it follows that for $i \in \mathcal{I}$ and all z^k sufficiently close to z^*,

$$p(\mu_k)E(z^k)_{ii} = p(\mu_k)\left(1 + (x_i^k - s_i^k)/\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}\right) = O((\mu_k)^3).$$

Then there exists a constant $r > 0$ such that for all z^k sufficiently close to z^*,

$$r_k = \min_{i\in\mathcal{I}} \frac{4(\mu_k)^2}{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + O((\mu_k)^3) \ge r(\mu_k)^2, \tag{5.19}$$

which implies $p(\mu_k)/r_k = O(\mu_k)$. Thus, from (5.17) and (5.19) we have for $j \in \mathcal{B}$ that

$$\begin{aligned}
\frac{D(z^k)_{jj}}{r_k} &\le \frac{1 - \frac{x_j^k - s_j^k}{\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2}}}{r(\mu_k)^2} = \frac{\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2} - (x_j^k - s_j^k)}{r(\mu_k)^2\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2}} \\
&= \frac{4}{r\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2}\left(\sqrt{(x_j^k - s_j^k)^2 + 4(\mu_k)^2} + (x_j^k - s_j^k)\right)} \\
&= O(1)
\end{aligned}$$

and for any $j \in \mathcal{N}$ that

$$\frac{E(z^k)_{jj}}{r_k} \leq \frac{4}{r\sqrt{(s_j^k - x_j^k)^2 + 4(\mu_k)^2}\left(\sqrt{(s_j^k - x_j^k)^2 + 4(\mu_k)^2} + (s_j^k - x_j^k)\right)} = O(1).$$

Hence, there exists a constant $C_4 > 0$ such that for all z^k sufficiently close to z^*,

$$\|\varpi(z^k)\|/r_k \leq C_4(\|x^k - x^{k*}\| + \|s^k - s^{k*}\|) \leq 2C_2C_4\|H(z^k)\|, \tag{5.20}$$

where the second inequality is due to (5.4) and (5.11). On the other hand, by (5.15) we have

$$\begin{aligned}
\varrho_i(z^k) &= 2\left[\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} - \sqrt{(x_i^k - s_i^k)^2}\right] + d(z^k)_i\mu_k - \hat{p}_i(z^k) \\
&= \frac{8(\mu_k)^2}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + \sqrt{(x_i^k - s_i^k)^2}} - \frac{4(\mu_k)^2}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}} - \hat{p}_i(z^k) \\
&= \frac{4(\mu_k)^2\left[\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} - \sqrt{(x_i^k - s_i^k)^2}\right]}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}\left(\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + \sqrt{(x_i^k - s_i^k)^2}\right)} - \hat{p}_i(z^k) \\
&= \frac{16(\mu_k)^4}{\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2}\left(\sqrt{(x_i^k - s_i^k)^2 + 4(\mu_k)^2} + \sqrt{(x_i^k - s_i^k)^2}\right)^2} - \hat{p}_i(z^k),
\end{aligned}$$

which, together with (5.17) and (5.10), implies that for all z^k sufficiently close to z^*,

$$\frac{|\varrho_i(z^k)|}{r_k} \leq \frac{|\varrho_i(z^k)|}{r(\mu_k)^2} = O((\mu_k)^2) + O(\mu_k) = O(\mu_k) = O(\|H(z^k)\|).$$

Hence, there exists a constant $C_5 > 0$ such that for all z^k sufficiently close to z^*,

$$\|\varrho(z^k)\|/r_k \leq C_5\|H(z^k)\|. \tag{5.21}$$

By combining (5.16) with (5.20) and (5.21), we further obtain that

$$\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \leq \|\varpi(z^k)\|/r_k + \|\varrho(z^k)\|/r_k \leq (2C_2C_4 + C_5)\|H(z^k)\|. \tag{5.22}$$

Since $\|E(z^k)\| \leq 2$, (5.4) and (5.11) imply that for all z^k sufficiently close to z^*,

$$\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \geq [\|\Delta x^k\| - \|x^k - x^{k*}\|]/\|E(z^k)\| \geq \|\Delta x^k\| - C_2\|H(z^k)\|/2.$$

This, together with (5.22), implies that for all z^k sufficiently close to z^*,

$$\|\Delta x^k\| = O(\|H(z^k)\|). \tag{5.23}$$

Thirdly, we estimate $\|\Delta s^k\|$. From (5.13) and (5.20)–(5.22) we have that for all z^k sufficiently close to z^*,

$$\begin{aligned}
\|s^k + \Delta s^k - s^{k*}\| &\leq \|[D(z^k) + p(\mu_k)I]^{-1}E(z^k)^{-1}\|\|D(z^k) + p(\mu_k)I\|\|\alpha(z^k)\| \\
&\quad + \|D(z^k) + p(\mu_k)I\|\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \\
&\leq 3\frac{\|\varpi(z^k)\| + \|\varrho(z^k)\|}{\min_{i\in\mathcal{I}}(D(z^k)_{ii} + p(\mu_k))E(z^k)_{ii}} + 3\|E(z^k)^{-1}(x^k + \Delta x^k - x^{k*})\| \\
&\leq 6(2C_2C_4 + C_5)\|H(z^k)\|,
\end{aligned} \tag{5.24}$$

where the second inequality is due to $\|D(z^k) + p(\mu_k)I\| \leq 2 + p(\mu_k)$ and $\alpha(z^k) = \varpi(z^k) + \varrho(z^k)$ and the last inequality is due to (5.22). Hence, from (5.4) and (5.24) we have that for all z^k sufficiently close to z^*,

$$\|\Delta s^k\| \leq \|s^k + \Delta s^k - s^{k*}\| + \|s^k - s^{k*}\| = O(\|H(z^k)\|). \tag{5.25}$$

Now, by combining (5.10) with (5.23) and (5.25), we obtain that there exists a constant $C_3 > 0$ such that (5.6) holds for all z^k sufficiently close to z^*. □

Lemma 5.4 *Suppose that all the conditions assumed in Lemma 5.3 are satisfied. Then there exists a constant $C_6 > 0$ such that for all z^k sufficiently close to z^*, $z^{k+1} = z^k + \Delta z^k$ and*

$$\|H(z^{k+1})\| \leq C_6 \|H(z^k)\|^2. \tag{5.26}$$

Proof. From Lemma 1.2 in [36] we have

$$\|\phi''(\mu, a, b)\| \leq 4/\sqrt{(a-b)^2 + 4\mu^2}$$

for any $(\mu, a, b) \in \Re_+ \times \Re^{2n}$ and $(a-b)^2 + 4\mu^2 \neq 0$. For any $k \geq 0$ and $i \in \mathcal{I}$, let $y_i^k := (\mu_k, x_i^k, s_i^k)$ and $\Delta y_i^k := (\Delta\mu_k, \Delta x_i^k, \Delta s_i^k)$. Then by using the strict complementarity condition we can obtain a positive number $\bar{C}$ such that $\|\phi''(y_i^k)\| \leq \bar{C}$ for all z^k sufficiently close to z^*. Hence, from Lemma 5.3 we have that for all z^k sufficiently close to z^*,

$$|\phi(y_i^k + \Delta y_i^k) - \phi(y_i^k) - \phi'(y_i^k)\Delta y_i^k| \leq \int_0^1 t \int_0^1 \|\phi''(y_i^k + ts\Delta y_i^k)\| ds dt \|\Delta y_i^k\|^2 \leq \bar{C}\|\Delta y_i^k\|^2.$$

Furthermore, from the definition of $H(\cdot)$ (see (2.1)), it is not difficult to show that there exists a constant $C_7 > 0$ such that for all z^k sufficiently close to z^*,

$$\|H(z^k + \Delta z^k) - H(z^k) - H'(z^k)\Delta z^k\| \leq C_7 \|\Delta z^k\|^2. \tag{5.27}$$

Hence, from Lemmas 5.2 and 5.3, (5.27) and (2.15) we have that for all z^k sufficiently close to z^*,

$$\begin{aligned} \|H(z^k + \Delta z^k)\| &= \|(H(z^k + \Delta z^k) - H(z^k) - H'(z^k)\Delta z^k + H(z^k) + H'(z^k)\Delta z^k\| \\ &\leq \|H(z^k + \Delta z^k) - H(z^k) - H'(z^k)\Delta z^k\| + \|H(z^k) + H'(z^k)\Delta z^k\| \\ &\leq C_7\|\Delta z^k\|^2 + \sqrt{(\gamma\mu_0\|H(z^k)\|^2)^2 + \|u(z^k)\|^2} \\ &\leq C_7(C_3)^2\theta(z^k)^2 + \sqrt{(\gamma\mu_0\|H(z^k)\|^2)^2 + \|u(z^k)\|^2}. \end{aligned} \tag{5.28}$$

Lemma 2.1 and the strict complementarity condition imply that there exists a constant $C_8 > 0$ such that for all z^k sufficiently close to z^*,

$$\|u(z^k)\| \leq C_8(\mu_k)^2 + n\gamma\mu_0\|H(z^k)\|^2,$$

which, together with (5.28), implies that there exists a constant $C_6 > 0$ such that for all z^k sufficiently close to z^*,

$$\|H(z^k + \Delta z^k)\| \leq C_6\|H(z^k)\|^2.$$

Hence, by the definition of $\psi(\cdot)$, $\psi(z^k + \Delta z^k) \leq C_6^2 \psi(z^k)^2$, i.e., . (5.26) holds, for all z^k sufficiently close to z^*. This further implies that $z^{k+1} = z^k + \Delta z^k$ for all z^k sufficiently close to z^*. The proof is completed. □

For any $\varepsilon > 0$, define $N(z^*, \varepsilon) := \{z \in \Re_+ \times \Re^{2n} : \|z - z^*\| \leq \varepsilon\}$. Since H is locally Lipschitz continuous around z^*, it follows that $\|H(y^1) - H(y^2)\| \leq \mathcal{L}\|y^1 - y^2\|$ for some constant $\mathcal{L} > 0$ and any $y^1, y^2 \in N(z^*, \varepsilon)$. Let

$$\bar{\varepsilon} := \min\{\varepsilon/[2 + 4C_3\mathcal{L}], \quad 1/[2C_6\mathcal{L}]\}, \tag{5.29}$$

where C_3 and C_6 are given by (5.6) and (5.26), respectively.

The following lemma, which to some extent is motivated by a result in Yamashita and Fukushima [48] on the Levenberg-Marquardt method, is on the convergence of the whole iteration sequence $\{z^k\}$.

Lemma 5.5 *Suppose that all the conditions assumed in Lemma 5.3 are satisfied. Let $\bar{\varepsilon}$ be defined by (5.29). If for some k the iterate $z^k \in N(z^*, \bar{\varepsilon})$ and ε is sufficiently small, then $z^{k+q} \in N(z^*, \varepsilon/2)$ for all $q = 0, 1, 2, \ldots$ and $\{z^{k+q}\}_{q=1}^{\infty}$ is a convergent sequence.*

Proof. Suppose that for some k, $z^k \in N(z^*, \bar{\varepsilon})$. Then, it is obvious that $z^k \in N(z^*, \bar{\varepsilon}) \subseteq N(z^*, \varepsilon/2)$ by (5.29). By reducing ε if necessary, we have from Lemmas 5.4 and 5.3 and (5.29) that

$$\begin{aligned} \|z^{k+1} - z^*\| &= \|z^k + \Delta z^k - z^*\| \leq \|z^k - z^*\| + \|\Delta z^k\| \leq \bar{\varepsilon} + C_3\|H(z^k)\| \\ &= \bar{\varepsilon} + C_3\|H(z^k) - H(z^*)\| \leq \bar{\varepsilon} + C_3\mathcal{L}\|z^k - z^*\| \\ &\leq (1 + C_3\mathcal{L})\bar{\varepsilon} \leq \varepsilon/2, \end{aligned}$$

which implies $z^{k+1} \in \mathcal{N}(z^*, \varepsilon/2)$. Suppose that for some $q \geq 1$, $z^{k+1}, \cdots, z^{k+q} \in N(z^*, \varepsilon/2)$. We now show $z^{k+q+1} \in \mathcal{N}(z^*, \varepsilon/2)$. From Lemma 5.4, we have

$$\begin{aligned} \|H(z^{k+q})\| &\leq C_6\|H(z^{k+q-1})\|^2 \leq \cdots \leq (C_6)^{2^q-1}\|H(z^k)\|^{2^q} \\ &= (C_6)^{2^q-1}\|H(z^k) - H(z^*)\|^{2^q} \leq (C_6)^{2^q-1}\mathcal{L}^{2^q}\|z^k - z^*\|^{2^q} \\ &\leq (C_6)^{2^q-1}\mathcal{L}^{2^q}\left(\tfrac{1}{2C_6\mathcal{L}}\right)^{2^q-1}\bar{\varepsilon} = \mathcal{L}\bar{\varepsilon}(1/2)^{2^q-1} \leq \mathcal{L}\bar{\varepsilon}(1/2)^{2q-1}, \end{aligned} \tag{5.30}$$

which, together with (5.6), implies that

$$\sum_{i=1}^{q}\|\Delta z^{k+i}\| \leq C_3\sum_{i=1}^{q}\|H(z^{k+i})\| \leq C_3\mathcal{L}\bar{\varepsilon}\sum_{i=1}^{q}(1/2)^{2i-1} \leq C_3\mathcal{L}\bar{\varepsilon}\sum_{i=1}^{\infty}(1/2)^i = C_3\mathcal{L}\bar{\varepsilon}.$$

This further leads to

$$\|z^{k+q+1} - z^*\| \leq \|z^{k+1} - z^*\| + \sum_{i=1}^{q}\|\Delta z^i\| \leq (1 + C_3\mathcal{L})\bar{\varepsilon} + C_3\mathcal{L}\bar{\varepsilon} \leq \varepsilon/2.$$

Therefore, $z^{k+q+1} \in \mathcal{N}(z^*, \varepsilon/2)$.

From (5.30) and Lemma 5.4 we know that $\|\Delta z^{k+q}\| \le C_3 \mathcal{L} \bar{\varepsilon} (1/2)^{2q-1}$ holds for all $q \ge 0$. Hence, for any positive integers l, m with $l \ge m \ge q$,

$$\|z^l - z^m\| \le \sum_{i=m}^{l-1} \|\Delta z^i\| \le \sum_{i=m}^{\infty} \|\Delta z^i\| \le C_3 \mathcal{L} \bar{\varepsilon} \sum_{i=m}^{\infty} (1/2)^{2i-1} = \frac{1}{3} C_3 \mathcal{L} \bar{\varepsilon} (1/2)^{2m-3},$$

which indicates that the sequence $\{z^k\}$ is a Cauchy sequence. This implies that $\{z^k\}$ is a convergent sequence. The proof is completed. □

Theorem 5.1 *Suppose that M is a positive semidefinite matrix and Assumption 3.1 is satisfied. Let z^* be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 2.1. If (x^*, s^*) satisfies the strict complementarity condition $x^* + s^* > 0$, then the whole sequence $\{z^k\}$ generated by Algorithm 2.1 converges to z^*,*

$$\|H(z^{k+1})\| = O\left(\|H(z^k)\|^2\right) \quad \text{and} \quad \mu_{k+1} = O\left((\mu_k)^2\right).$$

Proof. This theorem follows directly from Lemmas 5.4 and 5.5. □

It is worth pointing out that in Theorem 5.1 only one accumulation point of the iteration sequence is assumed to satisfy the strict complementarity condition and the whole sequence is proved to converge to this accumulation point while in [46] all accumulation points are assumed to satisfy the strict complementarity condition uniformly and in [17, 18] the whole sequence is assumed to converge to a strict complementary solution. This indicates that even in this case our results are stronger than those in [46, 17, 18].

6 Some Final Remarks

It has been tested in [49] that the QSZ smoothing algorithm [38] performs very efficiently in practice for solving complementarity problems. In this paper, we have shown that a modified version of the QSZ smoothing algorithm for the P_0 and monotone LCP has better convergence properties than those appeared in [38] and [46, 17, 18]. Specifically, it is shown that the new method converges globally and i) locally sub-quadratically if M is a P_0 matrix and the LCP satisfies a nonsingularity condition; and ii) locally quadratically if M is positive semidefinite and there is an accumulation point that satisfies a strict complementarity condition.

We implemented Algorithm 2.1 for LCPs in Matlab in order to see the behavior of the new smoothing method. Theoretically, the new algorithm has better guaranteed convergence properties than the QSZ smoothing algorithm. Numerically, our testing shows that the new algorithm behaves quite similarly to the QSZ smoothing algorithm in practice. As an example, we considered the following convex quadratic programming problem:

$$\min \ c^T \bar{x} + \frac{1}{2} \bar{x}^T G \bar{x} \qquad \text{s.t.} \ \ A\bar{x} \le b, \ \bar{x} \ge 0\,, \tag{6.1}$$

where $c \in \Re^n, A \in \Re^{m\times n}, b \in \Re^m$, and $G \in \Re^{n\times n}$ is symmetric positive semidefinite. Then (6.1) is equivalent to the following LCP

$$s = Mx + q, \quad x^T s = 0, \quad (x, s) \geq 0,$$

with

$$s = \begin{pmatrix} y \\ v \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -A \\ A^T & G \end{pmatrix}, \quad x = \begin{pmatrix} u \\ \bar{x} \end{pmatrix}, \quad q = \begin{pmatrix} b \\ c \end{pmatrix}.$$

For any positive integers n_1 and n_2, let $rand(n_1, n_2)$ denote a matrix by $n_1 \times n_2$ whose each element is randomly chosen in $(0, 1)$. Assume that the problem is given by $A = rand(500, 220), G = BB^T$ and $q = rand(720, 1)$ where $B = rand(220, 200)$. Throughout the computational experiments, the parameters used in Algorithm 2.1 were chosen as $\sigma = 0.0001$, $\delta = 0.5$, $t = 0.9$, $\mu_0 = 100$, and $\kappa = 0.1$. Take $\tau = 1/(10\sqrt{n})$, $\gamma = 0.1 \min\{1, 1/\mu_0\}$, and $\eta = \gamma\mu_0 + \tau\sqrt{n}$. Let the starting point $x^0 = rand(720, 1)$ and set $y^0 := Mx^0 + q$. We used $\|H(z^k)\| \leq 10^{-12}$ as the stopping rule. This problem is tested ten times by using Algorithm 2.1 and the QSZ smoothing algorithm, respectively. The iteration numbers are listed in Table 1.

Table 1: The numerical results for random generated convex quadratic programming

QSZ's Algorithm	20	17	19	19	18	19	18	18	18	19
Algorithm 2.1	19	16	20	17	19	18	17	18	19	18

Table 1 shows that the new algorithm and the QSZ smoothing algorithm behave similarly for the above convex quadratic problem. We also observed similar results for other examples.

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